Chapter 2

Lagrange's Solution of a Linear Partial Differential Equation Pq + Qq = R

Relevant Information on

1. $Pp + Qq = R$ (P, Q, R are function of x, y, z) is a typical linear partial differential equation of first order.
2. Lagrange's solution of $Pp + Qq = R$ using Lagrange's Auxiliary equations: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.
3. Integral surfaces: Orthogonal surfaces.

2.1 The General Solution of a Linear Equation

We refer to **Chapter 1, Art. 1.3:** Given an arbitrary functional relation

$$\phi(u, v) = 0. \tag{2.1.1}$$

We can deduce a linear p.d.e. of first order in the form

$$Pp + Qq = R. \tag{2.1.2}$$

If (2.1.1) is deduced from (2.1.2), then we call (2.1.1), the **general solution** of (2.1.2). Since ϕ *is an arbitrary function,* the general solution (2.2.1) is more general than another solution of (2.1.2) that merely contains two arbitrary constants.

As for example, let $z = f(x^2 - y^2)$, where $f$is arbitrary. Then

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)2x, \; q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y)$$

which leads to $py + qx = 0$, a **linear p.d.e.**

We shall call $z = f(x^2 - y^2)$ the general solution of $py + qx = 0$.

See that we could write

$$\begin{aligned} z &= a(x^2 - y^2) + b(x^2 - y^2)^2 \quad (a, b \text{ are arbitrary constants}) \\ \text{or, } z &= a\sin(x^2 - y^2) + b \quad (a, b \text{ are arbitrary constants}) \end{aligned}$$

as the solution of $py + qx = 0$. But certainly $z = f(x^2 - y^2)$ is a more general solution; all these solutions with arbitrary constants are included in $z = f(x^2 - y^2)$.

2.2 An Equation that is Equivalent to $Pp + Qq = R$

$\therefore$ A general type of a linear p.d.e. in p and q is

$$Pp + Qq = R, \tag{2.2.1}$$

where P, Q, R are functions of x, y, z.

Suppose that $u(x, y, z) = c$ satisfies (2.2.1).

Differentiation with respect to x and y gives

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p &= 0 \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q &= 0 \end{aligned}\right\} p = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}}; \quad q = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}}.$$

Substituting these values of p and q in (2.2.1) we obtain

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0. \tag{2.2.2}$$

Therefore, if $u = c$ be an integral of (2.2.1), then $u = c$ also satisfies (2.2.2).

Conversely, if $u = c$ be an integral of (2.2.2), it is also an integral of (2.2.1).

Dividing (2.2.2) by $\frac{\partial u}{\partial z}$ and using the values of p and q, (2.2.2) reduces to $Pp + Qq = R$.

So we find equation (2.2.2) is equivalent to equation (2.2.1).

2.3 Lagrange's Method of Solving $Pp + Qq = R$

Lagrange's Rule: Statement: *The general solution of the linear partial differential equation*

$$Pp + Qq = R, \tag{2.3.1}$$

where P, Q, R *are functions of* x, y, z, *is given by*

$$\phi(u, v) = 0; \tag{2.3.2}$$

where ϕ *is an arbitrary function and*

$$\left.\begin{aligned} u(x, y, z) &= c_1 \\ v(x, y, z) &= c_2 \end{aligned}\right\} \tag{2.3.3}$$

are two independent solutions of the Auxiliary Equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{2.3.4}$$

This is known as **Lagrange's solution** of the linear equation.

Proof. Given $\phi(u, v) = 0$.

We consider z as dependent variable, and x and y as independent variables so that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial y}{\partial x} = 0, \quad \frac{\partial x}{\partial y} = 0.$$

Differentiating $\phi(u, v) = 0$ with respect to x, we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x}\right] = 0$$

$$\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}} = -\frac{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}}{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}} = -\frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} \quad \left(p = \frac{\partial z}{\partial x}\right).$$

Similarly, differentiating $\phi(u,v)=0$ with respect to y, we get

$$\frac{\frac{\partial\phi}{\partial u}}{\frac{\partial\phi}{\partial v}} = -\frac{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial v}{\partial z}} \quad \left(q=\frac{\partial z}{\partial y}\right)$$

Eliminating ϕ, we obtain,

$$\frac{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}} = \frac{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}}$$

$$\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)$$

$$\textbf{or, } Pp+Qq = R,$$

where

$$\left.\begin{aligned} P &= \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y},\\ Q &= \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z},\\ R &= \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} \end{aligned}\right\}.$$

Thus $\phi(u,v)=0$ is the general solution (integral) of $Pp+Qq=R$.

We now proceed to obtain u and v for substitution in $\phi(u,v)=0$.

Consider $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ where c_1 and c_2 are arbitrary constants. Taking differentials we get

$$du = 0 \text{ and } dv=0$$

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz = 0 \text{ and } \frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0.$$

Solving for dx, dy, dz,

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}.$$

Using the values of P, Q, R as given above, we may write

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}.$$

Thus we find that $u = c_1, v = c_2$ form a solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Hence $u = c_1$, $v = c_2$ determine u and v for substitution in $\phi(u, v) = 0$.

This is what we wished to prove.

Note 2.3.1 *Equations* $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ *are known as Lagrange's Auxiliary (or Subsidiary) equations.*

Working Rule

Solution of $Pp + Qq = R$ (Lagrange's Method)

1. Put the given linear equation in the form

$$Pp + Qq = R.$$

2. Write down Lagrange's Auxiliary Equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

3. Taking two of the ratios at a time, and by using method of solving ordinary differential equations, obtain

$$u(x, y, z) = c_1, \quad v(x, y, z) = c_2.$$

as two independent solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

4. The general solution of $Pp+Qq = R$ can be written as $\phi(u, v) = 0$ or in the form $u = f(v)$ or $v = F(u)$.

In Lagrange's Method of solving $Pp+Qq = R$, the most important part is to obtain $u = c_1$, $v = c_2$ from Lagrange's Auxiliary Equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

We ask our readers to remember **four types** of problems:

Solved Problems: (Type I – Type IV)

Type I

▶ **Example 2.3.1** *Solve:* $\frac{y^2 z}{x} p + xzq = y^2$.

