2. Find the centre and the radius of the sphere given by

(*i*) $2(x^2+y^2+z^2)-2x+4y-6z=15$; [*B. P. 1968*]

(*ii*) $x^2+y^2+z^2-2x+4y-6z=11$;

(*iii*) $x^2+y^2+z^2-6x+8y=0$.

3. (*a*) Find the equation of the sphere passing through the four points

(*i*) $(0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c)$; [*C. P. 1970*]

(*ii*) $(1, -1, -1), (3, 3, 1), (-2, 0, 5), (-1, 4, 4)$.

Find also the centre and the radius of the sphere.

(*b*) Obtain the equation of the sphere circumscribing the tetrahedron formed by the planes $x=y=z=0$ and

$$2x+3y+4z-12=0.$$ [*C. P. 1981*]

(*c*) The plane $x+2y+3z=6$ meets the co-ordinate axes in *A*, *B*, *C*. Find the equation of the sphere *OABC*, *O* being the origin; and determine the centre and the radius of the sphere. [*B. P. 1981, 2004*]

(*d*) Find the equation of the sphere passing through the points $(3, 1, -3), (-2, 4, 1), (-5, 0, 0)$ and whose centre lies on the plane $2x+y-z+3=0$.

(*e*) Show that the equation of the sphere passing through the points $(0, -2, -4), (2, -1, -1)$ and having its centre on the straight line $2x-3y=0=5y+2z$ is

$$x^2+y^2+z^2-6x-4y+10z+12=0.$$

4. (*a*) (*i*) Find the centre and the radius of the circle

$$x^2+y^2+z^2-2y-4z-11=0, \quad x+2y+2z=15.$$

[*C. P. 2006*]

(*ii*) Find the radius of the circle

$$3x^2+3y^2+3z^2+x-5y-2=0, \quad x+y=2.$$

[*C. P. 1981*]

(*iii*) Find the centre and the radius of the circle

$$(x-3)^2+(y+2)^2+(z-1)^2=100, \quad 2x-2y-z+9=0.$$

[*K. H. 1986*]

(b) Show that the circle

$$x^2+y^2+z^2-2x-4y-6z=2,\ x+2y+2z=20$$

has its centre at (2, 4, 5) and radius $\sqrt{7}$ units.

5. (a) Show that the point (2, −1, 3) lies outside the sphere $(x-3)^2+(y+1)^2+(z-1)^2=4$ and inside the sphere

$$x^2+y^2+z^2-4x+6y-8z+22=0.$$

(b) Find the radius and the co-ordinates of the centre of the sphere $x^2+y^2+z^2-4x+6y+9=0$. Determine whether the point $P(1,-2,1)$ lies inside or outside the sphere. Find the values of k for which the plane $x+y+kz=3$ is a tangent to the sphere. [*C. P. 1977*]

[For tangency of the plane, use the property that the radius of the sphere is equal to the perpendicular distance of the centre from the plane.]

(c) Determine the values of h for which the plane $x+y+z=h$ is a tangent plane to the sphere

$$x^2+y^2+z^2=48.$$ [*C. P. 1980*]

6. (a) Find the equation of the sphere described on the join of $P(2,-3,4)$ and $Q(-1,0,5)$ as diameter. [*V. P. 2002; C. P. 2003*]

(b) Find the equation of the sphere which has (3, 4, −1) and (−1, 2, 3) as the ends of a diameter and find its centre and radius. [*B. P. 1966*]

(c) If (1, 2, 3) be one end point of a diameter of the sphere $x^2+y^2+z^2-2x+4y-6z=2$, then find the other end point of the diameter. [*V. P. 1991*]

(d) A straight line passing through the centre of a sphere cuts the sphere in two points (1, 2, 3) and (5, 6, 5). Find the centre and the radius of the sphere.

7. (a) Find the equation of the sphere through the circle

$$x^2+y^2+z^2=9,\ 2x+3y+4z=5$$

and (i) the point (1, 2, 3); (ii) the origin.

(b) Determine the equation of the sphere passing through the circle

$$x^2+y^2+z^2+6x-8y-4z+4=0,\ x+2y+3z=6$$

and through the point (2, 3, 1). [*C. P. 1979*]

(c) Find the equation of the sphere passing through the point (1, 0, −3) and through the circle represented by $x^2+y^2+z^2-4x-6y+2z-16=0,\ 3x+y+3z-4=0$.

8. Show that the spheres

$$x^2+y^2+z^2=25$$

and $$x^2+y^2+z^2-18x-24y-40z+225=0$$

touch each other at the point $\left(\frac{9}{5},\frac{12}{5},4\right)$.

