Defⁿ: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$ be a cluster pt. of A.

(i) We say that lim f tends to infinity as $x \to c$, denoted by $\lim_{x \to c} f = \infty$ if for every $\alpha \in \mathbb{R}$, there exists a $\delta = \delta(\alpha) > 0$ s.t. ~~∀~~ for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$.

→ that means value of the funⁿ can't be bounded, it is unbounded.

(ii) $\lim_{x \to c} f = -\infty$ means if for every $\beta \in \mathbb{R}$ there exists a $\delta = \delta(\beta) > 0$ s.t. for all $x \in A$ $0 < |x - c| < \delta$, then $f(x) < \beta$.

Ex: $f(x) = \frac{1}{x^2}$, $\lim_{x \to 0} \frac{1}{x^2}$.

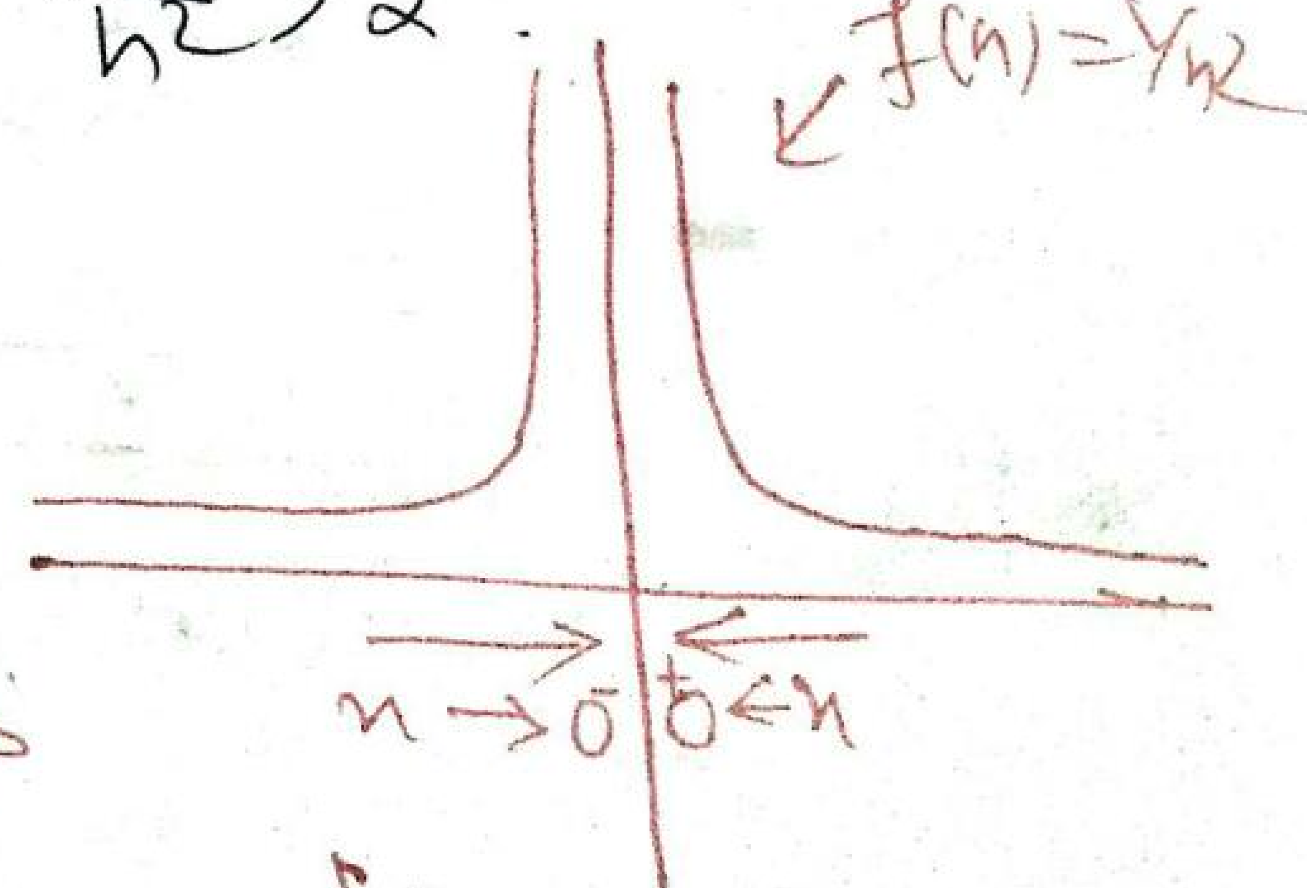
$\lim_{x \to 0} \frac{1}{x^2} = \infty$

To justify, if $\alpha > 0$ (given), let $\delta = \frac{1}{\sqrt{\alpha}}$.

So, δ-nbd of 0, $0 < |x| < \delta \Rightarrow x^2 < 1/\alpha$.

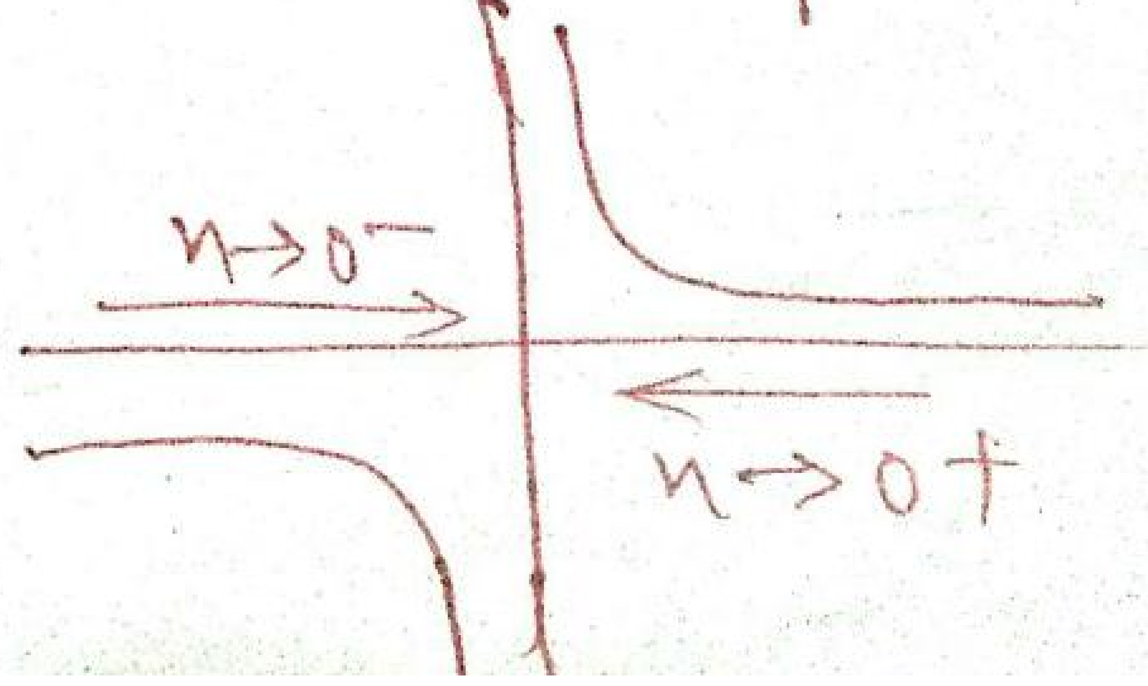
$\Rightarrow \frac{1}{x^2} > \alpha$.

$\Rightarrow \lim_{x \to 0} \frac{1}{x^2} = \infty$.



