SEMESTER-II

CORE COURSE C3-T

UNIT-I

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SEMESTER-II

HONOURS

CORE COURSE---C3-T

UNIT-I (MARKS-24)

UNIT-I **Dr. Pradip Kumar Gain**

Syllabus for Unit-I: Review of algebraic and order properties of R, $\varepsilon-$neighbourhood of a point in R. Idea of countable sets, uncountable sets and uncountability of R. Bounded above sets, bounded below sets, bounded sets, unbounded sets. Suprema and infima. Completeness property of R and its equivalent properties. The Archimedean property, density property of rational and irrational numbers in R. Intervals. Limit point of a set, isolated points, open set, closed set, derived set, illustrations of Bolzano-Weierstrass theorem for sets, compact set in R, Heine-Borel Theorem.

DEFINITION (Natural Numbers) : The numbers 1,2,3,..........that were discovered in the natural process of counting are known as ***natural numbers***.

The set of all natural numbers is denoted by N, i.e., $N = \{1,2,3,.........\}$.

The set of natural numbers can be determined in terms of some axioms, formulated by G. Peano in 1889.

PEANO'S AXIOMS :

Axiom 1. $1 \in N$. That is, 1 is a natural number.

Axiom 2. For every natural number $n \in N$, there exists a unique natural number $n' \in N$, called the successor of n and n is called the predecessor of n'.

Axiom 3. There exist no predecessor of $1 \in N$. That is, 1 is not successor of any natural number.

Axiom 4. If two natural numbers have the same successors then they are themselves equal. That is, $n' = m' \Rightarrow n = m$

Axiom 5. If G be any set of some natural numbers, such that (i) $1 \in G$ and (ii) $n \in G \Rightarrow n' \in G$, then G contains all natural numbers, i.e., $G = N$

(Axiom 5 is known as principle of finite or mathematical induction)

Well ordering principle of the set of natural numbers :

THEOREM 1.1: (**Well ordering principle**) ***Every non-empty subset of*** N ***has a least element***.

The set $N=\{1,2,3,.........\}$ of all natural numbers is closed under addition and multiplication but not closed under the subtraction and division operations, i.e., the difference between any two numbers of N are not always a member of N and quotient of any two natural numbers are not always a member of N . That is why it became necessary to extend the system of natural numbers by introducing negative of natural numbers and the number zero 0. Thus to every natural number(positive integer) $n \in N$ there is a unique integer $-n$, called additive inverse of n such that $n+(-n)=0$, where 0 has the property such that $n+0=n$ and $n.0=0$ for each integer n .

DEFINITION : (**Integers**) **The union of set of natural numbers(positive integer) and the set of all negative integers $\{........ -3,-2,-1\}$ and the singleton set $\{0\}$ is called the set of all *integers*. The set of all integers is denoted by Z . That is, $Z=\{........ -3,-2,-1,0,1,2,3,...........\}$.**

Clearly, $N \subset Z$.

Now the set $Z=\{........ -3,-2,-1,0,1,2,3,...........\}$ of all integers is closed under addition, multiplication and subtraction but not closed under the division operation. That is why it became necessary to extend the system of all integers.

DEFINITION : (**Rational Numbers**) **Any number which can be expressed as the ratio $\frac{p}{q}$ where $p \in Z$, $q \in N$ and p and q are prime to each other is called a *rational number*. The set of all rational numbers is denoted by Q .**

Clearly, $N \subset Z \subset Q$.

EXAMPLE : i) $5=\frac{5}{1}$, $5 \in Z, 1 \in N$ **(ii)** $-3=\frac{-3}{1}$, $-3 \in Z, 1 \in N$ **(iii)** $\frac{3}{4}=\frac{3}{4}$, $3 \in Z, 4 \in N$ **(iv)** $-\frac{7}{2}=\frac{-7}{2}$, $-7 \in Z, 1 \in N$ **. Clearly** 5, -3, $\frac{3}{4}$, $-\frac{7}{2}$ **are examples of rational numbers.**

Special Characteristics of rational numbers : **Any rational number can be transformed either into a finite decimal form or an infinite recurring decimal form. For example, let us consider the rational numbers $\frac{3}{4}$ and $\frac{4}{3}$. We see that $\frac{3}{4}=.75$ (a finite decimal form) and $\frac{4}{3}=1.33333333.....=1.\dot{3}$ (an infinite recurring decimal).**

Now the set Q of all rational numbers is closed under addition, multiplication and subtraction and division.

ORDER PROPERTIES OF Q :

(i) If a and b are two rational numbers, i.e., $a,b \in Q$ then either $a<b$ or $b<a$ (law of trichotomy)

(ii) If a, b and c are three rational numbers, i.e.,_ $a,b,c \in Q$ then $a<b$ and $b<c$ $\Rightarrow a<c$ (transitive law)

(iii) $a<b \Rightarrow a+c<b+c$ for any $a,b,c \in Q$.

(iv) $a<b$ and $c>0 \Rightarrow ac<bc$ for any $a,b,c \in Q$.

In view of the above order properties we say that the set Q of all rational numbers is an ***ordered set***.

DEFINITION : (Irrational Numbers) Any number which is not a rational number is called an ***irrational number.*** The set of all irrational numbers is denoted by I.

EXAMPLE : Show that $\sqrt{2}$ is not a rational number.

SOLUTION : If possible let $\sqrt{2}$ is a rational number. Then $\sqrt{2}=\frac{p}{q}$, where $p \in Z, q \in N$

Now $1<2<4 \Rightarrow 1<\sqrt{2}<2$. That is, $\sqrt{2}$ lies between two consecutive integers 1 and 2. Hence $\sqrt{2}$ cannot be an integer. So $q \neq 1$. So $\sqrt{2}=\frac{p}{q}=$ a proper fraction.

As per our assumption $\sqrt{2}=\frac{p}{q}$. That is, $2=\frac{p^2}{q^2}$. Or, $2q=\frac{p^2}{q}$ ……………………(1). Left hand side of the relation (1) is an integer whereas right hand side is a proper fraction($\because \frac{p}{q}$ is a proper fraction) and relation (1) shows a proper fraction is equal to an integer which is not possible at all. So our assumption, i.e., "$\sqrt{2}$ is a rational number" is not true. Hence $\sqrt{2}$ is not a rational number.

DEFINITION : (Real Numbers) The union of the set of all rational numbers and the set of all irrational numbers constitute the set of all ***real numbers***. The set of all real numbers is denoted by R. That is, $R=Q \cup I$.

Clearly, $N \subset Z \subset Q \subset R$.

ORDER PROPERTIES OF R :

(i) If a and b are two real numbers, i.e., $a,b \in R$ then either $a<b$ or $b<a$ (law of trichotomy)

(ii) If a, b and c are three real numbers, i.e.,_ $a,b,c \in R$ then $a<b$ and $b<c$ $\Rightarrow a<c$ (transitive law)

(iii) $a<b \Rightarrow a+c<b+c$ for any $a,b,c \in R$.