Solution: Lagrange's Auxiliary Equations are

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \left[\text{cf. } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\right]$$

From the **first two ratios** we see that the **variable** z **cancels out** and we are left with

$$\frac{dx}{y^2/x} = \frac{dy}{x} \quad \textbf{or,} \quad x^2 dx - y^2 dy = 0.$$

Integrating, we get

$$x^3 - y^3 = c_1. \quad [\text{cf. } u = c_1]$$

Taking **the first** and **last ratio** we see that y^2 **cancels out** and we obtain

$$\frac{dx}{z/x} = \frac{dz}{1} \quad \textbf{or,} \quad x\ dx - z\ dz = 0.$$

Integrating we get

$$x^2 - z^2 = c_2 \quad [\text{cf. } v = c_2]$$

$\therefore$ The **required general solution** of the given equation is

$$\phi(u, v) = 0$$

$$\text{i.e.,} \quad \phi(x^3 - y^3, x^2 - z^2) = 0$$

where ϕ is any arbitrary function of its arguments.

▶ **Example 2.3.2** *Solve:* $ap + aq = z$.

Solution: Lagrange's Auxiliary Equations are

$$\frac{dx}{a} = \frac{dy}{a} = \frac{dz}{z} \quad \left(\text{cf. } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\right)$$

First two ratios at once give $u = x - y = c_1$.

Again, Second and Third Ratios give $v = y - a\log z = c_2$.

$\therefore$ The **required general solution** is

$$\phi(u, v) = 0, \quad \text{i.e.,} \quad \phi(x - y, y - a\log z) = 0.$$

▶ **Example 2.3.3** *Solve:* $y^2p - xyq = x(z - 2y)$.

Solution: Lagrange's Auxiliary Equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}.$$

First two ratios give

$$\frac{dx}{y} = \frac{dy}{-x}, \quad \text{i.e.,} \quad x\,dx + y\,dy = 0 \Rightarrow u \equiv x^2 + y^2 = c_1.$$

Last two ratios give

$$\frac{dy}{-y} = \frac{dz}{z-2y},$$

i.e., $\dfrac{dz}{dy} + \dfrac{1}{y}z = 2$ $\left(\begin{array}{l}\text{ord. diff. eqn. Linear Form} \\ \text{with I.F.} = e^{\int \frac{1}{y}dy} = e^{\log y} = y\end{array}\right)$

Multiplying by the I.F. y and integrating we get

$$zy = y^2 + c_2, \quad \text{i.e.,} \quad u = zy - y^2 = c_2.$$

$\therefore$ The **required general solution** is $\phi(u, v) = 0$

$$\phi(x^2 + y^2, zy - y^2) = 0, \quad \phi \text{ being arbitrary.}$$

A

Try Yourself (*Examples for Practice*)

Solve the following linear equations by using Lagrange's Auxiliary Equations (Lagrange's Method)

1. $2p + 3q = 1$.
2. $p + q = \sin x$.
3. $xzp + yzq = xy$.
4. $p \tan x + q \tan y = \tan z$.
5. $zp + x = 0$.
6. $yzp + zxq = xy$.
7. $x^2p + y^2q + z^2 = 0$.
8. $x^2p + y^2q = z^2$.
9. $xp + yq = z$.

Answers

1. $\phi(x - 2z, y - 3z) = 0$. **2.** $\phi(x - y, z + \cos x) = 0$. **3.** $\phi\left(z^2 - xy, \frac{y}{x}\right) = 0$. **4.** $\frac{\sin x}{\sin y} = \phi\left(\frac{\sin y}{\sin z}\right)$. **5.** $x^2 + z^2 = \phi(y)$. **6.** $\phi(x^2 - y^2, x^2 - z^2) = 0$. **7.** $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right) = 0$. **8.** $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z} = 0\right)$. **9.** $\phi\left(\frac{x}{y}, \frac{x}{z}\right) = 0$.

Type II

Suppose that one integral of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ can be obtained easily by taking suitably chosen two ratios (as used in Type I Problems) and suppose that another integral **cannot be** obtained by this same method.

Then one integral known to us is used to find another integral (see the Solved Examples given below).

In the second integral the constant of integration of the first integral should be removed.

▶ **Example 2.3.4** *Solve:* $p + 3q = 5z + \tan(y - 3x)$.

Solution: The Lagrange's Auxiliary Equations are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}.$$

First see that one integral can be easily obtained from the first two ratios, namely $\frac{dx}{1} = \frac{dy}{3}$. Thus $y - 3x = c_1$ (*first integral*). The second integral cannot be obtained in this manner. However, we take first ratio and the last ratio and obtain

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y - 3x)} = \frac{dz}{5z + \tan c_1} \text{ (using the known integral)}$$

Writing

$$5dx = \frac{5dz}{5z + \tan c_1}$$

and integrating we obtain

$$5x - \log(5z + \tan c_1) = \text{arbitrary constant } c_2 \text{ (say)}.$$

$\therefore$ **The required general solution** is

$$5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x),$$

where ϕ is arbitrary.

▶ **Example 2.3.5** *Solve:* $xyp + y^2q = xyz - 2x^2$.

Solution: Lagrange's Auxiliary Equations are

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$$

Taking the first two ratios (cancelling y), we get $\frac{dx}{x} = \frac{dy}{y}$ and hence, on integration $\frac{x}{y} = c_1$.

From second and third ratios

$$\frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$$

$$= \frac{dz}{(c_1 y)yz - 2c_1^2 y^2} \quad \left(\because \frac{x}{y} = c_1 \text{ or, } x = c_1 y\right)$$

$$\Rightarrow \frac{dy}{y^2} = \frac{dz}{c_1 y^2 z - 2c_1^2 y^2}$$

$$\Rightarrow c_1 dy = \frac{dz}{(z - 2c_1)}$$

$$\Rightarrow c_1 y - \log(z - 2c_1) = c_2$$

$$\Rightarrow x - \log\left(z - \frac{2x}{y}\right) = c_2.$$

$\therefore$ **The required general solution is**

$$x - \log\left(z - \frac{2x}{y}\right) = \phi\left(\frac{x}{y}\right),$$

ϕ being an arbitrary function.

▶ **Example 2.3.6** *Solve:* $xzp + yzq = xy$.

