∵ λ_2 can be interpreted as the marginal utility of relaxing the constrained.

∴ $\lambda_2 = 0$

Find Kuhn Tucker condⁿ :-

Maximise $U = U(x,y)$

s. to $P_x x + P_y y \leq B$

$x \leq x_0$

$x, y \geq 0$

The langrangian fⁿ is

$$Z = U(x,y) + \lambda_1 (B - P_x x - P_y y) + \lambda_2 (x_0 - x)$$

& the Kuhn-Tucker conditions are

$$Z_x = U_x - P_x \lambda_1 - \lambda_2 \leq 0 \qquad x \geq 0 \;\&\; x Z_x = 0$$
$$Z_y = U_y - P_y \lambda_1 \leq 0 \qquad y \geq 0 \;\&\; y Z_y = 0$$
$$Z_{\lambda_1} = B - P_x x - P_y y \geq 0 \qquad \lambda_1 \geq 0 \;\&\; \lambda_1 Z_{\lambda_1} = 0$$
$$Z_{\lambda_2} = x_0 - x \geq 0 \qquad \lambda_2 \geq 0 \;\&\; \lambda_2 Z_{\lambda_2} = 0$$

It is useful to examine the implications of the 3rd column of the Kuhn-Tucker conditions. The condⁿ $\lambda_1 Z_{\lambda_1} = 0$, in particular, requires that,

$$\lambda_1 (B - P_x x - P_y y) = 0$$

∴ we must have either $\lambda_1 = 0$ or $B - P_x x - P_y y = 0$

If we interpret λ_1 as the marginal utility of budget money (income), & if the budget constraint is nonbinding (satisfied as an ~~ineq~~ inequality in the solⁿ, with money left over), the marginal utility of B should be zero ($\lambda_1 = 0$).

Similarly, the cond $\lambda_2 Z_{\lambda_2} = 0$ requires that

$$\lambda_2 = 0 \quad \text{or} \quad x_0 - x = 0$$

Since λ_2 can be interpreted as the marginal utility of relaxing the constraint, we see that if the ration constraint is nonbinding, the marginal utility of relaxing the constraint should be zero ($\lambda_2 = 0$).

This feature, referred to as complementary slackness plays an essential role in search for a solⁿ. We shall now illustrate this with a numeric example:

$$\text{Maximize } U = xy$$
$$\text{s.t.c } \quad x + y \leq 100$$
$$x \leq 40$$
$$x, y \geq 0$$

∴ The langrangian fⁿ is:

$$Z = xy + \lambda_1(100 - x - y) + \lambda_2(40 - x)$$

and the Kuhn-Tucker condⁿ becomes:

$$Z_x = y - \lambda_1 - \lambda_2 \leq 0 \qquad x \geq 0, \; xZ_x = 0$$
$$Z_y = x - \lambda_1 \leq 0 \qquad y \geq 0, \; yZ_y = 0$$
$$Z_{\lambda_1} = 100 - x - y \geq 0 \qquad \lambda_1 \geq 0, \; \lambda_1 Z_{\lambda_1} = 0$$
$$Z_{\lambda_2} = 40 - x \geq 0 \qquad \lambda_2 \geq 0, \; \lambda_2 Z_{\lambda_2} = 0$$

To solve a non-linear programming problem, the typical approach is one of trail & error. We can, for example, start by trying a zero value for choice variable. Setting a variable equal to zero always simplifies the marginal conditions by causing certain terms to drop out. If appropriate nonnegative values of Langrange multiplies can then be found that satisfy all the marginal inequalities, the zero solⁿ will be optimal. If, on the other hand, the zero solⁿ violates some of the inequalities, then we must let one or more choice variable be +ve. For every the choice variable, we may, by complementary slackness, convert a weak inequality marginal condⁿ into a strick equality. Properly solved such an equality will lead us either to a solⁿ, or a contradiction that would then compel us to try something else. If a solⁿ exists, such trials will eventually enable us to uncover it. We can also start by assuming one of the constraints to be nonbinding.

