Partial Differential Equations

1

Defn: An equation involving more than one independent variables and one dependent variable with its partial derivatives is called a partial diff. equation (PDE).

Ex: (i) $\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$

(ii) $\frac{\partial^2 z}{\partial x^2} = 2\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x}$.

Order of a PDE:- The order of a pde is the order of the highest partial derivative (or derivatives) occuring in the diff. equation.

Degree of a PDE:- The degree of a pde is the greatest exponent of the highest order derivative involved in the diff. eqn.

Linear PDE:- A linear pde is one which is linear in the dependent variable and all its partial derivatives occuring in the diff. eqn.

Quasi-linear PDE:- A pde which is linear in the highest derivative (or derivatives) occuring in the equation is called a quasi-linear eqn.

Semi-linear PDE:- An almost linear or half-linear or semi-linear pde is a quasi-linear eqn. in which the coefficients of the highest derivatives are functions of the independent variables only.

Non-linear PDE:- A pde is called non-linear if it does not come under the above three types.

Ex:- (i) $xp + yq = z \rightarrow$ pde of first order, linear

(ii) $r + s + 2t = 0 \rightarrow$ pde of second order, linear.

(iii) $(x^2 - y^2 - u^2)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 2xu$

$\hookrightarrow$ pde of first order, quasi-linear.

(iv) $x^2\frac{\partial^2 u}{\partial x^2} + 4xy\frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} + u^2 = 0 \rightarrow$ pde of 2nd order semi-linear.

(v) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1 \rightarrow$ pde of 1st order, non-linear.

* Some important PDEs:—

- Laplace Equation: $\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$.
- Heat Equation: $\nabla^2 w = \frac{1}{\sigma} \frac{\partial w}{\partial t}$
- Wave Equation: $\nabla^2 w = \frac{1}{a^2} \frac{\partial^2 w}{\partial t^2}$

III Construction of PDEs by the process of elimination of arbitrary Constants:-

Let $\phi(x,y,z,a,b)=0$ be a relation between three variables x,y,z and two arbitrary constants a,b. As usual, z is the dependent variable and x,y two independent variables. In order to eliminate a,b we require two other equations besides the given relation $\phi(x,y,z,a,b)=0$ —— ①

Diff. the given relation $\phi=0$ w.r.to x, and resp. y, we Obtain $\phi_x + \phi_z \frac{\partial z}{\partial x} = 0$ —— ②

and $\phi_y + \phi_z \frac{\partial z}{\partial y} = 0$ —— ③

Using the relations of ①, ② and ③, we can eliminate a,b and obtain $F(x,y,z,p,q)=0$, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. This is a pde of order one.

NOTE: ① If the no. of arbitrary constants = the no. of independent variables, then the elimination of arbitrary constants will give rise to a pde of first order.

Ex 1- Eliminate a,b from the relation $z = ax^2 + by^2 + ab$.

⇒ Given relation is $z = ax^2 + by^2 + ab$ —— ①

Diff. ① partially w.r.to x and y, respectively, we get

$$\frac{\partial z}{\partial x} = 2ax \quad \text{and} \quad \frac{\partial z}{\partial y} = 2by$$

$$\therefore a = \frac{p}{2x} \quad \text{and} \quad b = \frac{q}{2y}.$$

Substituting these values in the given relation, we obtain $4xyz = 2px^2y + 2qxy^2 + pq$.

NOTE: ② If the no. of arbitrary constants is less than the no. of independent variables, then elimination of the arbitrary constants will give rise two distinct partial differential equations of first order.

Ex! Eliminate the arbitrary constants 'a' from the given relation $z = a(x+y)$.

⇒ Given relation is $z = a(x+y)$ —①

diff. w.r.to x and y, resp. we have

$$p = a \quad \text{and} \quad q = a.$$

We thus get two distinct pde of order one

$$z = p(x+y) \quad \text{and} \quad z = q(x+y).$$

NOTE:- ③ If the no. of arbitrary constants is more than the no. of independent variables, then on elimination of the constants, a pde (or pdes) of order more than one can be obtained.

Ex! Eliminate the three arbitrary constants a, b, c from the relation $z = ax + by + cxy$.