(iv) $a<b$ and $c>0 \Rightarrow ac<bc$ for any $a,b,c \in R$.

In view of the above order properties we say that the set R of all real numbers is an ***ordered set***.

ALGEBRAIC PROPERTIES OF R :

Addition and multiplication are defined on the set R.

Under addition(+)

(i) $a+b \in R$ for all $a,b \in R$ (Closure property)

(ii) $(a+b)+c=a+(b+c)$ for all $a,b,c \in R$ (Associative property)

(iii) There exists an element 0 in (called the zero element or additive identity) such that $a+0=a$ for all $a \in R$.(existence of additive identity)

(iv) For each $a \in R$ there exists an element $-a \in R$ such that $a+(-a)=0$ (existence of additive inverse for each element)

(v) $a+b=b+a$ for all $a,b \in R$ (commutative property)

Under addition R is an abelian group.

Under multiplication(.)

(i) $a.b \in R$ for all $a,b \in R$ (Closure property)

(ii) $(a.b).c=a.(b.c)$ for all $a,b,c \in R$ (Associative property)

(iii) There exists an element 1 in (called the unity or multiplicative identity) such that $a.1=a$ for all $a \in R$.(existence of multiplicative identity)

(iv) For each $a \in R$ there exists an element $\frac{1}{a} \in R$ such that $a.\frac{1}{a}=1$ (existence of multiplicative inverse for each element)

(v) $a.b=b.a$ for all $a,b \in R$ (commutative property)

Under multiplication R is a commutative group.

(vi) $a.(b+c)=a.b+a.c$ for all $a,b,c \in R$ (left distributive property)

$(a+b).c=a.c+b.c$ for all $a,b,c \in R$ (right distributive property)

The set R of real numbers obeys all the field axioms. Also R is an ordered set.

Hence R is an ordered field.

NOTE : It should be noted that the set Q of all rational numbers also obeys all the field axioms as well as order properties and hence Q is also an ordered field.

DEFINITION : (Intervals) Let $a.b \in R$ such that $a < b$. We write

(i) $[a,b] = \{x : a \le x \le b\}$ (called *closed interval*)

(ii) $(a,b) = \{x : a < x < b\}$ (called *open interval*)

(iii) $[a,b) = \{x : a \le x < b\}$ (called *left closed right open interval*)

(iv) $(a,b] = \{x : a < x \le b\}$ (called *right closed left open interval*)

DEFINITION : (Bounded above set) A set S of real numbers is called ***bounded above*** if there exists a real number b such that $x \le b$, $\forall x \in S$. The number b is called ***upper bound* of S.**

An upper bound may or may not belong to the set S. Every real number greater than an upper bound of a set is also an upper bound of that set. So a bounded above set has infinite number of upper bounds.

EXAMPLE : Let us consider the following sets :

$S_1 = [1,2]$ is bounded above set. Here 2 is obviously an upper bound of S since $x \le 2$, $\forall x \in S$. Every real number greater than 2 is also an upper bound of the set S_1. 2 is the least of all upper bounds of the set S_1.

$S_2 = (1,2)$ is bounded above set. Here 2 is obviously an upper bound of S since $x < 2$, $\forall x \in S$. Every real number greater than 2 is also an upper bound of the set S_2. 2 is the least of all upper bound of the set S_2.

$S_3 = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ is bounded above set. Here 1 is obviously an upper bound of S_3 since $x \le 1$, $\forall x \in S$. Every real number greater than 1 is also an upper bound of the set S_3. 1 is the least of all upper bounds of the set S_3.

$S_4 = \phi$ is bounded above set. Here every real number is an upper bound of S_4.

Least of all upper bounds is called ***least upper bound*** or ***l.u.b*** or ***supremum*** of the set.

NOTE : A set may or may not contain ***l.u.b*** . In the above example S_1 & S_3 contain ***l.u.b*** but S_2 does not contain ***l.u.b***.

DEFINITION : (Bounded below set) A set S of real numbers is called ***bounded below*** if there exists a real number a such that $x \ge a$, $\forall x \in S$. The number a is called ***lower bound* of S.**

A lower bound may or may not belong to the set S. Every real number smaller than a lower bound of a set is also a lower bound of that set. So a bounded below set has infinite number of lower bounds.

EXAMPLE: Let us consider the following sets :

$S_1 = [1,2]$. Here 1 is obviously a lower bound of S since $x \geq 1$, $\forall x \in S$. Every real number smaller than 1 is also a lower bound of the set S_1. 1 is the greatest of all lower bounds of the set S_1.

$S_2 = (1,2)$. Here 1 is obviously a lower bound of S since $x \geq 1$, $\forall x \in S$. Every real number smaller than 1 is also a lower bound of the set S_2. 1 is the greatest of all lower bounds of the set S_2.

$S_3 = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots\ldots\right\}$. Here 0 is obviously a lower bound of S_3 since $x \geq 0$, $\forall x \in S$. Every real number smaller than 0 is also a lower bound of the set S_3. 0 is the greatest of all lower bounds of the set S_3.

$S_4 = \phi$ is bounded below set. Here every real number is a lower bound of S_4.

greatest of all lower bounds is called ***greatest lower bound*** or ***g.l.b*** or ***infimum*** of the set.

NOTE : A set may or may not contain *g.l.b*. In the above example S_1 contains *g.l.b* but S_2 & S_3 do not contain *g.l.b* .

DEFINITION : (**Bounded set**) A set S of real numbers is called ***bounded*** if it is both ***bounded above*** and ***bounded below***.

DEFINITION : (**Suprema**) Least of all upper bounds of a bounded above set is called ***least upper bound*** or ***l.u.b*** or ***supremum*** ***or*** ***suprema*** of the set.

EXAMPLE : Let us consider the following set :

$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots\ldots\right\}$ is bounded above set. Here 1 is obviously an upper bound of S since $x \leq 1$, $\forall x \in S$. Every real number greater than 1 is also an upper bound of the set S. The set of all upper bounds of S is $[1, \infty]$. Clearly 1 is the least of all upper bounds of the set S. Therefore, ***supremum*** or ***suprema*** of S is 1. That is, $\sup S = 1$

DEFINITION : (**Infima**) Greatest of all lower bounds of a bounded below set is called ***greatest lower bound*** or ***g.l.b*** or ***infimum*** or ***infima*** of the set.

EXAMPLE : Let us consider the following set :

$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots\ldots\right\}$ is bounded below set. Here 0 is obviously a lower bound of S since $x \geq 0$, $\forall x \in S$. Every real number smaller than 0 is also a lower bound of the set S. The set of all lower bounds of S is $[-\infty, 0]$. Clearly 0 is the greatest of all lower bounds of the set S. Therefore, infimum or infima of S is 0. That is, $\inf S = 0$

DEFINITION : (**Greatest member of a set**) A real number m is called the ***greastest member*** of a set S if (i) $m \in S$ (ii) m is an upper bound of S.