$f(x) = \frac{1}{x}$, $\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to 0^-} f(x) = -\infty$



Theorem:- Let $A \subseteq \mathbb{R}$ & let $f, g : A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster pt. of A. suppose $f(x) \leq g(x)$ $\forall x \in A$; $x \neq 0$

(a) If $\lim_{x \to c} f(x) = \infty$ then $\lim_{x \to c} g(x) = \infty$

(b) If $\lim_{x \to c} g(x) = -\infty$ then $\lim_{x \to c} f(x) = -\infty$.

Def^n:- Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$. If $c \in \mathbb{R}$ be a cluster pt. of the set $A \cap (c, \infty)$ $= \{x \in A \mid x > c\}$, then

$\lim_{x \to c^+} f = \infty$ means if for every $\alpha \in \mathbb{R}$ $\exists$ there is a $\delta = \delta(\alpha) > 0$ s.t. $\forall x \in A$ with $0 < x - c < \delta$, then $f(x) > \alpha$.

Def^n:- Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$. If $c \in \mathbb{R}$ be a cluster pt. of the set $A \cap (-\infty, c) = \{x \in A \mid x < c\}$, then

$\lim_{x \to c^-} f = -\infty$ means if for every $\alpha \in \mathbb{R}$ there is a $\delta = \delta(\alpha) > 0$ s.t. $\forall x \in A$ with $0 < c - x < \delta$, then $f(x) < \alpha$. [$\lim_{x \to c} f(x) = \infty\ (-\infty)$ iff $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = +\infty\ (-\infty)$]

Ex:- $g(x) = e^{1/x}$, $x \neq 0$

For any interval $(0, \delta)$, $\delta > 0$

$e^{1/x} \to \infty$ as $x \to 0+$

$\to 0$ as $x \to 0^-$

<u>Theorem</u>: Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ be a funⁿ. Let c be a limit pt. of A. Then ~~$\lim_{n\to c} f(n)$~~ the following statements are equivalent

(i) $\lim_{n\to c} f(n) = \infty \; (-\infty)$

(ii) For every sequence $\{x_n\}$ in A, $x_n \neq c \; \forall n$ converges to c, the sequence $\{f(x_n)\}$ diverges to $\infty \; (-\infty)$

① $\lim_{n\to 0} \cancel{\frac{1}{0}}$ $\lim_{n\to 0} f(n)$, $f(n) = 1/n$ does not exists.

$\{x_n\} = \cancel{\{\tfrac{1}{n}\}} \to$ $\qquad f(x_n) = n$

$= \{\frac{1}{n}\}$ $\quad \lim_{n\to\infty} x_n = 0 \qquad \{n\} \to \infty$

~~$f(x_n) = \{\tfrac{1}{n}\}$~~

$\{-\frac{1}{n}\} \to 0$ but $\{-n\} \to -\infty$.