⇒ Given relation is $z = ax + by + cxy$ —①

diff. w.r.to x and y, resp. we get -

$$p = a + cy \quad —② \qquad \text{and} \qquad q = b + cx \quad —③$$

Again diff. w.r.to x and y, we have

$$r = \frac{\partial p}{\partial x} = \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{and} \quad t = \frac{\partial q}{\partial y} = \frac{\partial^2 z}{\partial y^2} = 0.$$

Also, $\dfrac{\partial p}{\partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = s = c.$

Thus we have - $p = a + sy$ and $q = b + sx$

or, $a = p - sy$ and $b = q - sx$.

put this values in ① we have -

$$z = (p - sy)x + (q - sx)y + sxy$$

$$\text{or, } z = px + qy - sxy$$

∴ the required pdes are $r = 0, t = 0$ and $z = px + qy - sxy$

Construction of PDEs by the process of elimination of arbitrary functions :-

Suppose that u and v are two functions of x, y, z and suppose that there is a relation between u and v expressed

either $\phi(u,v)=0$ or $u=f(v)$ —— (1)

where ϕ and f are arbitrary functions.

Diff (1) partially w.r.to x and y, resp, we get,

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right]=0$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \quad \text{—— (2)}$$

Similarly, $\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$ —— (3)

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from (2) and (3), we get -

$$\frac{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}}{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}}=\frac{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}}{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}$$

or, $p\left[\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}\right]+q\left[\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}-\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial z}\right]=\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}$

, $pP+qQ=R$. where $P=\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}$ etc.

This is the required PDE of (1).

Ex! Eliminate the arbitrary functions f and ϕ from : $y=f(x-at)+\phi(x+at)$.

$\Rightarrow$ Given that $y=f(x-at)+\phi(x+at)$ —— (1)

Diff. (1) partially w.r.to x and t, we get,

$$\frac{\partial y}{\partial x}=f'(x-at)+\phi'(x+at)$$

$$\frac{\partial^2 y}{\partial x^2}=f''(x-at)+\phi'(x+at) \quad \text{—— (2)}$$

and $\frac{\partial y}{\partial t}=-af'(x-at)+a\phi'(x+at)$

$$\frac{\partial^2 y}{\partial t^2}=a^2f''(x-at)+a^2\phi'(x+at) \quad \text{—— (3)}$$

If we eliminate f and ϕ from (2) and (3), then

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}.$$

: Worked out Examples :

(I) Formation of PDE by elimating arbitrary constants :-

Ex.1 Form a PDE by the elimination of the arbitrary constants 'a' from $z = ax + y$.

Sol^n ⇒ Given $z = ax + y$ ——— (1)

diff. (1) partially w.r.to x and y, respectively -

$$\frac{\partial z}{\partial x} = a \quad \text{——— (2)} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 1 \quad \text{——— (3)}$$

Eliminate 'a' from (1) and (2), we have

$$z = x\frac{\partial z}{\partial x} + y$$

$$\therefore \; z = px + y.$$

which is the ~~diff.~~ PDE of (1).

Ex.2 Construct a PDE by eliminating a and p from $z = a e^{-p^2 t} \cos px$.

⇒ Given $z = a e^{-p^2 t} \cos px$ ——— (1)

diff. (1) partially w.r.to x and t, respectively -

$$\frac{\partial z}{\partial x} = -ap\, e^{-p^2 t} \sin px \quad \text{——— (2)}$$

and $$\frac{\partial z}{\partial t} = -ap^2 e^{-p^2 t} \cos px \quad \text{——— (3)}$$

Again diff. (2) w.r.to x, we get

$$\frac{\partial^2 z}{\partial x^2} = -ap^2 e^{-p^2 t} \cos px$$

$$\therefore \; \frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t} \quad [\text{using (3)}].$$

which is the required PDE of (1).

Ex.3 Obtain a PDE by eliminating 'a' and 'b' from $az + b = a^2 x + y$

⇒ Given eq^n is $az + b = a^2 x + y$ ——— (1)

diff. partially w.r.to x and y, we get

$$ap = a^2 \quad \text{——— (2)} \quad \text{and} \quad aq = 1 \quad \text{——— (3)}$$

Eliminating 'a' from (2) and (3) we get $pq = 1$.