Solution: The Lagrange subsidiary equations are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}.$$

Taking the first two ratios (concelling z), we get $\frac{dx}{x} = \frac{dy}{y}$ and this gives, on integration

$$\log x - \log y = \log c_1 \quad \textbf{or,} \quad \frac{x}{y} = c_1.$$

The second and third ratios are

$$\frac{dy}{yz} = \frac{dz}{xy}$$

or, $\dfrac{dy}{z} = \dfrac{dz}{x}$

or, $xdy = z\ dz$

or, $c_1 y\ dy = z\ dz$ (using the known integral $x = c_1 y$)

Hence, on integration we get

$$c_1\frac{y^2}{2} = \frac{z^2}{2} + \text{constant}$$

or, $c_1 y^2 - z^2 = c_2$

or, $xy - z^2 = c_2$.

$\therefore$ **The required general solution is** $\phi(u, v) = 0$,

or, $\phi\left(\dfrac{x}{y}, xy - z^2\right) = 0$, ϕ being arbitrary.

▶ **Example 2.3.7** *Solve:* $py + qx = xyz^2(x^2 - y^2)$.

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$$

From first two fractions we obtain $x^2 - y^2 = c_1$.

The last two fractions give

$$\frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$$

$$\Rightarrow \quad \frac{dy}{1} = \frac{dz}{yz^2 c_1} \quad (\because\ x^2 - y^2 = c_1)$$

$$\Rightarrow \quad y\ dy = \frac{1}{c_1}\frac{dz}{z^2}.$$

On integration,

$$\frac{y^2}{2} = \frac{1}{c_1}\left(-\frac{1}{z}\right) + \text{constant}$$

or, $c_1 y^2 = -\dfrac{2}{z} + \text{constant}$

or, $(x^2 - y^2)y^2 + \dfrac{2}{z} = c_2$.

Hence the required general solution is

$$(x^2 - y^2)y^2 + \frac{2}{z} = \phi(x^2 - y^2), \quad \phi \text{ is arbitrary.}$$

B

Try Yourself (*Examples for Practice*)
Solve by Lagrange's Method

1. $z(z^2 + xy)(px - qy) = x^4$. **2.** $p - q = \frac{z}{x+y}$.
3. $p - 2q = 3x^2 \sin(y + 2x)$. **4.** $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.
5. $z(p - q) = z^2 + (x + y)^2$.

Answers

1. $\phi(xy, x^4 - z^4 - 2xyz^2) = 0$. **2.** $x - (x + y)\log z = \phi(x + y)$. **3.** $x^3 \sin (y+2x) - z = \phi(y+2x)$. **4.** $\frac{x^2+y^2+z^2}{z} = \phi\left(\frac{y}{z}\right)$. **5.** $e^{2y}[z^2+(x+y)^2] = \phi(x+y)$.

Type III

We may required to write (using a well-known Rule of Ratio and Proportion)

Lagrange's Subsidiary Equations as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1dx + Q_1dy + R_1dz}{P_1P + Q_1Q + R_1R}.$$

Multipliers P_1, Q_1, R_1 should be chosen carefully. If $P_1P + Q_1Q + R_1R = 0$, then write $P_1dx + Q_1dy + R_1dz = 0$. Now, on integration of $P_1dx + Q_1dy + R_1dz = 0$ we may obtain an integral $u_1(x, y, z) = c_1$.

With the help of another set of suitably chosen multipliers we may obtain another integral $u_2(x, y, z) = c_2$.

Sometimes only one integral is possible by use of suitable multipliers and the other integral may be obtained by methods of Type I and Type II.

Then the required general solution is written as

$$\phi(u_1, u_2) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.8** *Solve:* $\frac{b-c}{a}yzp + \frac{c-a}{b}zxq = \frac{a-b}{c}xy.$

Solution: The Lagrange's subsidiary equations are

$$\frac{dx}{\left(\frac{b-c}{a}\right)yz} = \frac{dy}{\left(\frac{c-a}{b}\right)zx} = \frac{dz}{\left(\frac{a-b}{c}\right)xy}$$

$$\text{or,}\quad \frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy}. \qquad (1)$$

Multiply both Numerator and Denominator of first ratio by x, second ratio by y and third ratio by z.

Then each ratio of (1)

$$= \frac{ax\,dx + by\,dy + cz\,dz}{xyz(b-c+c-a+a-b)} = \frac{ax\,dx + by\,dy + cz\,dz}{0}.$$

Hence we may write

$$ax\,dx + by\,dy + cz\,dz = 0.$$

On integration we obtain

$$ax^2 + by^2 + cz^2 = \text{constant } c_1 \text{ (say)}. \qquad (2)$$

Take another set of multipliers ax (first ratio), by (second ratio) and cz (third ratio). Then each ratio of (1) becomes

$$= \frac{a^2x\ dx + b^2y\ dy + c^2z\ dz}{xyz[a(b-c) + b(c-a) + c(a-b)]} = \frac{a^2x\ dx + b^2y\ dy + c^2z\ dz}{0}.$$

Hence $a^2x\ dx + b^2y\ dy + c^2z\ dz = 0$. Now integrating we get

$$a^2x^2 + b^2y^2 + c^2z^2 = \text{constant } c_2 \text{ (say)}. \qquad (3)$$

From (2) and (3) we write the general solution of the given equation as

$$\phi(ax^2 + by^2 + cz^2, a^2x^2 + b^2y^2 + c^2z^2) = 0,$$

where ϕ is an arbitrary function.

▶ **Example 2.3.9** *Solve:* $(mz - ny)p + (nx - lz)q = ly - mx.$

Solution: The Lagrange's subsidiary equations corresponding to the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}. \tag{1}$$

Using multipliers x for the first ratio, y for the second and z for the third we write each ratio of (1)

$$= \frac{x\ dx + y\ dy + z\ dz}{(mz - ny)x + (nx - lz)y + (ly - mx)z} = \frac{x\ dx + y\ dy + z\ dz}{0}.$$

Hence we can write $x\ dx + y\ dy + z\ dz = 0$. On integration we get

$$x^2 + y^2 + z^2 = \text{constant } c_1 \text{ (say)}. \tag{2}$$

Next we choose l, m, n as multipliers of first, second and third ratios of (1). Then each ratio of (1) becomes

$$= \frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + ndz}{0}.$$

Hence

$$\begin{aligned} & ldx + mdy + ndz = 0 \\ \Rightarrow \quad & lx + my + nz = \text{constant } c_2 \text{ (say)}. \end{aligned} \tag{3}$$

The required general solution from (2) and (3) is

$$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0,$$

ϕ being an arbitrary function.

▶ **Example 2.3.10** *Solve:* $x(y^2 - z^2)(p - y(z^2 + x^2)q = z(x^2 + y^2)$.