$\Rightarrow$

② ~~$\lim_{n\to\pi/2} \tan n$~~

on,

$\lim_{n\to 0^+} \frac{1}{x} = \infty$, $\lim_{n\to 0^-} \frac{1}{n} = -\infty$

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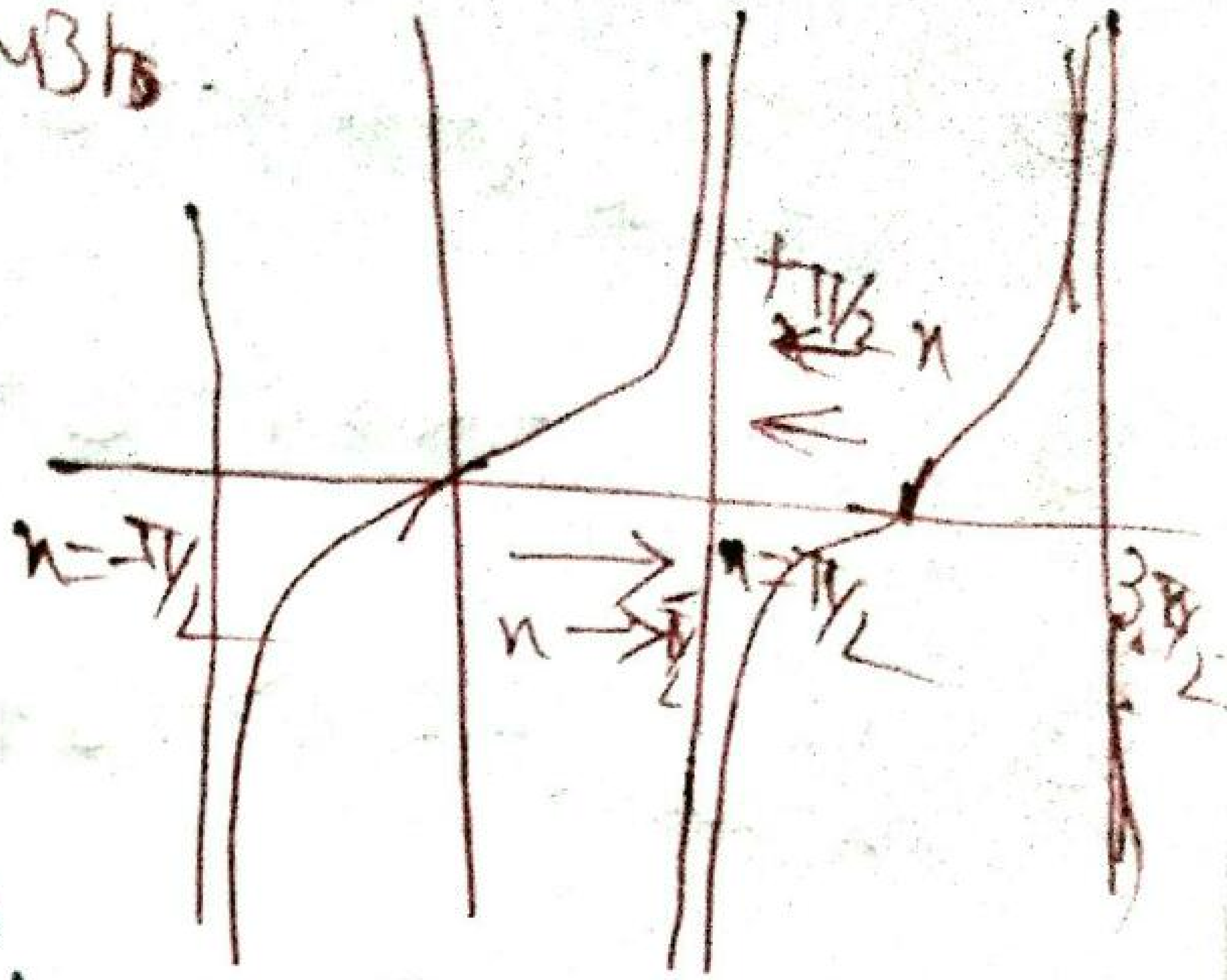
$\lim_{x \to \pi/2} \tan x$ does not exist.