EXAMPLE: Let us consider the set $S = [1,2]$. Clearly, 2 is the greatest member of the set S, since (i) $2 \in S$ and (ii) 2 is an upper bound of S.
Again if we consider the set $S = (1,2)$, we see that this set has no greatest member, for, 2 is obviously an upper bound of S since $x < 2$, $\forall x \in S$ but $2 \notin S$.

NOTE : An infinite bounded above set may or may not contain greatest member.
NOTE : Every finite set has always a greatest member.

DEFINITION : (**Smallest member of a set**) A real number n is called the ***smallest member*** of a set S if (i) $n \in S$ (ii) n is a lower bound of S.

EXAMPLE: Let us consider the set $S = [1,2]$. Clearly, 1 is the smallest member of the set S, since (i) $1 \in S$ and (ii) 1 is a lower bound of S.
Again if we consider the set $S = (1,2)$, we see that this set has no smallest member, for, 1 is obviously a lower bound of S since $x > 1$, $\forall x \in S$ but $1 \notin S$.

NOTE : An infinite bounded above set may or may not contain smallest member.
NOTE : Every finite set has always a smallest member.

THEOREM 1.2: *Let L and U be two proper subsets of R of real numbers such that*

(i) $L \cup U = R$
(ii) $\alpha \in L$, $\beta \in U \Rightarrow \alpha < \beta$

Then either the subset L has a greastest member or the subset U has a smallest member.

COMPLETENESS PROPERTY OF R

THEOREM 1.3: (**Least upper bound axiom**) *The set of all upper bounds of a bounded above set admits of a smallest member.*

Proof : Let S be the given bounded above set. Let R be the set of all real numbers. Let us consider two proper subsets L and U of R defined as follows:

$L = \{x : x$ is not an upper bound of $S\}$

$U = \{x : x$ is an upper bound of $S\}$

Clearly, $L \neq \phi$ and $U \neq \phi$. Also $L \cup U = R$ and $\alpha \in L$, $\beta \in U \Rightarrow \alpha < \beta$. Then by theorem 1.2, we can say that either the subset L has a greastest member or the subset U has a

smallest member. Let us suppose that L has a greastest member(say ξ). Now $\xi \in L$.Then $\xi \notin U \Rightarrow \xi$ is not an upper bound of S. Then there must exist a number $\alpha \in S$ such that $\xi < \alpha$. Let b is a real number such that $\xi < b < \alpha$. Now $\xi < b \Rightarrow b \in U$...............(1)

Again $b < \alpha \Rightarrow b$ is not an upper bound of S and this implies $b \notin U$................(2)

Here we see that (1) and (2) contradicts each other. So L cannot have greatest member. Therefore, by theorem 1.2, we can say that U has a smallest member.

THEOREM 1.4: (Greatest lower bound axiom) ***The set of all lower bounds of a bounded below set admits of a greatest member.***

Proof : Proof is left to the students.

In view of above axioms, i.e., theorem 1.3 & theorem 1.4 , the set R of all real numbers is *complete.*

ARCHIMEDEAN PROPERTY OF REAL NUMBERS

THEOREM 1.5: (Archimedean property) ***If $x, y \in R$ and $x > 0$,then there exists a positive integer n such that $nx > y$.***

Proof : If possible let $nx \leq y$. Let $A = \{nx : n = 1,2,3,..........\}$. Clearly, y is an upper bound of A. Therefore, A is bounded above set. Also $A \neq \phi$. Hence by l.u.b axiom, least upper bound i.e., supremum of A ($\sup A$) exists ($= m$ say), where $m \in R$.

Now $x > 0$

$\Rightarrow -x < 0$

$\Rightarrow m - x < m$

$\Rightarrow m - x$ is not an upper bound of A.

Therefore, $m - x < px$, for some integer p.

That is, $m < (p+1)x = kx$ (say) (taking $k = p+1$)

$\Rightarrow m$ is not an upper bound of A.

$\Rightarrow m$ is not $\sup A$ .... a contradiction.

Hence $nx > y$.

DENSITY PROPERTY OF R

THEOREM 1.6: (Density property) ***If x and y are two real numbers such that $x < y$, then there exists a rational number r where $x < r < y$.***

Proof : **Case-I** **Let us suppose that** $x > 0$ **and** $0 < x < y$**. Then** $y - x = z > 0$**. Then by Archimedean property, there exists a positive integer** n **such that** $nz > 1$**. That is,** $n(y - x) > 1$**. Which implies** $\frac{1}{n} < (y - x)$**.**

Let $A = \left\{ m : m \in N, \frac{m}{n} > x \right\}$**. Clearly,** $A \neq \phi$ **.(by Archimedean property). By well-ordering principle of natural numbers, we know that every non-empty subset of natural numbers has a least element. Let** A **has the least element** $p > 1$**. Then** $\frac{p}{n} > x$ **but** $\frac{p-1}{n} \leq x$**. Thus** $\frac{p}{n} \leq x + \frac{1}{n} < x + (y - x)$**. This implies** $\frac{p}{n} < y$**. Already we have** $\frac{p}{n} > x$**. So** $x < \frac{p}{n} < y$**.**

Therefore, there exists a rational number $\frac{p}{n} = r$ **(say) contained in** $0 < x < y$**.**

Case-II **Let us suppose that** $x \leq 0 < y$**. Then by Archimedean property, there exists a positive integer** n **such that** $ny > 1$**. Therefore,** $\frac{1}{n} < y$**. This implies** $x < \frac{1}{n} < y$**. Therefore, there exists a rational number** $\frac{1}{n} = r$ **(say) contained in** $x \leq 0 < y$**.**

Case-III **Lastly, let** $x < y \leq 0$**. Then** $0 \leq -y < -x$**. Then by case-I, there is rational number** r **such that** $-y < r < -x$**. This implies** $x < -r < y$**.**

This completes the proof.

CONCLUSION: **The existence of one rational number between** x **and** y **implies the existence of infinitely many rational numbers between** x **and** y**.**

Hence R **is dense with rational numbers.**

COUNTABILITY : COUNTABLE SETS

DEFINITION : **(Equivalent set) For any two sets** A **and** B**, if there exists a one-to-one mapping from the set** A **onto the set** B**, we say that** A **is *equivalent* to** B**.**
Symbolically, $A \sim B$**.**

DEFINITION : (Enumerable/Denumerable set) An infinite set A is said to be ***enumerable*** or ***denumerable*** if A is equivalent to the set N of all natural numbers. In other words A is said to be ***enumerable*** if there exists a bijective mapping $f : N \to A$.

DEFINITION : (**Countable set/ *Uncountable set***) A set which is either empty or finite or enumerable is called ***countable set***. Otherwise it is ***uncountable***.

NOTE : An enumerable set is sometimes called countably infinite set.