9. Show that the circle, in which the spheres

$$x^2+y^2+z^2-2x-4y-11=0$$

and $$x^2+y^2+z^2+2x-y+12z+5=0$$

intersect each other, lies in the plane $4x+3y+12z+16=0$. Show also that this plane is perpendicular to the straight line joining the centres of these spheres. [*C. P. 1978*]

10. Determine the positions of the points

(i) (4, 0, 1) and

(ii) (2, 3, 4) with respect to the sphere

$$x^2+y^2+z^2-4x+6y-2z=2.$$

11. (a) Show that the shortest distance of $P(-2,\ 6,\ -3)$ to the sphere $x^2+y^2+z^2=4$ is 5 units and that of $Q(1,\ -1,\ 3)$ from the sphere $x^2+y^2+z^2-6x+4y-10z-62=0$ is 7 units.

(b) Show that the greatest distance of the point (1, −1, 2) from the sphere $x^2+y^2+z^2-4x+6y-8z=71$ is 13 units.

12. On the sphere $(x-1)^2+(y+2)^2+(z-3)^2=25$, find a point P nearest to the plane $3x-4z+19=0$ and show that the distance of P from the plane is 3 units.

13. (a) Find the equation of the sphere having the circle

(i) $x^2+y^2+z^2-2x+4y-6z+7=0,\ 2x-y+2z=5$;

(ii) $x^2+y^2+z^2=9,\ x+y+z+3=0$ [N. B. P. 1999]

as a great circle.

(b) Find the equation of the sphere having the circle $x^2+y^2+z^2=9,\ x-2y+2z=5$ for a great circle; determine its centre and radius.

14. A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by

$$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2.$$ [B. H. 1985]

15. A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the locus of the centroid of the triangle ABC is the sphere

$$9(x^2+y^2+z^2)=4k^2.$$ [B. H. 1983; V. P. 1997]

Answers

1. (i) $x^2+y^2+z^2-4x+6y-8z+4=0$.
   (ii) $x^2+y^2+z^2+2x-4y-6z+5=0$.
   (iii) $x^2+y^2+z^2=14$.
2. (i) $\left(\frac{1}{2}, -1, \frac{3}{2}\right)$, $\sqrt{11}$ units. (ii) $(1, -2, 3)$, 5 units.
   (iii) $(3, -4, 0)$, 5 units.
3. (a) (i) $x^2+y^2+z^2-ax-by-cz=0$, $\left(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c\right)$, $\frac{1}{2}\sqrt{a^2+b^2+c^2}$.
   (ii) $x^2+y^2+z^2-2y-4z=9$, $(0, 1, 2)$, $\sqrt{14}$ units.
   (b) $x^2+y^2+z^2-6x-4y-3z=0$.
   (c) $x^2+y^2+z^2-6x-3y-2z=0$; $\left(3, \frac{3}{2}, 1\right)$; $\frac{7}{2}$ units.
   (d) $(x-1)^2+(y+2)^2+(z-3)^2=49$.
4. (a) (i) $(1, 3, 4)$, $\sqrt{7}$ units, (ii) $\frac{1}{\sqrt{2}}$ unit. (iii) $(-1, 2, 3)$, 8 units.
5. (b) 2 units; $(2, -3, 0)$; inside; $\pm\sqrt{2}$. (c) ± 12.

6. (*a*) $x^2 + y^2 + z^2 - x + 3y - 9z + 18 = 0$.

(*b*) $x^2 + y^2 + z^2 - 2x - 6y - 2z + 2 = 0$, (1, 3, 1), 3 units.

(*c*) (1, −6, 3).

(*d*) (3, 4, 4) ; 3 units.

7. (*a*) (*i*) $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$.

(*ii*) $5(x^2 + y^2 + z^2) - 18x - 27y - 36z = 0$.

(*b*) $5(x^2 + y^2 + z^2) + 28x - 44y - 26z + 32 = 0$.

(*c*) $5(x^2 + y^2 + z^2) - 44x - 38y - 14z - 48 = 0$.

10. (*i*) inside. (*ii*) outside.

12. $P(-2, -2, 7)$.

13. (*a*) (*i*) $9(x^2 + y^2 + z^2) + 2x + 26y - 34z + 13 = 0$.

(*ii*) $x^2 + y^2 + z^2 + 2x + 2y + 2z = 3$.

(*b*) $9(x^2 + y^2 + z^2) - 10x + 20y - 20z - 31 = 0$;

$(5/9, -10/9, 10/9)$, $\frac{2}{3}\sqrt{14}$ units.

... line and a sphere.