Solution: The Lagrange's Auxiliary Equations corresponding to the given equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}. \tag{1}$$

Using x, y, z as multipliers for first, second and third ratio of (1), we write

$$\begin{aligned} \text{each ratio of (1)} &= \frac{x\ dx + y\ dy + z\ dz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} \\ &= \frac{x\ dx + y\ dy + z\ dz}{0} \\ \therefore\ x\ dx + y\ dy + z\ dz &= 0, \text{ whence } x^2 + y^2 + z^2 = c_1. \end{aligned} \tag{2}$$

Using $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ as multipliers of ratios of (1) we again find each ratio of (1) becomes

$$= \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)}$$

$$= \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0}$$

$\therefore\ \frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0$ which yields on integration,

$$\log x - \log y - \log z = \log c_2, \quad \textbf{or,} \quad \frac{x}{yz} = c_2. \tag{3}$$

$\therefore$ The required general solution [from (2) and (3)] is

$$\phi\left(x^2 + y^2 + z^2, \frac{x}{yz}\right) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.11** *Solve:* $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$.

Solution: The Lagrange's subsidiary equations corresponding to the given equation are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}. \tag{1}$$

Choosing $x, y, -1$ as multipliers of ratios of (1)

$$\text{each ratio of (1)} = \frac{x\ dx + y\ dy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)}$$

$$= \frac{x\ dx + y\ dy - dz}{0}.$$

Hence $x\ dx + y\ dy - dz = 0$ which, on integration, gives

$$x^2 + y^2 - 2z = c_1. \tag{2}$$

Again, choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers of ratios of (1) we obtain, each ratio of (1) becomes

$$= \frac{dx/x + dy/y + dz/z}{y^2 + z - (x^2 + z) + x^2 - y^2} = \frac{dx/x + dy/y + dz/z}{0}.$$

Hence $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$ or, $\log xyz = \log c_2$

$$\text{i.e.,} \quad xyz = c_2. \tag{3}$$

∴ From (2) and (3) the required general solution is

$$\phi(x^2 + y^2 - 2z, xyz) = 0,$$

ϕ being an arbitrary function.

▶ **Example 2.3.12** *Solve:* $(y^2 + z^2)p - xyq = -zx$. **[I.A.S. 1990]**

Solution: The Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}. \tag{1}$$

Using multipliers of the ratios of (1) as x, y, z respectively, we obtain

$$\text{each ratio of (1)} = \frac{x\,dx + y\,dy + z\,dz}{x(y^2 + z^2) - xy^2 - xz^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

∴ $x\,dx + y\,dy + z\,dz = 0$ which gives, on integration

$$x^2 + y^2 + z^2 = c_1. \tag{2}$$

Again, from last two ratios of (1) we get

$$\frac{dy}{-xy} = \frac{dz}{-xz} \quad \text{or,} \quad \frac{dy}{y} = \frac{dz}{z}.$$

On integration this gives

$$\log y - \log z = \log c_2$$
$$\text{or,} \quad \frac{y}{z} = c_2. \tag{3}$$

∴ The required general solution [from (2) and (3)] is

$$\phi\left(x^2 + y^2 + z^2, \frac{y}{z}\right) = 0$$

where ϕ is an arbitrary function.

C

Try Yourself (*Examples for Practice*)
Solve by Lagrange's Method of Solution

1. $z(x+y)p+z(x-y)q=x^2+y^2$.
2. $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$.
3. $x(y-z)p+y(z-x)q=z(x-y)$.
4. $(y-zx)p+(x+yz)q=x^2+y^2$.
5. $(x+2z)p+(4zx-y)q=2x^2+y$.
6. $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$.
7. $(z^2-2yz-y^2)p+(xy+zx)q=xy-zx$.
8. $z(xp-yq)=y^2-x^4$.
9. $(y^3x-2x^4)p+(2y^4-x^3y)q=9z(x^3-y^3)$.
10. $(3x+y-z)p+(x+y-z)q=2(z-y)$.
11. $x(x^2+3y^2)p-y(3x^2+y^2)q=2z(y^2-x^2)$.

[Delhi B.Sc. Hons. 2000]

[Here Lagrange's subsidiary equations are

$$\frac{dx}{x(x^2+3y^2)}=\frac{dy}{-y(3x^2+y^2)}=\frac{dz}{2z(y^2-x^2)}.$$

Multipliers $\frac{1}{x},\frac{1}{y},-\frac{1}{z}$

$$\Rightarrow \frac{\frac{1}{x}dx+\frac{1}{y}dy-\frac{1}{z}dz}{x^2+3y^2-3x^2-y^2-2(y^2-x^2)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy-\frac{1}{z}dz}{0}.$$

$\therefore\ \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$ will give $\frac{xy}{z}=c_1$

Now take the first two ratios $\frac{dx}{x(x^2+3y^2)}=\frac{dy}{-y(3x^2+y^2)}$

$$\text{or, } \frac{dy}{dx}=-\frac{y(3x^2+y^2)}{x(x^2+3y^2)} \quad \text{(Homogeneous Equation)}$$

$$=-\frac{y}{x}\left[\frac{3+\left(\frac{y}{x}\right)^2}{1+3\left(\frac{y}{x}\right)^2}\right].$$

Put $y = vx$. Then $v + x\frac{dv}{dx} = -v\frac{(3+v^2)}{1+3v^2}$

$$\text{or,}\quad x\frac{dv}{dx} = -v\left[\frac{3+v^2}{1+3v^2}+1\right] = -\frac{4(1+v^2)v}{1+3v^2}$$

$$\text{or,}\quad 4\frac{dx}{x} + \frac{1+3v^2}{v(1+v^2)}dv = 0$$

$$\text{or,}\quad 4\frac{dx}{x} + \left(\frac{1}{v} + \frac{2v}{1+v^2}\right) dv = 0.$$

On integration, $4\log x + \log v + \log(1+v^2) = \text{constant}$

$$\text{or,}\quad x^4 v(1+v^2) = \text{constant}$$

$$\text{or,}\quad x^4\frac{y}{x}\left(1+\frac{y^2}{x^2}\right) = \text{constant}$$

$$\text{or,}\quad xy(x^2+y^2) = \text{constant}$$

$$\text{or,}\quad c_1 z(x^2+y^2) = \text{constant}$$

$$\text{or,}\quad z(x^2+y^2) = c_2 \text{ (say)}.$$

$\therefore$ The required solution is

$$\phi\left(z(x^2+y^2), \frac{xy}{z}\right) = 0, \quad \phi \text{ being arbitrary.}]$$

12. $(y-z)p + (z-x)q = x - y$.