Then the related langrange multiplier will be zero by complementary slackness & we have thus eliminated a variable. If this assumption leads to a contradiction, then, we must treat the said constraint as a strict equality & proceed on that basis.

$$Z_x = Z_y = 0$$

$$y - \lambda_1 - \lambda_2 = x - \lambda_1 (= 0)$$

$$\Rightarrow y - \lambda_2 = x$$

Now, assume the ration constraint to be nonbinding in the soln, which implies that $\lambda_2 = 0$. Then we have $x = y$, & the given budget $B = 100$ yields the trail soln $x = y = 50$. But this soln violates the ration constraint $x \leq 40$. Hence we must adopt the alternative assumption that the ration constraint is binding with $x^* = 40$. The budget constraint the allows the consumer to have $y^* = 60$. Moreover, since complementary slackness dicates that, $Z_x = Z_y = 0$, we can readily calculate the $\lambda_1^* = 40$, & $\lambda_2^* = 20$

Q. Minimize : $C = (x_1 - 4)^2 + (x_2 - 4)^2$

s.t c : $2x_1 + 3x_2 \geq 6$

$-3x_1 - 2x_2 \geq -12$

$x_1, x_2 \geq 0$

The langrangain fn is:

$$Z = (x_1 - 4)^2 + (x_2 - 4)^2 + \lambda_1 (6 - 2x_1 - 3x_2) + \lambda_2 (-12 + 3x_1 + 2x_2)$$

∴ The Kuhn Tucker conditions are:-

$$\frac{\delta Z}{\delta x_1} = 2(x_1 - 4) - 2\lambda_1 + 3\lambda_2 \geq 0$$

$$\frac{\delta Z}{\delta x_2} = 2(x_2 - 4) - 3\lambda_1 + 2\lambda_2 \geq 0$$

$$\frac{\delta z}{\delta \lambda_1} = 6 - 2x_1 - 3x_2 \leq 0$$

$$\frac{\delta z}{\delta \lambda_2} = 3x_1 + 2x_2 - 12 \leq 0$$

plus the nonnegativity & complementary-slackness condⁿ

let $\lambda_1 > 0$ & $\lambda_2 > 0$.

$$\Rightarrow \frac{\delta z}{\delta \lambda_1} = \frac{\delta z}{\delta \lambda_2} = 0.$$

Thus, $2x_1 + 3x_2 = 6$ & $3x_1 + 2x_2 = 12$

$$\Rightarrow x_1 = \frac{6 - 3x_2}{2} \qquad \Rightarrow 3\left(\frac{6 - 3x_2}{2}\right) + 2x_2 = 12$$

$$\Rightarrow \frac{18 - 9x_2}{2} + 4x_2 = 24$$

$$\Rightarrow -5x_2 = 24 - 18$$

$$\Rightarrow x_2 = -6/5 \text{ or } -1\tfrac{1}{5}$$

$$\Rightarrow x_1 = \frac{6 - 3\left(-6/5\right)}{2} = \frac{6 + 18/5}{2} = 24/5 = 4\,4/5$$

This violates the nonnegativity restriction on x_2.

$\therefore$ let $x_1 > 0$ & $x_2 > 0$

$$\Rightarrow \frac{\delta z}{\delta x_1} = \frac{\delta z}{\delta x_2} = 0 \text{ by complementary slackness.}$$

$$\Rightarrow \underbrace{2(x_1 - 4) - 2\lambda_1 + 3\lambda = 0}_{\times 2} \ \& \ \underbrace{2(x_2 - 4) - 3\lambda_1 + 2\lambda = 0}_{\times 3}$$

$$\Rightarrow 4x_1 - 16 - 4\lambda_1 + \cancel{6\lambda}$$
$$\underline{-6x_2 \mp 24 \mp 9\lambda_1 \pm \cancel{6\lambda}}$$
$$4x_1 - 6x_2 + 8 + 5\lambda_1 = 0$$

