

Equation of any sphere through the intersection of two given sphere:-

Consider two spheres as,

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0.$$

The general sphere through the intersection of spheres S_1 & S_2 is given by

$$S_1 + \lambda S_2 = 0, \text{ where } \lambda \text{ is an arbitrary constant.}$$

Intersection of two spheres:-

The curve result from the intersection of two sphere is a plane.

Consider two spheres as,

$$S_1: x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$S_2: x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

For any coordinate of points common to S_1 & S_2 we have $S_1 - S_2 = 0$.

$$S_1 - S_2 = 2x(u_1 - u_2) + 2y(v_1 - v_2) + 2z(w_1 - w_2) + (d_1 - d_2) = 0$$

This equation is in first degree and represents a plane.

∴ the equation of the sphere is $x^2 + y^2 + z^2 - ax - by - cz = 0$.

2. Prove that the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ cuts the sphere $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ in a great circle, if $2(uu' + vv' + ww') = 2r'^2 + d + d'$, where r' is the radius of the second sphere.

The equation of the plane of the circle through the given spheres is

$$2(u - u')x + 2(v - v')y + 2(w - w')z + d - d' = 0.$$

If this circle is a great circle of the second sphere, then the centre $(-u', -v', -w')$ lies on it.

$$\therefore -2(u - u')u' - 2(v - v')v' - 2(w - w')w' + d - d' = 0,$$

$$\text{or } 2(uu' + vv' + ww') = 2(u'^2 + v'^2 + w'^2 - d') + d + d',$$

$$\text{or } 2(uu' + vv' + ww') = 2r'^2 + d + d'.$$

3. Find the greatest and the least distance from the point $(2, -1, 1)$ to the sphere $x^2 + y^2 + z^2 - 8x + 4y - 6z + 4 = 0$.

The centre and radius of the sphere are $(4, -2, 3)$ and 5. The distance between the point $(2, -1, 1)$ and the centre

$$= \sqrt{(4 - 2)^2 + (-2 + 1)^2 + (3 - 1)^2} = 3.$$

Since $3 < 5$, the point $(2, -1, 1)$ is within the sphere.

The equation of the line through the point and the centre is

$$\frac{x - 2}{2} = \frac{y + 1}{-1} = \frac{z - 1}{2} = r \quad (\text{say}).$$

If the line meets the sphere at $(2r + 2, -r - 1, 2r + 1)$ then

$$(2r + 2)^2 + (r + 1)^2 + (2r + 1)^2 - 8(2r + 2) - 4(r + 1) - 6(2r + 1) + 4 = 0,$$

$$\text{or } 9r^2 - 18r - 16 = 0, \quad \text{or } r = \frac{8}{3}, -\frac{2}{3}.$$

the line meets the sphere at points $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$ and $\left(\frac{22}{3}, -\frac{11}{3}, \frac{19}{3}\right)$. These points lie on the diameter through the given points. Hence the least distance = $\sqrt{(2 - 2/3)^2 + (-1 + 1/3)^2 + (1 + 1/3)^2} = 2$ and the greatest distance = $10 - 2 = 8$.

4. Prove that the equation to a sphere circumscribing the tetrahedron whose faces are

$$\frac{y}{b} + \frac{z}{c} = 0, \quad \frac{z}{c} + \frac{x}{a} = 0, \quad \frac{x}{a} + \frac{y}{b} = 0, \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{is} \quad \frac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0.$$

$$\begin{aligned} \frac{y}{b} + \frac{z}{c} &= 0 \quad \dots (1), & \frac{z}{c} + \frac{x}{a} &= 0 \quad \dots (2), \\ \frac{x}{a} + \frac{y}{b} &= 0 \quad \dots (3), & \frac{x}{a} + \frac{y}{b} + \frac{z}{c} &= 1 \quad \dots (4). \end{aligned}$$

If we solve these four equations taken three at a time, then we get the vertices of the tetrahedron. These are $(0, 0, 0)$, $(-a, b, c)$, $(a, -b, c)$ and $(a, b, -c)$.

Let the equation of the sphere through these four points be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

The vertices will be on the sphere, if $d = 0$,

$$a^2 + b^2 + c^2 - 2ua + 2vb + 2wc = 0, \quad (5)$$

$$a^2 + b^2 + c^2 + 2ua - 2vb + 2wc = 0, \quad (6)$$

$$a^2 + b^2 + c^2 + 2ua + 2vb - 2wc = 0. \quad (7)$$

Adding (5) and (6), $2w = -\frac{a^2 + b^2 + c^2}{c}$.

Similarly $2v = -\frac{a^2 + b^2 + c^2}{b}$, $2u = -\frac{a^2 + b^2 + c^2}{a}$.

the required equation is

$$x^2 + y^2 + z^2 - (a^2 + b^2 + c^2) \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \right) = 0,$$

$$\text{or} \quad \frac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0.$$

5. Find the equation of the sphere which passes through the origin and touches the sphere $x^2 + y^2 + z^2 = 56$ at the point $(2, -4, 6)$.