13. $(y+zx)p - (x+yz)q + y^2 - x^2 = 0$.
[Choose multipliers (i) x, y and $-z$ and (ii) y, x and 1.]

14. $y^2p + x^2q = x^2y^2z^2$.

Answers

1. $\phi(x^2-y^2-z^2, 2xy-z^2) = 0$; **2.** $\phi(x^2+y^2+z^2, xyz) = 0$; **3.** $\phi(x+y+z, xyz) = 0$; **4.** $\phi(x^2-y^2+z^2, xy-z) = 0$; **5.** $\phi(xy-z^2, x^2-y-z) = 0$; **6.** $\phi\left(xyz, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 0$; **7.** $\phi(y^2-z^2-2yz, x^2+y^2+z^2) = 0$; **8.** $\phi(x^2+y^2+z^2, xy) = 0$; **9.** $\phi\left(xyz^{1/3}, \frac{y}{x^2}+\frac{x}{y^2}\right) = 0$; **10.** $\phi\left(x-3y-z, \frac{x-y+z}{\sqrt{x+y-z}}\right) = 0$; **11.** $\phi(x+y+z, xy+yz+zx) = 0$; **12.** $\phi(x^2+y^2-z^2, xy+z) = 0$; **13.** $\phi\left(x^3-y^3, x^3+y^3+\frac{6}{z}\right) = 0$.

Type IV

As in Type III, with one set of multipliers P_1, Q_1, R_1 we write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}.$$

Now, if we see $P_1 dx + Q_1 dy + R_1 dz$ is an exact differential of the denominator $P_1 P + Q_1 Q + R_1 R$, then

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}$$

may be combined with **a suitable ratio** of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and this will give one integral.

Choose another set of multipliers, say P_2, Q_2, R_2, then

$$\text{each ratio} = \frac{P_2 dx + Q_2 dy + R_2 dz}{P_2 P + Q_2 Q + R_2 R}$$

Suppose that the numerator is again an exact differential of $P_2 P + Q_2 Q + R_2 R$. The two ratios

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \quad \text{and} \quad \frac{P_2 dx + Q_2 dy + R_2 dz}{P_2 P + Q_2 Q + R_2 R}$$

are then combined to give a second integral.

See the Worked Examples given below:

▶ **Example 2.3.13** *Solve:* $(y+z)p + (z+x)q = (x+y)$. **[I.A.S. 1997]**

Solution: The Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}. \qquad (1)$$

Choose multipliers $1, -1, 0$ for the first, second and third ratio of (1). Then, each ratio of (1)

$$= \frac{dx - dy}{y + z - z - x} = \frac{dx - dy}{y - x} = \frac{d(x-y)}{-(x-y)}.$$

[See that multipliers are so chosen that the numerator $dx - dy$ is an exact differential, namely $d(x-y)$]

Again, choosing multipliers as $1, 1, 1$, each ratio of (1)

$$= \frac{dx + dy + dz}{2(x + y + z)} = \frac{1}{2}\frac{d(x + y + z)}{x + y + z}.$$

Lastly choose $0, 1, -1$ as multipliers, then each ratio of (1)

$$= \frac{dy - dz}{z + x - (x + y)} = \frac{d(y - z)}{-(y - z)}$$

So we may write

$$-\frac{d(x - y)}{(x - y)} = \frac{1}{2}\frac{d(x + y + z)}{x + y + z} = -\frac{d(y - z)}{y - z}. \tag{2}$$

First two ratio of (2) give, on integration

$$-\log(x - y) = \frac{1}{2}\log(x + y + z) + \text{constant}$$

i.e., $\quad 2\log(x - y) + \log(x + y + z) = \text{constant}$

or, $\quad (x - y)^2(x + y + z) = c_1.$

Taking first and last ratio of (2), and integrating we easily obtain

$$\frac{x - y}{y - z} = c_2$$

$\therefore$ The required general solution is

$$\phi\left((x - y)^2(x + y + z), \frac{x - y}{y - z}\right) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.14** *Solve:* $y^2(x - y)p + x^2(y - x)q = z(x^2 + y^2)$.

[I.A.S. 1996]

Solution: Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{y^2(x - y)} = \frac{dy}{x^2(y - x)} = \frac{dz}{z(x^2 + y^2)}. \tag{1}$$

From the first two ratios we at once get $x^3 + y^3 = c_1$. Choosing $1, -1, 0$ as multipliers each ratio of (1)

$$= \frac{dx - dy}{y^2(x-y) - x^2(y-x)} = \frac{dx - dy}{(x-y)(x^2+y^2)}$$

Combining it with the third ratio of (1) we write

$$\frac{dx - dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)}$$

$$\text{or,} \quad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

On integration we get $\log(x-y) - \log z = \log c_2$ (say)

$$\text{or,} \quad \frac{x-y}{z} = c_2$$

$\therefore$ The required general solution is

$$\phi\left(x^3 + y^3, \frac{x-y}{z}\right) = 0.$$

where ϕ is any arbitrary function.

► **Example 2.3.15** *Solve:* $(x^2 - y^2 - z^2)p + 2xyq = 2xz.$

[I.A.S. 1973, W.B.C.S. 2001]

Solution: Lagrange's subsidiary equations corresponding to the given equation are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}. \qquad (1)$$

Taking the last two ratios we easily get, on integration $y/z = c_1$. Choosing x, y, z as multipliers of the ratios,

$$\text{each ratio of (1)} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 - y^2 - z^2) + 2xy^2 + 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}.$$

Combining this with third ratio of (1), we write

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz}$$

$$\text{or,} \quad \frac{d(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2} = \frac{dz}{z}.$$

On integration we get $\log(x^2+y^2+z^2) - \log z = \log c_2$

$$\text{or,}\quad \frac{x^2+y^2+z^2}{z} = c_2.$$

Hence the general solution is

$$\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.16** *Solve:* $\cos(x+y)p + \sin(x+y)q = z$.