DEFINITION : (**Atmost countable set**) A set A is called ***atmost countable*** if A is either finite or a countable set.

EXAMPLES OF COUNTABLE SETS

(A) **The set** $S=\{2,4,6,.........\}$ **of all even natural numbers is enumerable(i.e., countable) because there exists a a one-to-one onto mapping(bijective)** $f:N\to S$ **defined by** $f(n)=2n,\ \ n\in N$ **(In other words** $S\sim N$ **)**

(B) **The set** $S=\{0,2,4,6,.........\}$ **is enumerable(i.e., countable) because there exists a a one-to-one onto mapping(bijective)** $f:N\to S$ **defined by** $f(n)=2n,-2\ \ n\in N$ **(In other words** $S\sim N$ **)**

(C) **The set** $S=\{1^2,2^2,3^2.........\}$ **is enumerable(i.e., countable) because there exists a a one-to-one onto mapping(bijective)** $f:N\to S$ **defined by** $f(n)=n^2,\ \ n\in N$ **(In other words** $S\sim N$ **)**

(D) **The set** $N=\{1,2,3,.........\}$ **is enumerable(i.e., countable) because there exists a a one-to-one onto mapping(bijective)** $f:N\to N$ **defined by** $f(n)=n,\ \ n\in N$ **(In other words** $N\sim N$ **)**

(E) **The set** $Z=\{0,\pm1,\pm2,\pm3,.........\}$ **is enumerable(i.e., countable) because there exists a a one-to-one onto mapping(bijective)** $f:N\to Z$ **defined by**

$f(n)=\dfrac{n}{2}$ **if** n **is even**

$=\dfrac{1-n}{2}$ **if** n **is odd (In other words** $Z\sim N$ **)**

THEOREMS ON COUNTABILITY

THEOREM : 1.7 **An infinite subset of an enumerable set is enumerable.**

COROLLARY : **A subset of an enumerable set is either finite or enumerable.**

THEOREM : 1.8 **The union of a finite set and an enumerable set is enumerable.**

THEOREM : 1.9 **The union of two enumerable sets is enumerable.**

THEOREM : 1.10 **The union an enumerable numbers of enumerable sets is enumerable.**

THEOREM : 1.11 **Every superset of an uncountable set is uncountable.**

THEOREM : 1.12 **If** $f:A\to B$ **is a bijective mapping(i.e.,one-to one onto) and** B **is enumerable then** A **is also enumerable**

PROBLEMS :

PROBLEM-I : ***Show that the set*** Q ***of rational numbers is countable***.

Solution: Let Q^+ **be the set of all positive rational numbers,** Q^- **be the set of all negative rational numbers. Then obviously,** $Q = Q^+ \cup Q^- \cup \{0\}$**.**

Let us consider the following collection of sets:

$$A_1 = \left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \dots\dots\dots, \frac{n}{1}, \dots\dots\dots\dots\right\},$$

$$A_2 = \left\{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \dots\dots\dots, \frac{n}{2}, \dots\dots\dots\dots\right\},$$

$$\vdots$$

$$A_n = \left\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots\dots\dots, \frac{n}{n}, \dots\dots\dots\dots\right\}$$

$$\vdots$$

Clearly, $Q^+ = A_1 \cup A_2 \cup A_3 \cup \dots\dots\dots \cup A_n \cup \dots\dots\dots = \bigcup_{n=1}^{\infty} A_n$**. Now** A_1 **is enumerable since there exists a bijective mapping (i.e.,one-to one onto)** $f : N \to A_1$ **defined by** $f(n) = \frac{n}{1}$**,** $n \in N$**. Similarly,** A_2 **is enumerable since there exists a bijective mapping (i.e.,one-to one onto)** $f : N \to A_2$ **defined by** $f(n) = \frac{n}{2}$**,** $n \in N$**.**

Thus we can see that each of the sets $A_1, A_2, A_3, \dots\dots\dots, A_n, \dots\dots\dots$ **is an enumerable set. Also the collection** $A_1, A_2, A_3, \dots\dots\dots, A_n, \dots\dots\dots$ **is an enumerable collection sets.**

Therefore, $Q^+ = A_1 \cup A_2 \cup A_3 \cup \dots\dots\dots \cup A_n \cup \dots\dots\dots = \bigcup_{n=1}^{\infty} A_n$ **is the union of an enumerable collection of enumerable sets. Hence** Q^+ **is enumerable. Similarly, it can be shown that** Q^- **is enumerable. [Alternative proof : Since** Q^+ **is enumerable and there exists a bijective mapping (i.e.,one-to one onto)** $f : Q^- \to Q^+$ **defined by** $f(x) = -x$**,** $x \in Q^-$**,** Q^- **is also enumerable.]**

Therefore, $Q^{+} \cup Q^{-}$ is enumerable because the union of two enumerable sets is enumerable.

Since $Q^{+} \cup Q^{-}$ is enumerable and $\{0\}$ is finite set, $Q^{+} \cup Q^{-} \cup \{0\}$ is enumerable. That is, $Q \; \left(= Q^{+} \cup Q^{-} \cup \{0\}\right)$ is enumerable.

PROBLEM-II : *Show that the set $(0,1)$ is not enumerable. Hence show that the set R of real numbers is not enumerable.*

Solution: If possible let the set $(0,1)$ is enumerable. Then the numbers in the set $(0,1)$ can be described as $x_1, x_2, x_3, \ldots\ldots\ldots\ldots$. That is, they can be arranged in a succession. Also each real number in $(0,1)$ can be written as infinite decimal forms in the following way.

$$x_1 = 0.\, a_1 a_2 a_3 a_4 a_5 \ldots\ldots\ldots\ldots$$
$$x_2 = 0.\, b_1 b_2 b_3 b_4 b_5 \ldots\ldots\ldots\ldots$$
$$x_3 = 0.\, c_1 c_2 c_3 c_4 c_5 \ldots\ldots\ldots\ldots$$
$$\cdot$$
$$\cdot$$
$$\cdot$$

Let us consider a real number $x_\lambda = 0.\, pqrst\ldots\ldots\ldots\ldots$

where $p = 6(say)$ **if** $a_1 \neq 6$ $\qquad q = 5(say)$ **if** $b_2 \neq 5$

$= 7(say)$ **if** $a_1 = 6$ $\qquad = 6(say)$ **if** $b_2 = 5$ $\qquad$ **and so on.**

According to construction of x_λ in the way described above, $x_\lambda \notin (0,1)$. But According to the numerical value of x_λ, $x_\lambda \in (0,1)$. We thus arrived at a contradiction.

Therefore, the set $(0,1)$ is not enumerable.

2nd part : $(0,1) \subset R$. R i.e., the superset of a non-enumerable set $(0,1)$ is not enumerable.

COROLLARY : ***The set I of all irrational numbers is non-enumerable.***

Proof : Let I be the set of all irrational numbers. Then $Q \cup I = R$. We know tha the union of two enumerable sets is enumerable. Since R is not enumerable, I cannot be enumerable although Q is enumerable.