Ex: 4 Form a PDE by the elimination of the arbitrary constants a and k from $z = ae^{kt}\sin kx$.

⇒ Given eqn is $z = ae^{kt}\cdot\sin kx$ —— ①

Diff ① partially w.r.to x and y, respectively –

$$\frac{\partial z}{\partial x} = ake^{kt}\cdot\cos kx \quad \text{and} \quad \frac{\partial z}{\partial t} = ake^{kt}\cdot\sin kx$$

Again diff., we get.

$$\frac{\partial^2 z}{\partial x^2} = -ak^2\sin kx\cdot e^{kx} \text{ —— ②} \quad \text{and} \quad \frac{\partial^2 z}{\partial t^2} = ak^2e^{kt}\cdot\sin kx \text{ —— ③}$$

Adding ② and ③, we have –

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0.$$

Ex: 5 Form the PDE by eliminating the arbitrary constants h and k from the eqn $(x-h)^2+(y-k)^2+z^2=\lambda^2$.

⇒ Given eqn is $(x-h)^2+(y-k)^2+z^2=\lambda^2$. —— ①

diff. partially w.r.to x and y, resp. we get.

$$(x-h) + z\frac{\partial z}{\partial x} = 0 \text{ —— ②} \quad \text{and} \quad (y-k) + z\frac{\partial z}{\partial y} = 0 \text{ —— ③}$$

Eliminating h and k from ①, ② and ③, we get.

$$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 + z^2 = \lambda^2$$

$$\Rightarrow z^2(p^2+q^2+1) = \lambda^2.$$

Ex: 6 Find the PDE of the set of all right-circular cones whose axes coincide with z-axis.

⇒ The general eqn of the set of all right-circular cones whose axes coincide with z-axis, having semi-vertical angle α and vertex at $(0,0,c)$ is given by –

$$x^2+y^2=(z-c)^2\tan^2\alpha \text{ —— ①}$$

Diff. partially w.r.to x and y, resp, we get –

$$x = (z-c)\tan^2\alpha\cdot\frac{\partial z}{\partial x} \quad \text{and} \quad y = (z-c)\tan^2\alpha\,\frac{\partial z}{\partial y}$$

$$\Rightarrow xy = y(z-c)\tan^2\alpha\cdot p \quad \text{and} \quad xy = x(z-c)\tan^2\alpha\cdot q$$

If we eliminate α and c from above relations we get –

$$y(z-c)\tan^2\alpha \cdot p = x(z-c)\tan^2\alpha \cdot q$$

or, $yp = xq$.

Which is the required PDE of ①.

Ex! Show that the DE of all cones which have their vertex at the origin is $px+qy=z$, verify that $yz+zx+xy=0$ is a surface satisfying the above equation.

⇒ The equⁿ of a cone with vertex at origin is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0. \quad \text{①}$$

Where a,b,c,f,g,h are arbitrary constants.

Diff. ① partially w.r.to x and y, resp. we have

$$ax+gz+hy+p(cz+gx+fy)=0 \quad \text{②}$$

and

$$by+fz+hx+q(cz+fy+gx)=0 \quad \text{③}$$

Multiplying ② by x and ③ by y and then adding them, we get –

$$(ax^2+by^2+gzx+fyz+2hxy)+(cz+fy+gx)(px+qy)=0$$

or, $-(cz^2+fyz+gzx)+(cz+fy+gx)(px+qy)=0$, [using ①]

or, $(cz+fy+gx)(px+qy-z)=0$

; $px+qy=z$. $\quad [\because cz+fy+gx\neq 0]$

Which is the required PDE.

2nd part ⇒ Given surface is $yz+zx+xy=0 \quad \text{④}$

Diff. partially, w.r.to x and y, resp.

$$yp+px+z+y=0 \quad \text{and} \quad z+qy+xq+x=0$$

$$\therefore p=-\frac{z+y}{x+y} \quad \text{and} \quad q=-\frac{z+x}{x+y}$$

Then, $$px+qy-z=-\left(\frac{z+y}{x+y}\right)x-\left(\frac{z+x}{x+y}\right)y-z$$

$$=-\frac{2(xy+yz+zx)}{x+y}$$

$$=0 \quad [\text{by ④}]$$

Hence ④ is a surface satisfying the PDE.