Solution: Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}. \tag{1}$$

Choosing $1, 1, 0$ as multipliers of the ratios of (1), each ratio of (1)

$$= \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \tag{2}$$

Choosing $1, -1, 0$ as multipliers of the ratios of (1), each ratio of (1)

$$= \frac{d(x-y)}{\cos(x+y)-\sin(x+y)}. \tag{3}$$

From (1), (2) and (3) we write

$$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{d(x-y)}{\cos(x+y)-\sin(x+y)}. \tag{4}$$

Taking the first two fractions of (4)

$$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}.$$

Putting $x+y=t$ on the right side we write

$$\frac{dz}{z} = \frac{dt}{\cos t + \sin t} = \frac{dt}{\sqrt{2}\left\{\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right\}}$$

$$= \frac{dt}{\sqrt{2}\sin\left(\frac{\pi}{4}+t\right)}$$

$$\text{or,}\quad \sqrt{2}\frac{dz}{z} = \operatorname{cosec}\left(\frac{\pi}{4}+t\right)dt.$$

Hence on integration,

$$\log z^{\sqrt{2}} = \log\left\{\tan\left(\frac{\pi}{8}+\frac{t}{2}\right)\right\} + \log c_1$$

$\therefore \quad z^{\sqrt{2}} = c_1 \tan\left(\frac{\pi}{8}+\frac{t}{2}\right)$, where $t = x+y$

$$\text{or,} \quad z^{\sqrt{2}}\cot\left(\frac{\pi}{8}+\frac{t}{2}\right) = c_1. \tag{5}$$

Taking the last two fractions of (4) we get

$$d(x-y) = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y)$$

On the right side put $x+y=t$ and then integrate

$$x-y = \int \frac{\cos t - \sin t}{\cos t + \sin t}dt + \text{constant}$$

$$\text{or,} \quad x-y = \log(\sin t + \cos t) + \log c_2' = \log c_2'(\sin t + \cos t)$$

$$\text{or,} \quad e^{x-y} = c_2'[\sin(x+y)+\cos(x+y)]$$

$$\text{or,} \quad [\sin(x+y)+\cos(x+y)]e^{y-x} = \frac{1}{c_2'} = c_2 \text{ (say)}.$$

Then the general solution can be written as

$$\phi\left(z^{\sqrt{2}}\cot\left(\frac{\pi}{8}+\frac{x+y}{2}\right), e^{y-x}\{\sin(x+y)+\cos(x+y)\}\right) = 0$$

where ϕ is an arbitrary function.

▶ **Example 2.3.17** *Solve:*

$$px(x+y) - qy(x+y) + (x-y)(2x+2y+z) = 0.$$

Solution: Given equation may be written as

$$x(x+y)p - y(x+y)q = -(x-y)((2x+2y+z)$$

Lagrange's subsidiary equations are

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)}. \tag{1}$$

The first two ratios of (1) at once give, on integration $xy = c_1$.
Again each fraction of (1) becomes

$$= \frac{dx + dy}{x^2 + y^2} = \frac{dx + dy + dz}{x(x+y) - y(x+y) - (x-y)(2x+2y+z)}$$

or,
$$\frac{d(x+y)}{(x-y)(x+y)} = \frac{dx + dy + dz}{(x-y)[x+y-2x-2y-z]}$$
$$= \frac{d(x+y+z)}{-(x+y+z)}.$$

Hence, on integration, we get

$$(x+y)(x+y+z) = c_2$$

$\therefore$ The required general solution is

$$\phi(xy, (x+y)(x+y+z)) = 0$$

where ϕ is an arbitrary function.

D

Try Yourself (*Examples for practice*)
Solve the following linear partial differential equations

1. $(1+y)p + (1+x)q = z$.
2. $xzp + yzq = xy$.
3. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.
4. $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x-y)$.
5. $xp + yq = z - a\sqrt{x^2 + y^2 + z^2}$.
6. $(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2z(x^2 + y^2)$. **[I.A.S. 1993]**
7. $p + q = x + y + z$.
8. $(2x^2 + y^2 + z^2 - 2yz - zx - xy)p + (x^2 + 2y^2 + z^2 - yz - 2zx - xy)q = x^2 + y^2 + 2z^2 - yz - zx - 2xy$. **[I.A.S. 1992]**
9. $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x+y)$.
10. $\cos(x+y)p + \sin(x+y)q = z + \frac{1}{z}$.

3. $xyp + y(2x - y)q = 2xz.$

4. $x(y^m - z^m)p + y(z^m - x^m)q = z(x^m - y^m).$

5. $p - qy \log y = z \log y.$

6. $x^2p + y^2q = x + y.$

7. $z(p+q) = z^2 + (x-y)^2.$

8. $(xz + y^2)p + (yz - 2x^2)q + 2xy + z^2 = 0.$

9. $(x^2 + 2y^2)p - xyq = xz.$

Answers

1. $\phi(y-x, e^{-2x}y+x) = 0$ **2.** $\phi\left(xy, xe^{\frac{1}{z-xy}}\right) = 0$. **3.** $\phi(xy-x^2, z/xy) = 0$. **4.** $x^m+y^m+z^m = \phi(xyz)$. **5.** $\phi(yz, e^x \log y) = 0$. **6.** $\phi\left[\frac{1}{y} - \frac{1}{x}, e^{-z}(x-y)\right] = 0$. **7.** $\log\{z^2 + (x-y)^2\} - 2x = \phi(x-y)$. **8.** $\phi(yz + x^2, 2xz - y^2) = 0$. **9.** $\phi(x^2y^2 + y^4, yz) = 0$.

2.4 Integral Surfaces Through a Given Curve

Working Rule:-

Given: $Pp + Qq = R$, a linear p.d.e. of first order. We obtain two independent solutions

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \tag{2.4.1}$$

by using Lagrange's auxiliary equations.

Suppose now we wish to obtain the integral surface which passes through a curve whose parametric equations are

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

where t is a parameter.

(2.4.1) is then written as

$$u[x(t), y(t), z(t)] = c_1 \quad \text{and} \quad v[x(t), y(t), z(t)] = c_2$$

We eliminate t and get a relation involving c_1 and c_2. Finally, putting $c_1 = u$ and $c_2 = v$ we get the required integral surface passing through the given curve.