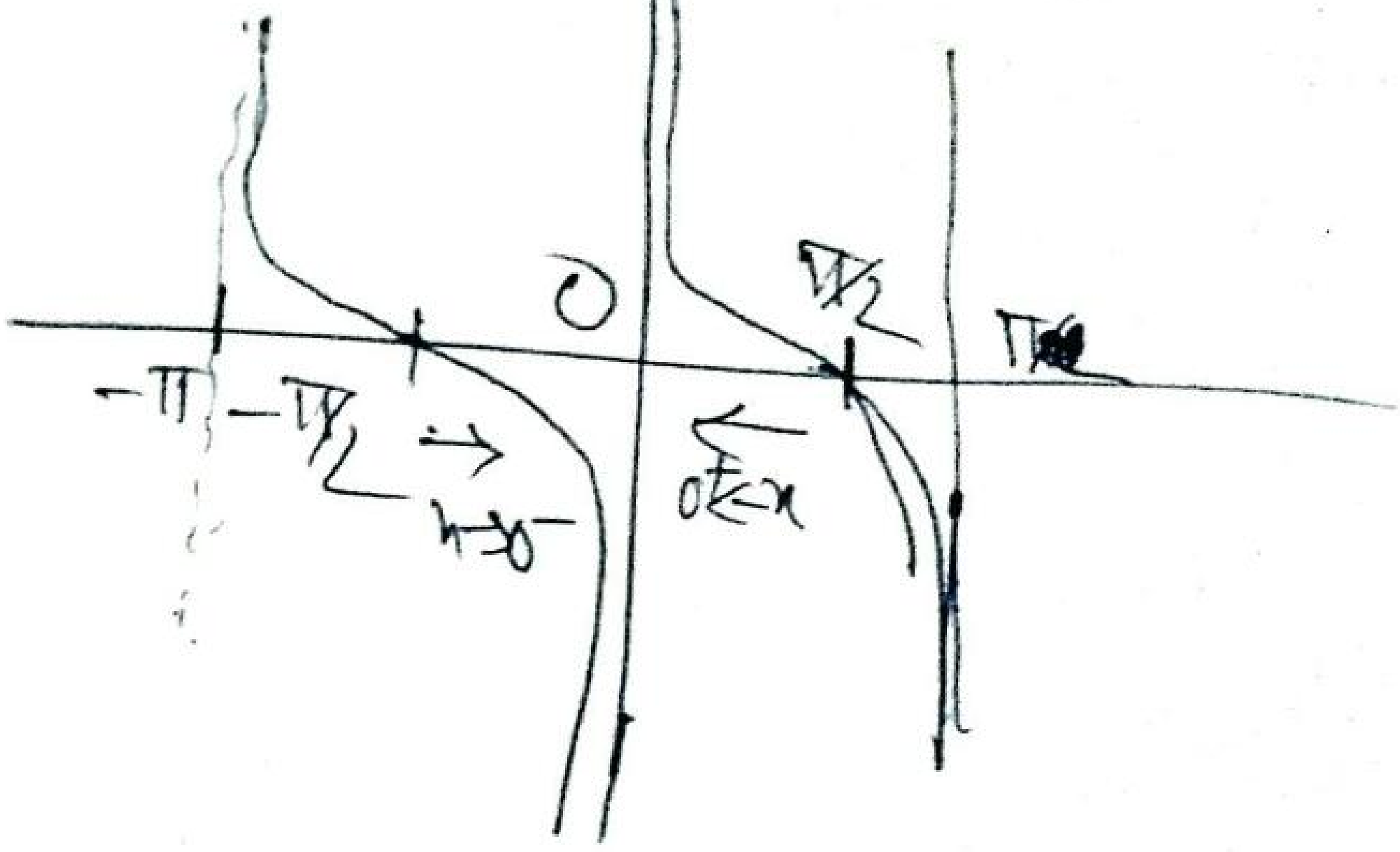


$\lim_{x \to \pi/2^+} \tan x = -\infty$

$\lim_{x \to \pi/2^-} \tan x = \infty$

$\Rightarrow \lim_{x \to \pi/2} \tan x$ does not exist.

$\lim_{x \to 0} \cot x$ does not exist.



$\lim_{x \to 0^+} \cot x = \infty$

$\lim_{x \to 0^-} \cot x = -\infty$

Limits at Infinity

Def^n:- Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. suppose that $(a,\infty) \subseteq A$ for some $a \in \mathbb{R}$. Then $\lim_{x\to\infty} f = L$ means for given $\varepsilon > 0$, there exists $K \equiv K(\varepsilon) > a$ s.t. for any $x > K$, $|f(x) - L| < \varepsilon$.

Theorem:- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and suppose that $(a,\infty) \subseteq A$ for some $a \in \mathbb{R}$. Then following statements are equivalent.