Ex: Find a PDE by eliminating a, b, c from the family of ellipsoids : $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

$\Rightarrow$ Given eqn is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1)

Diff. (1) partially, w.r.to x and y, resp. we get -

$$c^2x+a^2z.p=0 \text{ —— (2) and } c^2y+b^2z.q=0 \text{ —— (3)}$$

Again diff. w.r.to x and y, resp.

$$c^2+a^2p^2+a^2z.r=0 \text{ —— (4) and } c^2+b^2q^2+b^2z.t=0 \text{ —— (5)}$$

Eliminating 'c' from (2) and (4), we have

$$zxr+xp^2-zp=0 \text{ —— (6)}$$

Similarly, eliminate 'c' from (3) and (5), we have

$$zyt+yq^2-zq=0. \text{ —— (7)}$$

If we diff. (2) partially, w.r.to y, then

$$0+a^2\{q.p+z.s\}=0 \quad [\because s=\frac{\partial^2 z}{\partial x\partial y}]$$

$$\therefore pq+zs=0 \text{ —— (8)}$$

Therefore, the required p.d.e's are obtained in (6), (7) and (8).

Ex: Form the PDE by eliminating the arbitrary constants a and b from $\log_e(az-1)=x+ay+b$.

$\Rightarrow$ Given eqn is $\log_e(az-1)=x+ay+b$ —— (1)

Diff. (1) partially, w.r.to x and y, resp, we have -

$$\left(\frac{a}{az-1}\right).p=1 \text{ —— (2) and } \left(\frac{a}{az-1}\right)q=1.a$$

We elimate 'a' from (2) & (3) as, $az-1=q$

$$\therefore a=\frac{1+q}{z} \text{ —— (3)}$$

then $\left(\frac{1+q}{q.z}\right).p=1$

$\therefore (1+q)p=qz$

which is the required pde of given eqn.

(II) Formation of PDE by eliminating arbitrary functions:-

Ex: Obtain the PDE which has its general solution $u = f(\sqrt{x^2+y^2})$, where f is an arbitrary function.

$\Rightarrow$ The equation is $u = f(\sqrt{x^2+y^2})$ —— (1)

diff. (1) partially w.r.t x, and y, resp. we get.

$$\frac{\partial u}{\partial x} = f'(\sqrt{x^2+y^2}) \cdot \frac{x}{\sqrt{x^2+y^2}} \quad \text{—— (2)} \quad \text{and} \quad \frac{\partial u}{\partial y} = f'(\sqrt{x^2+y^2}) \cdot \frac{y}{\sqrt{x^2+y^2}} \quad \text{—— (3)}$$

From (2) and (3), we eliminate f', then $y\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial y}$.

which is the required PDE of (1).

Ex! Form a PDE by eliminating the function ϕ, from $z = e^{ax+by} \cdot \phi(ax-by)$.

$\Rightarrow$ Given eqn is, $z = e^{ax+by} \cdot \phi(ax-by)$. —— (1)

diff. (1) partially w.r.t x and y, resp. we get -

$$p = ae^{ax+by}\{\phi'(ax-by) + \phi(ax-by)\} \quad \text{—— (2)}$$

and $$q = be^{ax+by}\{-\phi'(ax-by) + \phi(ax-by)\} \quad \text{—— (3)}$$

Multiplying (2) by b and (3) by a, then adding them, we get, $bp + aq = 2ab\, e^{ax+by}$

$\therefore$ $bp + aq = 2ab\, z$ [using (1)].

Ex! Form PDE by eliminating arbitrary functions f and g from $z = f(x^2-y) + g(x^2+y)$.

$\Rightarrow$ Given relation is, $z = f(x^2-y) + g(x^2+y)$ —— (1)

diff. (1) partially w.r.t. x and y, resp. we get -

$$\frac{\partial z}{\partial x} = 2x\{f'(x^2-y) + g'(x^2+y)\} \quad \text{—— (2)}$$

$$\frac{\partial^2 z}{\partial x^2} = 4x^2\{f''(x^2-y) + g''(x^2+y)\} + 2\{f'(x^2-y) + g'(x^2+y)\} \quad \text{—— (3)}$$

and $$\frac{\partial z}{\partial y} = -f'(x^2-y) + g'(x^2+y) \quad \text{—— (4)}$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y). \quad \text{—— (5)}$$

Using (2) and (5), we have from (2)

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x}\frac{\partial z}{\partial x} + 4x^2\frac{\partial^2 z}{\partial y^2}.$$

Ex: Eliminate the arbitrary functions φ and ψ from $z = \varphi(x+iy) + \psi(x-iy)$, where $i^2 = -1$.