DEFINITION : (Nested intervals) If $\{I_1, I_2, I_3, \ldots\ldots\ldots\ldots\}$ be a family of intervals such that $I_{n+1} \subset I_n$, $n \in N$ then the family $\{I_1, I_2, I_3, \ldots\ldots\ldots\ldots\}$ is said to be a family of ***nested intervals***.

EXAMPLE : Let $I_1 = (0,1)$, $I_2 = \left(0, \frac{1}{2}\right)$, $I_3 = \left(0, \frac{1}{3}\right)$, …………… be a family of intervals. Clearly, $I_1 \supset I_2 \supset I_3 \supset \ldots\ldots\ldots\ldots$ Therefore, the family $\{I_1, I_2, I_3, \ldots\ldots\ldots\ldots\}$ is a family of nested intervals.

THEOREM : 1.13 If $\{[a_1,b_1],[a_2,b_2],[a_3,b_3],..........\}$ be a family of nested closed and bounded intervals then $[a_1,b_1]\cap[a_2,b_2]\cap[a_3,b_3]\cap..............\neq\phi$

PROBLEM-III : ***Show that the closed interval $[a,b]$ is not enumeable. Hence show that the set R of real numbers is not enumerable.***

Solution: If possible let the set $I=[a,b]$ is enumerable. Then the elements I of can be described as $x_1,x_2,x_3,...............$. That is, they can be arranged in a succession. That is, $I=[a,b]=\{x_1,x_2,x_3,..........\}$. Let us divide the interval $I=[a,b]$ into three subintervals $[a,c]$, $[c,d]$, $[d,b]$(say). Atleast one these three subintervals does not contain x_1. Let us call that subinterval by I_1. Again Let us divide the interval I_1 into three subintervals such that atleast one these three subintervals of I_1 does not contain x_2. Let us call that subinterval by I_2. So I_2 does not contain both x_1 and x_2. Continuing this process we can obtain a family of closed and bounded subintervals I_3, I_4, I_5,...such that I_1 does not contain x_1

I_2 does not contain x_1 and x_2

I_3 does not contain x_1, x_2, x_3

.

.

I_n does not contain x_1, x_2, x_3,................., x_n

.

.

.

Now by the construction of I_1, I_2, I_3,......, $I_1\cap I_2\cap I_3\cap........\cap I_n\cap..............=\phi$ ....(1)

Again as $I_1\supset I_2\supset I_3\supset..............................$, by the theorem of nested interval , we must have $I_1\cap I_2\cap I_3\cap........\cap I_n\cap..............\neq\phi$ ....(2). Clearly, (1) and (2) contradicts each other. Hence $I=[a,b]$ is not enumerable.

2nd part : $[a,b]\subset R$. R i.e., the superset of a non-enumerable set $[a,b]$ is not enumerable.

PROBLEM-IV : ***Show that the set R of all real numbers is not enumerable*(i.e., not countable).**

Solution: If possible let the set R is enumerable (i.e., countable). Then the elements R of can be described as $x_1,x_2,x_3,...............$. That is, they can be arranged in a succession. That is, $R=\{x_1,x_2,x_3,..........\}$. Let us consider the following open intervals :

$$I_1=\left(x_1-\frac{1}{2^2},x_1+\frac{1}{2^2}\right)$$

$$I_2 = \left(x_2 - \frac{1}{2^3}, x_2 + \frac{1}{2^3} \right)$$

.

.

$$I_n = \left(x_n - \frac{1}{2^{n+1}}, x_n + \frac{1}{2^{n+1}} \right)$$

.

.

.

Now $x_1 \in I_1 \Rightarrow \{x_1\} \subset I_1$

$x_2 \in I_2 \Rightarrow \{x_2\} \subset I_2$

.

.

.

$x_n \in I_n \Rightarrow \{x_n\} \subset I_n$

.

.

.

Therefore, $\{x_1\} \cup \{x_2\} \cup \{x_3\} \cup \ldots\ldots \cup \{x_n\} \cup \ldots\ldots\ldots\ldots \subset I_1 \cup I_2 \cup I_3 \cup \ldots\ldots \cup I_n \cup \ldots\ldots\ldots\ldots$

That is, $R \subset I_1 \cup I_2 \cup I_3 \cup \ldots\ldots \cup I_n \cup \ldots\ldots\ldots\ldots$ **...............(A).**

Length of $I_1 = \frac{1}{2}$**, length of** $I_2 = \frac{1}{2^2}$**,........., length of** $I_n = \frac{1}{2^n}$**,.................. . The total length of** I_1**,** I_2 **,** I_3**,......** $= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots\ldots = \dfrac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n} = 1$ **when** $n \to \infty$

Relation (A) shows that the whole real line (whose length is infinite) is contained in the union of intervals whose length add upto 1 which is clearly, not possible.

Hence R **is not enumerable (i.e., not countable).**

DEFINITION : ($\varepsilon -$**neighbourhood) Let** ξ **be a real number. Then any open interval of which** ξ **is a member is called a** ***neighbourhood*** **of** ξ**.**

In particular, for any $\varepsilon > 0$**, the open interval** $(\xi - \varepsilon, \xi + \varepsilon)$ **is called** $\varepsilon -$***neighbourhood*** **of** ξ**.**

$\varepsilon -$**neighbourhood of** ξ **is denoted by** $N_\zeta(\varepsilon)$

EXAMPLE : By $.01-$**neighbourhood of** 5 **we mean the open interval (5-.01 , 5+.01), i.e., we mean the open interval (4.99 , 5.01). Symbolically,** $N_5(.01) =$ **(4.99 , 5.01).**

DEFINITION : (**Interior point**) Let A be any non empty linear point set. Any element $x \in A$ is said to be an ***interior point*** of A if there exists some neighbourhoods of x which lie wholly in A.

EXAMPLE : Every point of $(2,3)$ is an interior point. Where as every point of $[2,3]$ except 2 and 3 is an interior point.

DEFINITION : (**Interior of a set**) The set of all interior points of a set A is called interior of A.

DEFINITION : (**Open set**) A linear point set A is called an ***open set*** if and only if every point of A is an interior point of A.

EXAMPLES : (i) Any open interval is an open set. (ii) ϕ is an open set. (iii) The set $R = (-\infty, \infty)$ an open set. (iv) A finite set is not an open set.

(v) The set Q of all rational numbers is not an open set.

PROBLEM-V ***Prove that the set Q of all rational numbers is not an open set.***

Solution: Let $x \in Q$. Let $N_x(\varepsilon)$ be $\varepsilon -$neighbourhood of x, where $\varepsilon > 0$, ε is arbitrary. Since $N_x(\varepsilon)$ contains rational and as well as irrational numbers, $N_x(\varepsilon) \not\subset Q$. Since ε is arbitrary , there is no such neighbourhood of x which lie wholly in Q. So, x is not an interior point of Q. Hence Q is not an open set.