(i) $\lim_{x\to\infty} f = L$

(ii) for every sequence $\{x_n\}$ in $A \cap (a,\infty)$ s.t. $\lim_{n\to\infty} x_n = \infty$, then $\{f(x_n)\}$ converges to L

① $\lim_{x\to\infty} \frac{ax^2+bx+c}{dx^2+ex+f}$ → divide by heighest exponent

$$= \lim_{x\to\infty} \frac{a + b/x + c/x^2}{d + e/x + f/x^2}$$

$$= a/d.$$

② $\lim_{x\to\infty} \frac{\sqrt{3x^2-1} - \sqrt{2x^2-1}}{4x+3}$

$$= \lim_{x\to\infty} \frac{\sqrt{3 - 1/x^2} - \sqrt{2 - 1/x^2}}{4 + 3/x} = \frac{\sqrt{3} - \sqrt{2}}{4}$$

③ $\lim_{x\to\infty} \frac{\sqrt{x^2+1} - \sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1} - \sqrt[5]{x^4+1}}$

$$= \lim_{x\to\infty} \frac{\sqrt{1+1/x^2} - \sqrt[3]{1 + 1/x^3}}{\sqrt[4]{1+1/x^4} - \sqrt[5]{1/x + 1/x^5}} = \frac{1-1}{1-0} = 0$$

(4) $\lim_{n\to\infty} \left(\sqrt{n^2+n+1} - \sqrt{n^2+1}\right)$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}+n+\cancel{1}-\cancel{n^2}-\cancel{1}}{\sqrt{n^2+n+1}+\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n+1}+\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n+1/n^2}+\sqrt{1+1/n^2}} = 1.$$

(5) $\lim_{n\to\infty} \left(\frac{1+2+3+\cdots+n}{n^2}\right)$ $\left[= \lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\right) \neq 0\right]$

$$= \lim_{n\to\infty} \frac{n(n+1)}{2n^2}$$

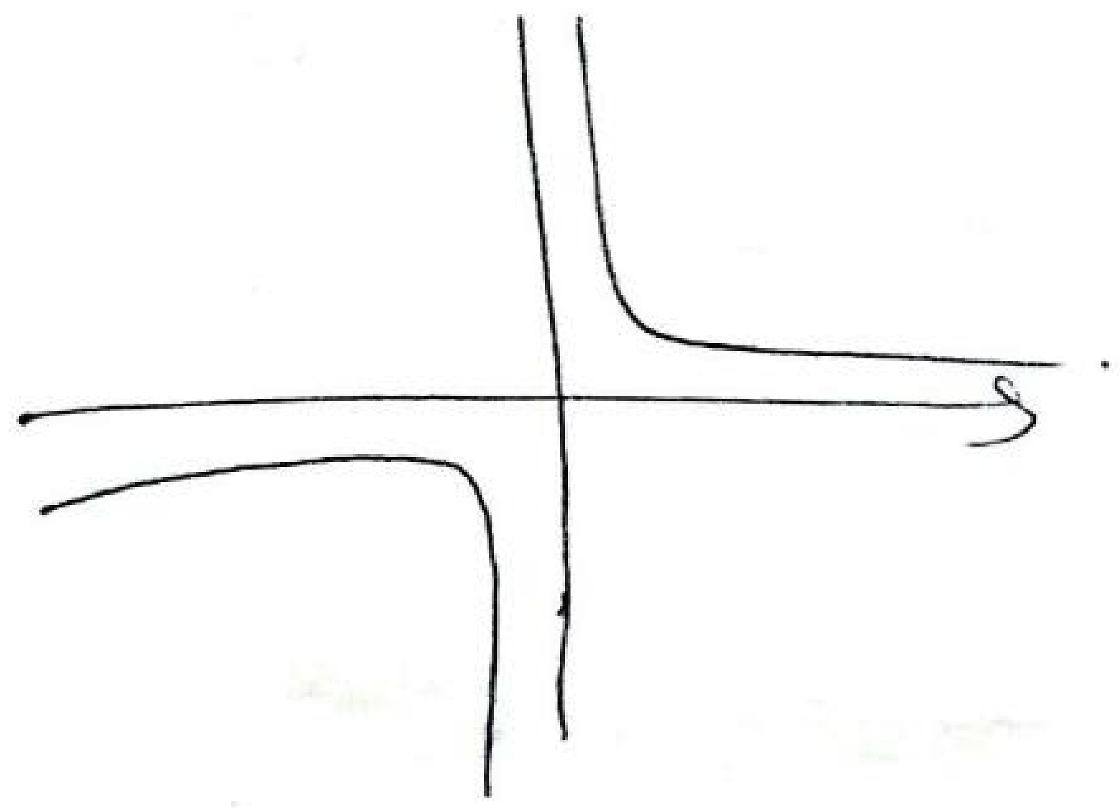
$$= \lim_{n\to\infty} \frac{1}{2}(1+1/n)$$