$\Rightarrow$ Given eqn is $z = \varphi(x+iy) + \psi(x-iy)$ —— ①

Diff. ① partially w.r.to x and y, resp. we get -

$$\frac{\partial z}{\partial x} = \varphi'(x+iy) + \psi'(x-iy)$$

and $$\frac{\partial z}{\partial y} = i\varphi'(x+iy) - i\psi'(x-iy)$$

Again diff. w.r.to x and y, resp. -

$$\frac{\partial^2 z}{\partial x^2} = \varphi''(x+iy) + \psi''(x-iy) \quad \text{—— ②}$$

and $$\frac{\partial^2 z}{\partial y^2} = -\{\varphi''(x+iy) + \psi''(x-iy)\} \quad \text{—— ③}$$

From ② and ③ we have $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Ex: Form a PDE by eliminating the function φ, from $lx + my + nz = \varphi(x^2+y^2+z^2)$.

$\Rightarrow$ The eqn is $lx + my + nz = \varphi(x^2+y^2+z^2)$ —— ①

Diff. ① partially w.r.to x and y resp, we get,

$$l + np = \varphi'(x^2+y^2+z^2) \cdot (2x + 2zp) \quad \text{—— ②}$$

and $$m + nq = \varphi'(x^2+y^2+z^2) \cdot (2y + 2zq) \quad \text{—— ③}$$

Dividing ② and ③, we get -

$$\frac{l+np}{m+nq} = \frac{x+zp}{y+zq}$$

$\therefore \; y(l+np) + z(lq - mp) = (m+nq) \cdot x$.

Ex: Form a PDE by eliminating the arbitrary function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

$\Rightarrow$ Diff. partially w.r.to x and y, resp. we have

$$-x^2 p = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{—— ①} \qquad \text{and} \qquad y(q-2y) = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{—— ②}$$

We eliminate f from ① and ②, we get -

$$y(q-2y) = -x^2 p$$

$$\therefore \; px^2 + qy = 2y^2. \qquad \text{Ans}$$

EXERCISE

① Form the PDE, by eliminating arbitrary constants from the following relations:-

(a) $z = ax + by + ab$

(b) $z = a(x+y) + b$

(c) $z = ax + (1-a)y + b$.

(d) $ax^2 + by^2 + cz^2 = 1$

(e) $z = a(x+y) + b(x-y) + abt + c$.

(f) $z = px + qy + p^2 + q^2$.

(g) $z = (x-a)^2 + (y-b)^2$.

② Form the PDE, by eliminating the arbitrary functions:-

(a) $z = f(x^2 - y^2)$

(b) $z = f(y + ax) + g(y + bx)$; $a \neq b$.

(c) $y = f(x - at) + g(x + at)$

(d) $f(x+y+z, \; x^2+y^2-z^2) = 0$

(e) $z = e^{ay} f(x-y)$.

(f) $z = e^{ax+by} f(ax+by)$.

(g) $y = f(x-at) + x\,g(x-at) + x^2 h(x-at)$

(h) $z = f(x\cos\alpha + y\sin\alpha - at) + \varphi(x\cos\alpha + y\sin\alpha + at)$.

WH-19

Ex! Find the differential equation of all spheres of radius λ, having centre in the xy-plane.

Hints ⇒ The eqn of any sphere of radius λ, having its centre at $(\alpha, \beta, 0)$ in the xy-plane is given by
$$(x-\alpha)^2 + (y-\beta)^2 + (z-0)^2 = \lambda^2.$$
where α, β are arbitrary constants.

Ex) Find the diff. equation of all surfaces of revolution having z-axis as the axis of revolution.

HINTS! ⇒ The equation of any surfaces of revolution having z-axis as the axis of rotation may be taken as $z = f(\sqrt{x^2+y^2})$, where f is an arbitrary constant.