DEFINITION : (**Boundary point**) A real number x is said to be a ***boundary point*** or a ***frontier point*** of a set A if every neighbourhood of x contains atleast one point of A and atleast one point that does not belong to A.

EXAMPLE : 2 and 3 are two boundary points of the set $A = [2,3]$.

DEFINITION : **(Exterior point)** A real number x is said to be a ***exterior point*** of a set A if there exists some neighbourhood of x which entirely lie outside of the set A.

PROPERTIES OF OPEN SETS

THEOREM 1.14: ***The union of an arbitrary collection of open sets is an open set.***

Proof : Let $S_1, S_2, S_3, \ldots\ldots\ldots\ldots$ be an arbitrary collection of open sets. Let $S = S_1 \cup S_2 \cup S_3 \cup \ldots\ldots \cup S_n \cup \ldots\ldots\ldots = \bigcup_n S_n$. We shall show that S is open set. Let

$x \in S$. Then x be a member of S_n for some n. Let $x \in S_k$ ($n = k$). Since every S_n is open set, S_k is also open. Since $x \in S_k$, x is an interior point of S_k. So there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset S_k$. As each $S_k \subset S$, $(x-\varepsilon, x+\varepsilon) \subset S$. Thus x is an interior point of S. Since x is an arbitrary point of S, every member of S is an interior point of S. Consequently, S is open.

THEOREM 1.15: ***The intersection of finite number of open sets is an open set.***

Proof : Let $S_1, S_2, S_3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots S_n$ be a finite collection of open sets. Let $S = S_1 \cap S_2 \cap S_3 \cap \ldots\ldots \cap S_n \cap \ldots\ldots = \bigcap_{k=1}^{n} S_k$. We shall show that S is open set. If S is empty, i.e., $S = \phi$ then there is nothing to prove because empty set is an open set. Let $S \neq \phi$. Let $x \in S$. Then $x \in S_k$ for every $k = 1,2,3,\ldots\ldots, n$. So is x an interior point of S_k for every $k = 1,2,3,\ldots\ldots, n$. Therefore, there must exist $\varepsilon_k > 0, k = 1,2,3,\ldots\ldots, n$, such that $(x-\varepsilon_k, x+\varepsilon_k) \subset S_k$ for every $k = 1,2,3,\ldots\ldots, n$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots\ldots, \varepsilon_n\}$. Then clearly, $(x-\varepsilon, x+\varepsilon) \subset S_k$ for every $k = 1,2,3,\ldots\ldots, n$. Hence $(x-\varepsilon, x+\varepsilon) \subset \bigcap_{k=1}^{n} S_k = S$. This shows that x is an interior point of S. Since x is an arbitrary point of S, every member of S is an interior point of S. Consequently, S is open.

PROBLEM-VI : ***Show by an example that intersection of an arbitrary collection of open sets may not be open.***

Solution: Let $S_1 = \left(\frac{-1}{1}, \frac{1}{1}\right)$, $S_2 = \left(\frac{-1}{2}, \frac{1}{2}\right)$, $S_3 = \left(\frac{-1}{3}, \frac{1}{3}\right)$,…………………. be an arbitrary collection of open sets. Then $S = S_1 \cap S_2 \cap S_3 \cap \ldots\ldots\ldots\ldots\ldots\ldots = \{0\}$ which is a finite set and hence S is not open.

Again let $S_1 = \left(\frac{-1}{2}, \frac{1}{2}\right)$, $S_2 = \left(\frac{-2}{3}, \frac{2}{3}\right)$, $S_3 = \left(\frac{-3}{4}, \frac{3}{4}\right)$,…………………. be an arbitrary collection of open sets. Then $S = S_1 \cap S_2 \cap S_3 \cap \ldots\ldots\ldots\ldots\ldots\ldots = \left(\frac{-1}{2}, \frac{1}{2}\right)$ which is an open interval and hence open.

DEFINITION : (Limit point) A real number x, may or may not belong to a set A is said to be a *limit point* or a *accumulation point or a cluster point* of a set A, if every neighbourhood of x contains atleast one point of A other than x.

Infact, every neighbourhood of x contains infinitely many point of the set A.

EXAMPLES :

A) Every member of the set $[2,3]$ is a limit point of this set. There is no other limit point of this lying outside of this set.

B) Every member of the set $(2,3)$ is a limit point of this set. There are another two limit points of this lying outside of this set, namely, 2 and 3.

C) The set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$ has only one limit point, namely, 0 but 0 does not belong this set.

D) The set $\left\{\frac{3}{2}, \frac{-4}{3}, \frac{5}{4}, \frac{-6}{5}, \frac{7}{6}, \frac{-8}{7}, \ldots\ldots\ldots\right\}$ has two limit points, namely, -1 and 1 but they do not belong this set.

E) A finite set has no limit point but it does not necessarily mean that an infinite set has always a limit point. The set $N = \{1,2,3\ldots\ldots\}$ of all natural numbers or the set $Z = \{\ldots\ldots\ldots -3,-2,-1,0,1,2,3,\ldots\ldots\ldots\}$ of all integers has no limit point.

DEFINITION : (Isolated point) A real number $x \in S$ is said to be an ***isolated point*** of S if there exists a neighbourhood of x which contains x but no other point of S.

EXAMPLE :

- A finite set contains isolated points only.
- Each point of the set of all integers is an isolated point.
- The set $[2,3]$ or the set $(2,3)$ has no isolated point.

DEFINITION : (Derived set) The set of all limit points of a set S is called ***derived set*** of that set. The derived set of a set S is denoted by S'.

DEFINITION : (Closed set) A set S is called ***closed*** if and only if every limit point of S is member of S. That is, $S' \subset S$.

EXAMPLES :.

- **The derived set of the set $S = \left\{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$ is $S' = \{0\}$ (the only limit point is 0). Clearly, $S' \subset S$. Hence the set S is closed. Whereas the set $S = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$ is not closed, since the only limit point of S is 0 and $S' = \{0\} \not\subset S$.**
- **The derived set of the set $S = (2,3)$ is $S' = [2,3]$. As $S' \not\subset S$, the set S is not closed.**
- **The derived set of the set $S = [1,2]$ is $S' = [1,2]$. As $S' \subseteq S$, the set S is closed.**

- **The empty set ϕ is closed because $\phi' \subset \phi$.**
- **The set R of all real numbers is closed because every limit point of R is a member of R. That is, $R' \subset R$.**
- **The set Q of all rational numbers is not closed because every rational number as well as every irrarional number is a limit point of Q and irrarional limit points do not belong to Q. That is, $Q' \not\subset Q$. Infact, $Q' = R \not\subset Q$.**

DEFINITION : (Dense-In-Itself set) A set S is called ***Dense-In-Itself*** if and only if every member of S is a limit point of S. That is, $S \subset S'$.

EXAMPLES :.