▶ **Example 2.4.1**. *Find the integral surface of the linear partial differential equation*

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains the straight line $x+y=0, z=1$. **[I.A.S. 1998]**

Solution: Lagrange's auxiliary equations corresponding to the given p.d.e. are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}. \quad (1)$$

Each ratio of (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z^2-(x^2+z)+(x^2-y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}.$$

$\therefore$ $\frac{1}{x}dx + \frac{1}{y}dy + \frac{dz}{z} = 0$, i.e., $\log x + \log y + \log z = \log c_1$

$$\text{or, } xyz = c_1. \quad (2)$$

Each ratio of (1) (Multipliers $x, y, -1$)

$$= \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - (x^2-y^2)z} = \frac{x\,dx + y\,dy - dz}{0}$$

$$\text{or} \quad x\,dx + y\,dy - dz = 0$$

$$\text{i.e.,} \quad x^2 + y^2 - 2z = c_2. \quad (3)$$

Now the straight line in the parametric forms:

$$x = t, \quad y = -t, \quad z = 1.$$

From (2) we then write $-t^2 = c_1$.
From (3) we write $2t^2 - 2 = c_2$.

$\therefore$ The required integral surface can be obtained as $-2c_1 - 2 = c_2$

$$\text{or, } -2(xyz) - 2 = x^2 + y^2 - 2z$$

$$\text{or, } 2xyz + x^2 + y^2 - 2z + 2 = 0.$$

▶ **Example 2.4.2** *Find the equation of the integral surface of the linear differential equation*

$$2y(z-3)p + (2x-z)q = y(2x-3)$$

which passes through the circle $x^2+y^2=2x, z=0$.

Solution: Lagrange's auxiliary equations corresponding to the given equation are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}. \quad (1)$$

From first and the last ratio we get by integration

[i.e., by integrating $(2x-3)dx = 2(z-3)dz$]

We get

$$x^2 - 3x - z^2 + 6z = c_1. \quad (2)$$

Again each ratio of (1)

$$= \frac{\frac{1}{2}dx + y\ dy - dz}{y(z-3) + y(2x-z) - y(2x-3)} \quad \left[\begin{array}{l}\text{Multipliers}\\ \text{are } \frac{1}{2}, y, -1\end{array}\right]$$

$$= \frac{\frac{1}{2}dx + y\ dy - dz}{0}.$$

$$\therefore \quad \frac{1}{2}dx + y\ dy - dz = 0 \quad \Rightarrow \quad \frac{x}{2} + \frac{y^2}{2} - z = \text{constant}$$

$$\Rightarrow \quad x + y^2 - 2z = c_2. \quad (3)$$

Now the given circle in the parametric form is

$$x = t, \quad y = \sqrt{2t - t^2}, \quad z = 0$$

From (2) and (3) we obtain

$$t^2 - 3t = c_1 \quad \text{and} \quad t + (2t - t^2) = c_2$$

Eliminating t, we get $c_1 + c_2 = 0$

$$\therefore \quad x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$$

i.e., $x^2 + y^2 - z^2 - 2x + 4z = 0$ **Reqd. Integral Surface**

We need not always write the equation of the given curve in parametric form. See the example given below:

Example 2.4.3 *Find the integral surface of the linear partial differential equation $(x-y)p+(y-x-z)q = z$ through the circle $x^2+y^2 = 1$, $z = 1$.*

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Solution: Given equation: $(x-y)p+(y-x-z)q=z$

Lagrange's auxiliary equations are:

$$\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z}. \tag{1}$$

Each ratio $=\frac{dx+dy+dz}{(x-y)+(y-x-z)+(z)}=\frac{dx+dy+dz}{0}$

$$\therefore\quad dx+dy+dz=0 \quad \textbf{or,}\quad x+y+z=c_1. \tag{2}$$

Taking the last two ratios of (1) we get $\frac{dy}{y-x-z}=\frac{dz}{z}$

$$\textbf{or,}\quad \frac{dy}{y-x-(c_1-x-y)}=\frac{dz}{z},\quad \text{using (2)}$$

$$\textbf{or,}\quad \frac{dy}{2y-c_1}=\frac{dz}{z}.$$

On integration this gives $\frac{1}{2}\log(2y-c_1)=\log z+\text{constant}$

$$\textbf{or,}\quad \log(2y-c_1)-2\log z=\log c_2$$

$$\textbf{or,}\quad \frac{2y-c_1}{z^2}=c_2$$

$$\textbf{or,}\quad \frac{2y-(x+y+z)}{z^2}=c_2\quad \text{[using (2)]}$$

$$\textbf{or,}\quad y-x-z=c_2z^2. \tag{3}$$

Given curve: $x^2+y^2=1, z=1$

Using $z=1$, in (2) and (3)

$$x+y=c_1-1,\quad (y-x)=c_2+1$$

$$\therefore\quad (x+y)^2+(y-x)^2=(c_1-1)^2+(c_2+1)^2$$

or, $2(x^2+y^2)=(c_1-1)^2+(c_2+1)^2$

or, $c_1^2+c_2^2-2c_1+2c_2=0$ (using $x^2+y^2=1$ of the curve).

From (2) and (3) it follows

$$(x+y+z)^2+\frac{(y-x-z)^2}{z^4}-2(x+y+z)+2\frac{(y-x-z)}{z^2}=0$$

i.e., the required surface is

$$z^4(x+y+z)^2+(y-x-z)^2-2z^4(x+y+z)+2z^2(y-x-z)=0.$$

Example 2.4.4 *Find the integral surface of*

$$x^2p + y^2q + z^2 = 0,$$

which passes through the hyperboloid $xy = x + y, z = 1$. **[I.A.S. 1994]**

Solution: Given equation: $x^2p + y^2q + z^2 = 0$

$$\text{or, } x^2p + y^2q = -z^2. \quad (1)$$

Given curve:

$$xy = x + y, \quad z = 1. \quad (2)$$

Lagrange's auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}. \quad (3)$$

Taking the first and third ratio

$$x^{-2}dx + z^{-2}dz = 0$$

Integrating $-\frac{1}{x} - \frac{1}{z} = -c_1$ (say)

$$\text{or, } \frac{1}{x} + \frac{1}{z} = c_1. \quad (4)$$

Taking the second and third ratio $y^{-2}dy + z^{-2}dz = 0$

Integrating $-\frac{1}{y} - \frac{1}{z} = -c_2$ (say)

$$\text{or, } \frac{1}{y} + \frac{1}{z} = c^2. \quad (5)$$

Adding (4) and (5) $\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = c_1 + c_2$

$$\textbf{or, } \frac{x+y}{xy} + \frac{2}{z} = c_1 + c_2$$

$$\textbf{or, } \frac{xy}{xy} + \frac{2}{1} = c_1 + c_2 \quad \text{[using (2)]}$$

$$\textbf{or, } c_1 + c_2 = 3. \quad (6)$$

Substituting the values of c_1 and c_2 from (4) and (5) in (6), we obtain

$$\frac{1}{x} + \frac{1}{z} + \frac{1}{y} + \frac{1}{z} = 3$$

$$\textbf{or, } yz + 2xy + xz = 3xyz. \quad \textbf{Ans.}$$

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Surfaces orthogonal to a given system of surfaces

Let

$$f(x, y, z) = c \tag{2.4.2}$$

represent a system of surfaces with c as a parameter. We wish to obtain a system of surfaces which cut each surface of (2.4.2) at right angles.