$$= 1/2$$

(6) $\lim_{n\to\infty} 1/n = 0$

$\lim_{n\to\infty^+} \frac{1}{n} = 0$

$\lim_{n\to\infty^-} \frac{1}{n} = 0$



Find the following limits

(1) $\lim_{n\to\infty} 2^n$, ~~lim~~ (2) $\lim_{n\to\infty^+} \left(\frac{2}{3}\right)^n$ (3) $\lim_{n\to\infty} \frac{2^{n+3}}{3^{n+1}}$

(4) $\lim_{n\to\infty} \frac{2^{n+3}}{3^{n+1}}$ (5) $\lim_{n\to\infty^-} \left(\frac{2}{3}\right)^n$ (6) $\lim_{n\to\infty} \frac{2^{2n+1}}{3^{n-1}}$

(7) lim

Defn:- Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in A$. We say f tends to ∞ as $n \to \infty$ i.e.,

$\lim_{n\to\infty} f = \infty$ $\left(\lim_{n\to\infty} f = -\infty\right)$

If given any $\alpha \in \mathbb{R}$, there exists $k = k(\alpha) > a$ such that for any $n > k$, then $f(n) > \alpha$
$(f(n) < \alpha)$.

Theorem:- Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:

(i) $\lim_{n\to\infty} f = \infty$ $\left(\lim_{n\to\infty} f = -\infty\right)$

(ii) for any sequence $\{x_n\}$ in (a, ∞) s.t. $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty} f(x_n) = \infty$. $\left(\lim_{n\to\infty} f(x_n) = -\infty\right)$.

(a) $\lim_{n\to\infty} n^n = \infty$

(b) $\lim_{n\to\infty} \left(-2n^3 + 1 - \frac{5}{n} + \frac{12}{n^4}\right) = -\infty$

(c) $\lim_{n\to\infty} \frac{-5n^3 - 2n + 4}{n^2} = \lim_{n\to\infty} -5n - \frac{2}{n} + \frac{4}{n^2}$
$= -\infty$

(d) $\lim_{n\to\infty} \frac{-5n^3 - 2n + 4}{n^3}$ (e) $\lim_{n\to\infty}$

(e) $\lim_{n\to\infty} (-2n^5 - 8n^4 + 7n^3 - 10)$

$= \lim_{n\to\infty} (-2n^5)\left(1 + \frac{8n^4}{2n^5} + \frac{7n^3}{2n^5} + \frac{10}{2n^5}\right)$

$= \lim_{n\to\infty} (-2n^5)\left[1 + \frac{4}{n} - \frac{7}{2}\frac{1}{n^2} + \frac{5}{n^5}\right) = -\infty$

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(f) $\lim_{n \to -\infty} (-2n^5 - 8n^4 + 7n^3 - 10)$

(g) $\lim_{n \to \infty} (-25n^5 - 8n^4 + 7n^3)$

(h) $\lim_{n \to \infty} (-2n^5 + 8n^6)$

(i) $\lim_{n \to -\infty} \frac{n^3 - 9n + 1}{3n^2 - 2n - 15}$

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