- The set $S = (2,3)$ is Dense-In-Itself because $S' = [2,3]$ and $S \subset S'$.
- The set $S = [1,2]$ is Dense-In-Itself because $S' = [1,2]$ and $S \subseteq S'$.
- The set Q of all rational numbers is Dense-In-Itself because $Q' = R$ and $Q \subset R = Q'$.
- The set R of all real numbers is Dense-In-Itself because every member of R is a limit point of R. That is, $R \subset R'$.

DEFINITION : (Perfect set) A set S is called ***Perfect set*** if $S = S'$. That is, S is both closed and Dense-In-Itself ($S' \subset S$ & $S \subset S'$).

EXAMPLES :.

- The set R of all real numbers is perfect because $R' \subset R$ & $R \subset R'$.
- The set $S = [1,2]$ is perfect because $S' \subset S$ & $S \subset S'$.

DEFINITION : (Closure of a set) The union of a set S and its derived set S' is called ***closure*** of S. The closure of a set is denoted by $\bar{S}$. That is, $\bar{S} = S \cup S'$.
In case of closed set S, $\bar{S} = S$.

THEOREM 1.16: ***If a set S is open then its complement S^c is closed. Conversely, if a set S is closed then its complement S^c is open.***

Proof : Let *a set* S is open. We are to prove S^c is closed. That is, we shall prove $S^{c'} \subset S^c$. Let $x \in S^{c'}$. We have show $x \in S^c$. If possible let, $x \notin S^c$. Then $x \in S$. Since S is open, x is an interior point of S. That is, there exists a neighbourhood of x lie wholly in S. So all the elements of that particular neighbourhood of x will be in the set S not in S^c…………………(i).
Again, $x \in S^{c'}$ (assumption) implies x is a limit point of S^c. That is, every neighbourhood of x will contain at least one element of S^c other than x. Which implies that some points of every neighbourhood of x will be in the set S^c…………………(ii). Clearly, the statements

(i) & (ii) contradicts each other. So, our assumption was wrong. Hence $x \in S^c$. Therefore, $S^{c'} \subset S^c$. This implies S^c is closed.

Conversely, let the *set* S is closed. We are to prove S^c is open. Let $x \in S^c$. Then $x \notin S$. Since the *set* S is closed and $x \notin S$, x is not a limit point of S. So there exist some neighbourhood of x which do not contain any point of S other than x. That is, there exist some neighbourhood of x which lie entirely in S^c. This implies x is an interior point S^c. Since x is arbitrary it follows that every point in S^c is an interior point of S^c. Hence S^c is open.

PROPERTIES OF CLOSED SETS

THEOREM 1.17: ***The union of a finite collection of closed sets is a closed set.***

Proof : Let $S_1, S_2, S_3, ..., S_n$ be a finite collection of closed sets. Then $S_1^c, S_2^c, S_3^c, ..., S_n^c$ is a finite collection of open sets.

Therefore, $(S_1 \cup S_2 \cup S_3 \cup .... \cup S_n)^c = S_1^c \cap S_2^c \cap S_3^c \cap .... \cap S_n^c$ (By De Morgan's Law). Right hand side of the above relation is the intersection of finite number of open sets and hence the intersection is open (by theorem 1.15). Thus $(S_1 \cup S_2 \cup S_3 \cup .... \cup S_n)^c$ is open. Clearly this implies $S_1 \cup S_2 \cup S_3 \cup, ..., \cup S_n$ is closed.

THEOREM 1.18: ***The intersection of an arbitrary collection of closed sets is a closed set.***

Proof : Let $S_1, S_2, S_3, ...$ be an arbitrary collection of closed sets. Then $S_1^c, S_2^c, S_3^c, ...$ is an arbitrary collection of open sets.

Therefore, $(S_1 \cap S_2 \cap S_3 \cap ....)^c = S_1^c \cup S_2^c \cup S_3^c \cup ....$ (By De Morgan's Law). Right hand side of the above relation is the union of an arbitrary collection of open sets and hence the union is open (by theorem 1.14). Thus $(S_1 \cap S_2 \cap S_3 \cap ....)^c$ is open. Clearly this implies $S_1 \cap S_2 \cap S_3 \cap, ...$ is closed.

PROBLEM-VII : ***Show by an example that union of an arbitrary collection of closed sets may not be closed.***

Solution : Let $S_1 = [-1,1]$, $S_2 = \left[\frac{-1}{2}, \frac{1}{2}\right]$, $S_3 = \left[\frac{-1}{3}, \frac{1}{3}\right]$,.......... be an arbitrary collection of closed sets. Then $S_1 \cup S_2 \cup S_3 \cup, ... = [-1,1]$ which is a closed interval and hence closed.

Again let $S_1 = [1,2]$, $S_2 = \left[\frac{1}{2}, 2\right]$, $S_3 = \left[\frac{1}{3}, 2\right]$, ... be an arbitrary collection of closed sets. Then $S_1 \cup S_2 \cup S_3 \cup, ... = (0,2]$ which is not a closed set.

BOLZANO WEIERSTRASS THEOREM

Theorem 1.19: ***Every bounded infinite subset of the set of all real numbers has atleast one limit point.***

DEFINITION : (Covering of a set/ open covering/ subcovering) Let S be a set of some real numbers. A collection F of sets is said to be a *Covering* of S if every $x \in S$ is contained in some sets in F.

If all the sets in F are open sets, then the collection F is called *open covering* of S.

If a subcollection of F is also a covering of S then this subcollection is called *subcovering* of S.

EXAMPLE : **Let $S = (0,1)$. Let us consider the following collection of sets :**

$F = \left\{ \left(\frac{1}{2},1\right), \left(\frac{1}{3},\frac{2}{3}\right), \left(\frac{1}{4},\frac{2}{4}\right), ... \right\}$. **This collection of sets clearly covers the set S. It is also an open covering.**

HEINE-BOREL THEOREM

Theorem 1.20: (Heine-Borel Theorem) Every open covering of a closed and bounded set S of real numbers has a finite subcovering.

If a set S is such that every open covering of S has a finite subcovering then the set S is said to possess the **Heine-Borel property**.

DEFINITION : (Compact Set) A set S is said to be ***Compact*** if and only if the set possesses the Heine-Borel property. That is, the set S is ***Compact*** if and only if every open covering of S contains a finite subcovering.

NOTE : Heine-Borel Theorem is not true if the set S is not closed and bounded.