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EXERCISE – VI A

1. Find the centre and radius of the following sphere.
   (i) $x^2 + y^2 + z^2 - 8y + 10z - 10 = 0$.
   (ii) $2(x^2 + y^2 + z^2) - 2x + 4y - 6z = 15$.
2. (a) Find the equation of the sphere passing through the following points.
   (i) $(0, 0, 0)$, $(0, 1, -1)$, $(-1, 2, 0)$, $(1, 2, 3)$.
   (ii) $(1, 1, 1)$, $(-2, 1, 2)$, $(3, -3, 1)$, $(-1, 2, -1)$.
   (b) Is there a sphere passing through the points $(1, 2, 3)$, $(2, 5, -4)$, $(1, 4, -3)$ and $(4, 7, -6)$?
   [•• **Hints.** Three points $(2, 5, -4)$, $(1, 4, -3)$ and $(4, 7, -6)$ lie on the line $\dfrac{x-2}{1} = \dfrac{y-5}{1} = \dfrac{z+4}{-1}$. Hence there is no sphere passing through the given points.]
3. Find the equation of the sphere which has the line segment joining the points $(2, 3, 4)$ and $(0, -1, 2)$ as diameter.
4. Discuss the position of the point $(2, -3, 0)$ w.r.t. the sphere $x^2 + y^2 + z^2 + 2x - 4y - 4z + 8 = 0$.
5. Find the equation of the sphere which passes through the origion and makes equal intercepts of unit length of the axes.
6. Find the equation of the sphere circumscribing the tetrahedrom whose faces are
$$\frac{y}{3} + \frac{z}{4} = 0,\ \frac{z}{4} + \frac{x}{2} = 0,\ \frac{x}{2} + \frac{y}{3} = 0,\ \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1.$$
7. Show that the equation to the sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the point $(1, 2, 3)$ is $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$.
8. (a) Show that the equation of the sphere for which the circle $x^2+y^2+z^2+7y-2z+2 = 0$, $2x+3y+4z-8 = 0$ is a great circle is $x^2+y^2+z^2-2x+4y-6z+10 = 0$.
   (b) A sphere S has points $(0, 1, 0)$ and $(3, -5, 2)$ as the ends of a diameter. Show that the equation of the sphere on which the intersection of the plane $5x - 2y + 4z + 7 = 0$ with the given sphere S is a great circle is $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$.
9. Find the coordinates of the centre and radius of the circle $x^2+y^2+z^2-2y-4z-11 = 0$, $x + 2y + 2z = 15$.
10. Show that the equations of that circle on the sphere $x^2+y^2+z^2-2x+4y-6z+3 = 0$ whose centre is $(2, 3, -4)$ are $x^2+y^2+z^2-2x+4y-6z+3 = 0$ and $x+5y-7z-45 = 0$.
11. Find the equation of the sphere which passes through the circle $x^2 + y^2 = 4$, $z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3.
   [•• **Hints.** Let the sphere through the circle $x^2 + y^2 = 4$, $z = 0$ be $x^2 + y^2 + z^2 - 4 + 2\lambda z = 0$. The centre is $(0, 0, -\lambda)$ and radius $= \sqrt{\lambda^2 + 4}$. The distance of

the plane from the centre $= -\frac{2\lambda}{3}$. Here $(\lambda^2+4) - \frac{4\lambda^2}{9} = 9$, or $\lambda = \pm 3$.
Hence there are two spheres with the required property and the equations of them are $x^2+y^2+z^2+6z-4=0$ and $x^2+y^2+z^2-6z-4=0$.]

12. (i) A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the locus of the centroid of the triangle ABC is the sphere $9(x^2+y^2+z^2) = 4k^2$.

(ii) A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by $(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2}) = 4r^2$.

13. Show that the greatest and least distance from the point $(1,-1,2)$ to the sphere $x^2+y^2+z^2-4x+6y-8z-71=0$ are 13 and 7 respectively.

14. Show that the two circles

$$x^2+y^2+z^2+3x-4y+3z=0,\ x-y+2z-4=0$$

and
$$2(x^2+y^2+z^2)+8x-13y+17z-17=0,\ 2x+y-3z+1=0$$

lie on the same sphere and find its equation.

15. Find the equation of the circle passing through the points $(2,0,1)$, $(-2,1,0)$ and $(0,3,5)$.

[•• **Hints.** It is the plane section of the sphere passing through the given points and the origin by the plane through the given points.]

16. Find the equation of the sphere passing through the points $(2,0,0)$, $(0,2,0)$, $(0,0,2)$ and having the least possible radius.