$\Rightarrow$ Assuming $\lambda_1 = 0$, we get $2x_1 - 3x_2 + 4 = 0$

$$\Rightarrow x_1 = \frac{-4 + 3x_2}{2}$$

$\therefore -12 + 3x_1 + 2x_2 = 0$

$$\Rightarrow -12 + 3\left(\frac{-4 + 3x_2}{2}\right) + 2x_2 = 0 \Rightarrow \boxed{x_1 = \frac{28}{13} \ \& \ x_2 = \frac{36}{13}}$$

4. Is the Arrow-Enthoven constraint qualification satisfied, given that the constraints of a *maximization* problem are:
   (*a*) $x_1^2 + (x_2 - 5)^2 \le 4$ and $5x_1 + x_2 < 10$
   (*b*) $x_1 + x_2 \le 8$ and $-x_1 x_2 \le -8$ (*Note:* $-x_1 x_2$ is not convex.)

Maximum-Value Functions and the Envelope Theorem†

A maximum-value function is an objective function where the choice variables have been assigned their optimal values. These optimal values of the choice variables are, in turn, functions of the exogenous variables and parameters of the problem. Once the optimal values of the choice variables have been substituted into the original objective function, the function indirectly becomes a function of the parameters only (through the parameters' influence on the optimal values of the choice variables). Thus the maximum-value function is also referred to as the *indirect objective function*.

The Envelope Theorem for Unconstrained Optimization

What is the significance of the indirect objective function? Consider that in any optimization problem the direct objective function is maximized (or minimized) for a given set of parameters. The indirect objective function traces out all the maximum values of the objective function as these parameters vary. Hence the indirect objective function is an "envelope" of the set of optimized objective functions generated by varying the parameters of the model. For most students of economics the first illustration of this notion of an envelope arises in the comparison of short-run and long-run cost curves. Students are typically taught that the long-run average cost curve is an envelope of all the short-run average cost curves (what parameter is varying along the envelope in this case?). A formal derivation of this concept is one of the exercises we will be doing in this section.

To illustrate, consider the following unconstrained maximization problem with two choice variables x and y and one parameter ϕ:

$$\text{Maximize} \quad U = f(x, y, \phi) \tag{13.27}$$

The first-order necessary condition is

$$f_x(x, y, \phi) = f_y(x, y, \phi) = 0 \tag{13.28}$$

If second-order conditions are met, these two equations implicitly define the solutions

$$x^* = x^*(\phi) \qquad y^* = y^*(\phi) \tag{13.29}$$

If we substitute these solutions into the objective function, we obtain a new function

$$V(\phi) = f(x^*(\phi),\ y^*(\phi), \phi) \tag{13.30}$$

where this function is the value of f when the values of x and y are those that maximi $f(x, y, \phi)$. Therefore, $V(\phi)$ is *the maximum-value function* (or indirect objective functio

† This section of the chapter presents an overview of the envelope theorem. A richer treatment of th topic can be found in Chap. 7 of *The Structure of Economics: A Mathematical Analysis* (3rd ed.) by Eugene Silberberg and Wing Suen (McGraw-Hill, 2001) on which parts of this section are based.

If we differentiate V with respect to ϕ, its only argument, we get

$$\frac{dV}{d\phi} = f_x \frac{\partial x^*}{\partial \phi} + f_y \frac{\partial y^*}{\partial \phi} + f_\phi \tag{13.31}$$

However, from the first-order conditions we know $f_x = f_y = 0$. Therefore, the first two terms disappear and the result becomes

$$\frac{dV}{d\phi} = f_\phi \tag{13.31$'$}$$

This result says that, at the optimum, as ϕ varies, with x^* and y^* allowed to adjust, the derivative $dV/d\phi$ gives the same result as if x^* and y^* are treated as constants. Note that ϕ enters the maximum-value function (13.30) in three places: one direct and two indirect (through x^* and y^*). Equation (13.31′) shows that, at the optimum, only the direct effect of ϕ on the objective function matters. This is the essence of the envelope theorem. The envelope theorem says that only the direct effects of a change in an exogenous variable need be considered, even though the exogenous variable may also enter the maximum-value function indirectly as part of the solution to the endogenous choice variables.