Then the direction ratios of the normal at (x, y, z) to (2.4.2) which passes through the point are

$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y} \text{ and } \frac{\partial f}{\partial z}.$$

Let the surface

$$z = \phi(x, y) \tag{2.4.3}$$

cuts each surface of (2.4.2) at right angles. Then the normal at (x, y, z) to (2.4.3) has direction ratios

$$\frac{\partial z}{\partial x}, \ \frac{\partial z}{\partial y}, \ -1, \ \textbf{i.e.,} \ p, q, -1.$$

Since normals at (x, y, z) to (2.4.2) and (2.4.3) are at right angles, we have

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z} = 0, \tag{2.4.4}$$

which is of the form $Pp + Qq = R$.

Conversely, any solution of (2.4.4) is orthogonal to every surface of (2.4.2).

▶ **Example 2.4.5** *Find the surface which intersects the surface of the system $z(x+y) = c(3z+1)$ orthogonally and which passes through the circle $x^2 + y^2 = 1, z = 1$.* **[I.A.S. 1999]**

Solution: The given system of surfaces is given by

$$f(x, y, z) = \frac{z(x+y)}{3z+1} = c. \tag{1}$$

$$\frac{\partial f}{\partial x} = \frac{z}{3z+1}, \ \frac{\partial f}{\partial y} = \frac{z}{3z+1}, \ \frac{\partial f}{\partial z} = (x+y)\left[\frac{1}{(3z+1)^2}\right].$$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}$$

or $$\frac{z}{3z+1}p+\frac{z}{3z+1}q=\frac{x+y}{(3z+1)^2}$$

or $$z(3z+1)p+z(3z+1)q=x+y. \tag{2}$$

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{z(3z+1)}=\frac{dy}{3(3z+1)}=\frac{dz}{x+y}. \tag{3}$$

Taking the first two fractions of (3) we get $dx-dy=0$ so that

$$x-y=c_1. \tag{4}$$

Choosing $x, y, -z(3z+1)$ as multipliers, each fraction of (3)

$$=\frac{x\ dx+y\ dy-z(3z+1)dz}{0}$$

or, $$x\ dx+y\ dy-3z^2dz-z\ dz=0.$$

Integrating, $\frac{x^2}{2}+\frac{y^2}{2}-3\times\frac{z^3}{3}-\frac{1}{2}z^2=\frac{1}{2}c_2$ (say)

or, $$x^2+y^2-2z^3-z^2=c_2. \tag{5}$$

Hence any surface which is orthogonal to (1) has equation of the form

$$x^2+y^2-2z^3-z^2=\phi(x-y), \tag{6}$$

ϕ being any arbitrary function.

In order to get the required surface passing through the circle $x^2+y^2=1$, $z=1$ we must choose $\phi(x-y)=-2$. Thus the required particular surface is $x^2+y^2-2z^3-z^2=-2$.

▶ **Example 2.4.6** *What is the geometrical description of the solutions of $Pp+Qq=R$ and of the system of equations $dx/P=dy/Q=dz/R$ and establish the relationship between the two.*

▶ **Example 2.4.9** *Find the family of surfaces orthogonal to the family of surfaces given by the differential equation*

$$(y+z)p+(z+x)q=x+y.$$

Solution: Let

$$P=y+z,\quad Q=z+x,\quad R=x+y. \tag{1}$$

Then the given differential equation becomes

$$Pp+Qq=R. \tag{2}$$

Now the differential equation of the family of surfaces orthogonal to the given family is given by

$$Pdx+Qdy+Rdz=0$$
$$\text{or}\quad (y+z)dx+(z+x)dy+(x+y)dz=0$$
$$\text{or}\quad (ydx+xdy)+(ydz+zdy)+(zdx+xdz)=0.$$

Integrating $xy+yz+zx=c$ (c being a parameter) which is the required family of surfaces.

Examples II

1. Find the equation of the integral surface of the differential equation

$$(x^2-yz)p+(y^2-zx)q=z^2-xy$$

which passes through the line $x=1$, $y=0$.

[Following the solved examples of **art. 2.4** obtain $(x-y)(y-z)=c_1$ and $xy+yz+zx=c_2$. Using the equation of the line $x=1$, $y=0$ obtain $c_1c_2=-1$

∴ Required surface $(x-y)(y-z)(xy+yz+zx)+1=0$.]

2. Find the equation of the integral surface satisfying

$$4yzp+q+2y=0$$

and passing through $y^2+z^2=0, x+z=2$. **[I.A.S. 1997]**

3. Find the general integral of the partial differential equation

$$(2xy-1)p+(z-2x^2)q=2(x-yz)$$

and also the particular integral which passes through the line $x=1, y=0$. **[I.A.S. 1981]**

4. Find the equation of the integral surface of the P.D.E.

$$2y(z-3)p+(2x-z)q=y(2x-3)$$

which passes through the circle $z=0, x^2-2x+y^2=0$.

5. Find the general solution of the P.D.E.

$$2x(y+z^2)p+y(2y+z^2)q=z^2$$

and hence prove that $yz(z^2+xz-2y)=x^2$ is a solution.

6. Solve: $xp+yq=z$. Find a solution representing a surface meeting the parabola $y^2=4x, z=1$.

7. Find the surface which is orthogonal to the one-parameter system $z=cxy(x^2+y^2)$ and which passes through the hyperbola $x^2-y^2=a^2$, $z=0$.

Answers

2. $y^2+z^2+x+z-3=0$. **3.** $x^2+y^2+z-xz-y=1$. **4.** $x^2+y^2-2x=z^2-4z$. **6.** General solution $\phi\left(\frac{x}{2},\frac{y}{2}\right)=0$; surface $y^2=4xz$. **7.** $(x^2+y^2+4z^2)(x^2-y^2)^2=a^4(x^2+y^2)$.