[•• **Hints.** Let the sphere be $(x-a)^2+(y-b)^2+(z-c)^2 = r^2$.
If it passes through the given points then

$$(2-a)^2+b^2+c^2=r^2,\ a^2+(2-b)^2+c^2=r^2,\ a^2+b^2+(2-c)^2=r^2.$$

From these three equations $a=b=c$.

$$\therefore\ (2-a)^2+a^2+a^2=r^2, \text{ or } 3\left[\left(a-\frac{2}{3}\right)^2+\frac{8}{9}\right]=r^2.$$

Therefore the least value of r^2 is $\frac{8}{3}$ when $a=\frac{2}{3}$. Thus the required equation is $3(x^2+y^2+z^2)-4(x+y+z)-4=0$.]

17. Find the smallest sphere (i.e. the sphere of smaller radius) which touches the lines $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z-6}{1}$ and $\frac{x+3}{7} = \frac{y+3}{-6} = \frac{z+3}{1}$.

[•• **Hints.** Since $\begin{vmatrix} 2+3 & 1+3 & 6+3 \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix} \neq 0$, the lines are skewlines.
Hence the smallest sphere is described on the s.d. as diameter. The line of s.d. meets the lines at the points $(1,3,5)$ and $(-3,-3,-3)$. Thus the required sphere is $(x-1)(x+3)+(y-3)(y+3)+(z-5)(z+3)=0$.]

18. Obtain the equation of the sphere passing through four non-coplanar points (x_i, y_i, z_i), $i = 1, 2, 3, 4$.

[•• **Hints.** Let the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. Since the given points lie on the sphere,

$$x_i^2 + y_i^2 + z_i^2 + 2ux_i + 2vy_i + 2wz_i + d = 0, i = 1, 2, 3, 4.$$

Eliminating u, v, w, d from the above five equations, we have

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

It is the required equation.]

19. Show that if all the plane sections of a surface represented by the equation of second degree are circles, the surface must be a sphere.

[•• **Hints.** Let the equation of the surface be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

The section made by the plane $z = 0$ is the conic $ax^2+by^2+2hxy+2ux+2vy+d = 0$. Since it is a circle, $a = b$, $h = 0$.
Similarly for sections by the planes $x = 0$ and $y = 0$, we get $b = c, f = 0$ and $a = c, g = 0$.
$\therefore \quad a = b = c, \ f = g = h = 0.$
Consequently the equation of the surface reduces to

$$a(x^2 + y^2 + z^2) + 2ux + 2vy + 2wz + d = 0.$$

It is the equation of a sphere.]

20. Find the locus of the centre of the sphere which passes through the points $(0, 0, \pm c)$ and cuts the lines $y = \pm x \tan\alpha$, $z = \pm c$ at two points A and B where AB has a constant length $2a$.

[•• **Hints.** Let the sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. If it passes through $(0, 0, \pm c)$, $w = 0$ and $d = -c^2$.
Any points on the lines may be taken as $(r, r\tan\alpha, c)$ and $(r', -r'\tan\alpha, -c)$. If these points lie on the sphere then

$$r = -2(u + v\tan\alpha)\cos^2\alpha \quad \text{and} \quad r' = -2(u - v\tan\alpha)\cos^2\alpha,$$

i.e. $$r - r' = -4v\sin\alpha\cos\alpha \quad \text{and} \quad r + r' = -4u\cos^2\alpha.$$

Again $(r - r')^2 + (r + r')^2\tan^2\alpha + 4c^2 = 4a^2$.

$\therefore$ $$16v^2\sin^2\alpha\cos^2\alpha + 16u^2\sin^2\alpha\cos^2\alpha + 4c^2 = 4a^2,$$

$u^2 + v^2 = (a^2 - c^2)$ cosec$^2 2\alpha$. Hence the locus of the centre is $x^2 + y^2 = (a^2 - c^2)$ cosec^2 2α, $z = 0$.]

ANSWERS

1. (i) $(0, 4, -5)$, $\sqrt{51}$; (ii) $(1/2, -1, 3/2)$, $\sqrt{11}$.
2. (i) $7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0$;
   (ii) $15(x^2 + y^2 + z^2) + 38x + 79y + 24z - 186 = 0$.
3. $x^2 + y^2 + z^2 - 2x - 2y - 6z + 5 = 0$.
4. outside.
5. $x^2 + y^2 + z^2 - x - y - z = 0$.
6. $x^2 + y^2 + z^2 - 29\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{4}\right) = 0$.
9. $(1, 3, 4)$, $\sqrt{7}$.
14. $x^2 + y^2 + z^2 + 5x - 6y + 7z - 8 = 0$.
15. $2(x^2 + y^2 + z^2) - 3x - 16y - 4z = 0$, $7x + 18y - 10z - 4 = 0$.