$(2, -4, 6)$ is a point on the given sphere and the origin is the centre of it. Since the required sphere passes through the origin, it touches the given sphere internally. Moreover it is described on the line segment joining the points $(0, 0, 0)$ and $(2, -4, 6)$ as a diameter. Hence the required equation is

$$x(x - 2) + y(y + 4) + z(z - 6) = 0,$$

$$\text{or} \quad x^2 + y^2 + z^2 - 2x + 4y - 6z = 0.$$

∴ the coordinates of the centre are (2, 2, -3).
 7. Obtain the equation of the circle lying on the sphere

$$x^2 + y^2 + z^2 - 2x + 2y - 4z + 3 = 0$$

and having its centre at the point (2, 2, -3).

The equations of the circle are the given equation of the sphere and a plane passing through (2, 2, -3) and normal to the line joining (2, 2, -3) and the centre of the given sphere.

The centre of the sphere is (1, -1, 2). The d.rs. of the line joining the points (2, 2, -3) and (1, -1, 2) are 1, 3, -5.

∴ the required plane is

$$x - 2 + 3(y - 2) - 5(z + 3) = 0,$$

$$\text{or } x + 3y - 5z - 23 = 0.$$

Hence the equations of the circle are

$$x^2 + y^2 + z^2 - 2x + 2y - 4z + 3 = 0,$$

$$x + 3y - 5z - 23 = 0.$$

8. Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 2x - 4y + 2z + 5 = 0$, $x - 2y + 3z + 1 = 0$ is a great circle.

The equation of a sphere through the given circle is

$$x^2 + y^2 + z^2 + 2x - 4y + 2z + 5 + \lambda(x - 2y + 3z + 1) = 0,$$

where λ is a parameter,

$$\text{or } x^2 + y^2 + z^2 + 2(1 + \lambda/2)x - 2(2 + \lambda)y + 2(1 + 3\lambda/2)z + 5 + \lambda = 0.$$

$$\text{Its centre is } \left\{ -\left(1 + \frac{\lambda}{2}\right), 2 + \lambda, -\left(1 + \frac{3\lambda}{2}\right) \right\}.$$

If the circle is a great circle, this centre lies on the given plane.

$$\therefore -\left(1 + \frac{\lambda}{2}\right) - 2(2 + \lambda) - 3\left(1 + \frac{3\lambda}{2}\right) + 1 = 0,$$

$$\text{or } -7\lambda - 7 = 0, \quad \text{or } \lambda = -1.$$

Hence the equation is $x^2 + y^2 + z^2 + x - 2y - z + 4 = 0$.
 9. Show that the circles

$$\begin{aligned} & x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, \quad 5y + 6z + 1 = 0 \\ \text{and} \quad & x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0, \quad x + 2y - 7z = 0 \end{aligned}$$

lie on the same sphere and find its equation.

The equation of any sphere through the first circle is

$$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + \lambda_1 (5y + 6z + 1) = 0 \quad (1)$$

and that of any sphere through the second circle is

$$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 + \lambda_2 (x + 2y - 7z) = 0. \quad (2)$$

These equations will be identical, if

$$\begin{aligned} -2 &= \lambda_2 - 3 & (3), & & 3 + 5\lambda_1 &= 2\lambda_2 - 4 & (4), \\ 4 + 6\lambda_1 &= 5 - 7\lambda_2 & (5), & & \lambda_1 - 5 &= -6 & (6). \end{aligned}$$

From (3) and (6), $\lambda_1 = -1, \lambda_2 = 1$.

These values of λ_1 and λ_2 satisfy the equations (4) and (5). Thus the equations (3), (4), (5) and (6) are consistent. Consequently the two circles lie on the same sphere and the equation of it is

$$\begin{aligned} & x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 - (5y + 6z + 1) = 0, \\ \text{or} \quad & x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0. \end{aligned}$$

10. A plane passes through a fixed point (p, q, r) and cuts the axes in A, B, C . Show that the locus of the centre of the sphere $OABC$ is $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2$.

Let the equation of the plane ABC be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. (1)

The equation of the sphere $OABC$ is $x^2 + y^2 + z^2 - ax - by - cz = 0$.

If (α, β, γ) be the centre of this sphere, then

$$\alpha = \frac{a}{2}, \quad \beta = \frac{b}{2} \quad \text{and} \quad \gamma = \frac{c}{2}. \quad (2)$$

Since the plane (1) passes through the point (p, q, r) ,

$$\frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1. \quad (3)$$

Putting the values of a, b and c from (2) in (3), we have $\frac{p}{\alpha} + \frac{q}{\beta} + \frac{r}{\gamma} = 2$.

Hence the required locus is $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2$.