Defⁿ (bounded) :- Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ and let c be a cluster point of A. We say f is bounded on a nbd of c if there exists a δ-nbd of c, $N_\delta(c)$ and a constant M s.t. $|f(x)| \leq M \ \forall x \in A \cap N_\delta(c)$.

Theorem :- Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$ then f is bounded on some nbd of c.

Proof :- If $l = \lim_{x \to c} f(x)$.

For given $\varepsilon (=1)$, $\exists \ \delta > 0$ s.t. $|f(x) - l| < 1$ if $0 < |x - c| < \delta$, then

$$\Rightarrow |f(x)| < |l| + 1$$

Here $M = |l| + 1$

In case if $c \in A$, we can take

$M = \sup \{ |f(c)|, |l| + 1 \}$

then $|f(x)| \leq M$ for $x \in N_\delta(c) \cap A$.

$\boxed{//}$

Converse of this theorem may not be true i.e., f is a bdd funⁿ on some nbd of c then $\lim_{x \to c} f(x)$ may or may not exists.

e.g., $f(x) = \operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

Range of this funⁿ is a ~~bdd~~ bounded set.

So, in the nbd of 0, it is basically a bounded set.

i.e., $|f(x)| \leq 1 \quad \forall x \in N_\delta(0)$.

But $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist.

Note :- If $f(x)$ is not bounded in some nbd of c, then $\lim_{x \to c} f(x)$ will not exist.

e.g., $\lim_{x \to 0} \frac{1}{x}$ does not exist.

$x_n = \frac{1}{n} \in N_{1/n}(0)$ but $f(x_n) = n$

$\downarrow$ 0 as $n \to \infty$ ∴ $\{n\}$ is a divergent sequence.

Since $\frac{1}{x}$ is unbounded in the nbd of 0, so $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Defⁿ :- Let $A \subseteq \mathbb{R}$ and let f and g be funⁿs defined on A to $\mathbb{R}$. Then we define

$(f \pm g)(x) = f(x) \pm g(x) \quad \forall x \in A$

$(fg)(x) = f(x) g(x) \quad$ "

$(bf)(x) = b f(x) \quad$ where $b \in \mathbb{R}$.

If $h(x) \neq 0$ for $x \in A$ then

$\left(\frac{f}{h}\right)(x) = \frac{f(x)}{h(x)} \quad \forall x \in A$.

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Theorem: Let $A \subseteq \mathbb{R}$, let f and g are fun^ns from A to $\mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster pt. of A. Further let $b \in \mathbb{R}$, then

(a) If $\lim_{x \to c} f = l$ i.e., $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} g = l$

then $\lim_{x \to c} (f \pm g) = l \pm m$

$\lim_{x \to c} (fg) = lm$ & $\lim_{x \to c} (bf) = bl$.

(b) If $h: A \to \mathbb{R}$, if $h(x) \neq 0 \ \forall x \in A$, and if— $\lim_{x \to c} h = H \neq 0$ then $\lim_{x \to c} \frac{f}{h} = \frac{L}{H}$.

Note:- Let $A \subseteq \mathbb{R}$. Let $f_1, \ldots, f_m$ be fun^ns on A to $\mathbb{R}$ and having limiting values, say $l_1, l_2, \ldots, l_m$ respectively at c, cluster pt. of A.

Then $\lim_{x \to c} (f_1 + f_2 + \cdots + f_m) = l_1 + l_2 + \cdots + l_m$.

$\lim_{x \to c} (f_1 f_2 \cdots f_m) = (l_1 l_2 \cdots l_m)$

Ex:- 1. Find $\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1}$ ← $f(x)$

↖ $h(x)$

$h(x) \neq 0$ in the nbd of 2

$\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{2^3 - 4}{2^2 + 1} = 4/5$

2. $\lim_{x \to 2} \frac{x^2 - 4}{3x - 6}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 2} \frac{(x-2)(x+2)}{3(x-2)} = \lim_{x \to 2} \frac{x+2}{3} = \frac{4}{3}$

3. If p is a polynomial of degree n
$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, $x \in \mathbb{R}$
$a_0, a_1, \ldots, a_n \in \mathbb{R}$.
Then $\lim_{x \to c} p(x) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0$
$= p(c)$

4. If $p(x)$ and $q(x)$ are polynomial fun^n on $\mathbb{R}$ and if $q(c) \neq 0$, then
$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

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