The Profit Function

Let us now apply the notion of the maximum-value function to derive the profit function of a competitive firm. Consider the case where a firm uses two inputs: capital K and labor L. The profit function is

$$\pi = Pf(K, L) - wL - rK \tag{13.32}$$

where P is the output price and w and r are the wage rate and rental rate, respectively.

The first-order conditions are

$$\begin{aligned} \pi_L &= Pf_L(K, L) - w = 0 \\ \pi_K &= Pf_K(K, L) - r = 0 \end{aligned} \tag{13.33}$$

which respectively define the input-demand equations

$$\begin{aligned} L^* &= L^*(w, r, P) \\ K^* &= K^*(w, r, P) \end{aligned} \tag{13.34}$$

Substituting the solutions K^* and L^* into the objective function gives us

$$\pi^*(w, r, P) = Pf(K^*, L^*) - wL^* - rK^* \tag{13.35}$$

where $\pi^*(w, r, P)$ is the *profit function* (an indirect objective function). The profit function gives the maximum profit as a function of the exogenous variables w, r, and P.

Now consider the effect of a change in w on the firm's profits. If we differentiate the original profit function (13.32) with respect to w, holding all other variables constant, we get

$$\frac{\partial \pi}{\partial w} = -L \tag{13.36}$$

However, this result does not take into account the profit-maximizing firm's ability to make a substitution of capital for labor and adjust the level of output in accordance with profit-maximizing behavior.

In contrast, since $\pi^*(w, r, P)$ is the maximum value of profits for any values of w, r, and P, changes in π^* from a change in w takes all capital-for-labor substitutions into account. To evaluate a change in the maximum profit function caused by a change in w, we differentiate $\pi^*(w, r, P)$ with respect to w to obtain

$$\frac{\partial \pi^*}{\partial w} = (Pf_L - w)\frac{\partial L^*}{\partial w} + (Pf_K - r)\frac{\partial K^*}{\partial w} - L^* \tag{13.37}$$

From the first-order conditions (13.33), the two terms in parentheses are equal to zero. Therefore, the equation becomes

$$\frac{\partial \pi^*}{\partial w} = -L^*(w, r, P) \tag{13.38}$$

This result says that, at the profit-maximizing position, a change in profits with respect to a change in the wage rate is the same whether or not the factors are held constant or allowed to vary as the factor price changes. In this case, (13.38) shows that the derivative of the profit function with respect to w is the negative of the factor demand function $L^*(w, r, P)$. Following the preceding procedure, we can also show the additional comparative-static results:

$$\frac{\partial \pi^*(w, r, P)}{\partial r} = -K^*(w, r, P) \tag{13.39}$$

and

$$\frac{\partial \pi^*(w, r, P)}{\partial P} = f(K^*, L^*) \tag{13.40}$$

Equations (13.38), (13.39), and (13.40) are collectively known as *Hotelling's lemma*. We have obtained these comparative-static derivatives from the profit function by allowing K^* and L^* to adjust to any parameter change. But it is easy to see that the same results will emerge if we differentiate the profit function (13.35) with respect to each parameter while holding K^* and L^* constant. Thus Hotelling's lemma is simply another manifestation of the envelope theorem that we encountered earlier in (13.31′).

Reciprocity Condition

Consider again our two-variable unconstrained maximization problem

$$\text{Maximize} \qquad U = f(x, y, \phi) \qquad \text{[from (13.27)]}$$

where x and y are the choice variables and ϕ is a parameter. The first-order conditions a[re] $f_x = f_y = 0$, which imply $x^* = x^*(\phi)$ and $y^* = y^*(\phi)$.

We are interested in the comparative statics regarding the directions of change in $x^*(\phi)$ a[nd] $y^*(\phi)$ as ϕ changes and the effects on the value function. The maximum-value function

$$V(\phi) = f(x^*(\phi), y^*(\phi), \phi)$$

By definition, $V(\phi)$ giv...