SOME EXAMPLES ON DERIVED SET

EX : Find the derived set of the set $S=\left\{\frac{1}{n}:n\in N\right\}$**.**

SOLUTION : Let $\varepsilon>0$**. By Archimedean property there exist a positive integer** n **such that**

$$0<\frac{1}{n}<\varepsilon$$

$$\Rightarrow -\varepsilon<\frac{1}{n}<\varepsilon$$

$$\Rightarrow 0-\varepsilon<\frac{1}{n}<0+\varepsilon$$

$\Rightarrow \frac{1}{n}\in(0-\varepsilon,0+\varepsilon)\Rightarrow \varepsilon-$**neighbourhood of** 0 **contains points of the set** S**. Since** ε **is arbitrary, we can say that any neighbourhood of** 0 **contains points of the set** S**. So,** 0 **is the only limit point of the set** S**. Therefore, the derived set of the set** S **is** $S'=\{0\}$**.**

EX : Find the derived set of the set $S=\left\{\frac{(-1)^m}{m}+\frac{1}{n}:m\in N,n\in N\right\}$**.**

SOLUTION : Let $\varepsilon>0$**. By Archimedean property there exist a positive integer** m **such that** $0<\frac{1}{m}<\varepsilon$

$$\Rightarrow -\varepsilon<\frac{1}{m}<\varepsilon$$

$$\Rightarrow -\varepsilon<\frac{(-1)^m}{m}<\varepsilon$$

$$\Rightarrow \frac{1}{n}-\varepsilon<\frac{(-1)^m}{m}+\frac{1}{n}<\frac{1}{n}+\varepsilon$$

$\Rightarrow \frac{(-1)^m}{m}+\frac{1}{n}\in\left(\frac{1}{n}-\varepsilon,\frac{1}{n}+\varepsilon\right)\Rightarrow \varepsilon-$**neighbourhood of** $\frac{1}{n}$ **contains points of the set** S**.**

Since ε **is arbitrary, we can say that any neighbourhood of** $\frac{1}{n}$ **contains points of the set** S**.**

So, $\frac{1}{n}$ **,** $n\in N$ **is limit point of the set** S**.**

Again let $\varepsilon>0$**. By Archimedean property there exist a positive integer** n **such that**

$$0<\frac{1}{n}<\varepsilon$$

$$\Rightarrow -\varepsilon<\frac{1}{n}<\varepsilon \quad \Rightarrow \frac{(-1)^m}{m}-\varepsilon<\frac{(-1)^m}{m}+\frac{1}{n}<\frac{(-1)^m}{m}+\varepsilon$$

$\Rightarrow \frac{(-1)^m}{m}+\frac{1}{n} \in \left(\frac{(-1)^m}{m}-\varepsilon, \frac{(-1)^m}{m}+\varepsilon\right) \Rightarrow \varepsilon-$**neighbourhood of** $\frac{(-1)^m}{m}$ **contains points of the set** S**. Since** ε **is arbitrary, we can say that any neighbourhood of** $\frac{(-1)^m}{m}$ **contains points of the set** S**. So,** $\frac{(-1)^m}{m}$ **is a limit point of the set** S**.**

Again let $\varepsilon > 0$**. By Archimedean property there exist a positive integer** n **such that**

$0 < \frac{(-1)^m}{m}+\frac{1}{n} < \varepsilon$

$\Rightarrow -\varepsilon < \frac{(-1)^m}{m}+\frac{1}{n} < \varepsilon \quad \Rightarrow 0-\varepsilon < \frac{(-1)^m}{m}+\frac{1}{n} < 0+\varepsilon$

$\Rightarrow \frac{(-1)^m}{m}+\frac{1}{n} \in (0-\varepsilon, 0+\varepsilon) \Rightarrow \varepsilon-$**neighbourhood of** 0 **contains points of the set** S**. Since** ε **is arbitrary, we can say that any neighbourhood of** 0 **contains points of the set** S**. So,** 0 **is a limit point of the set** S**.**

Therefore, the derived set of the set S **is** $S' = \left\{\frac{1}{n}, n \in N\right\} \cup \left\{\frac{(-1)^m}{m}, m \in N\right\} \cup \{0\}$**.**

That is, $\quad S' = \left\{0, \pm 1, \frac{1}{2}, \pm\frac{1}{3}, \ldots\ldots\ldots\right\}$**.**

EX : Find the derived set of the set $S = \left\{\frac{2}{m}+\frac{3}{n} : m \in N, n \in N\right\}$**.**

SOLUTION : Left as an exercise.

EX : Find the derived set of the set $S = \left\{(-1)^m + \frac{1}{n} : m \in N, n \in N\right\}$**.**

SOLUTION : Clearly, $S = \left\{(-1)^{2k} + \frac{1}{n} : m \in N, n \in N\right\} \cup \left\{(-1)^{2k-1} + \frac{1}{n} : m \in N, n \in N\right\}$

$= A \cup B$**.**

Since, $\left\{\frac{1}{n}, n \in N\right\}$ **has only limit point** 0**, the set** A **has only limit point** 1 **and the set** B **has only limit point** -1**. Therefore, the derived set of the set** S **is** $S' = (A \cup B)' = A' \cup B' = \{-1, 1\}$**.**

EX : Find the derived set of the set $S=\left\{(-1)^{n-1}\dfrac{n}{n-1}:n\in N\right\}$**.**

SOLUTION : Clearly, $S=\left\{(-1)^{n-1}+\dfrac{(-1)^{n-1}}{n-1}:n\in N\right\}$**. Let** $\varepsilon>0$**. By Archimedean property there exist a positive integer** n **such that** $0<\dfrac{1}{n-1}<\varepsilon$

$$\Rightarrow -\varepsilon<\frac{1}{n-1}<\varepsilon$$

$$\Rightarrow -\varepsilon<\frac{(-1)^{n-1}}{n-1}<\varepsilon$$

$\Rightarrow (-1)^{n-1}-\varepsilon<(-1)^{n-1}+\dfrac{(-1)^{n-1}}{n-1}<(-1)^{n-1}+\varepsilon$. **Therefore,** $\varepsilon-$**neighbourhood of** $(-1)^{n-1}$ **contains points of the set** S**. So,** $(-1)^{n-1}$ **, i.e.,** 1 **and** -1 **are the limit points of the set** S **(according as** n **is odd or even). Therefore, the derived set of the set** S **is** $S'=\{-1,1\}$**.**

QUESTIONS

- Define "limit point" of a set. Show that a finite set cannot have any limit point. Give an example of an infinite set which does not have any limit point.
- State Bolzano Weierstrass theorem. Verify this theorem for the set $S=\left\{1+\dfrac{(-1)^n}{n},n\in N\right\}$.
- Define interior of a set. Find the interior of the set $S=\{x\in R:1<x<3\}$.
- State Peano's axioms for natural numbers. Using this axioms prove that the set of all natural numbers has no upper bound.
- Prove that $\sqrt{17}$ is not rational number.
- Prove that the set Q of all rational numbers is not complete.
- If $x,y\in R$ and $x>0$, then show that there exists a positive integer n such that $nx>y$. (Archimedean property)
- If x and y are two real numbers such that $x<y$, then show that there exists a rational number r where $x<r<y$. (Density property)
- Define enumerable set. Prove that the set Q of all rational numbers is enumerable(countable).
- Is the set $Z=\{0,\pm1,\pm2,\pm3,.........\}$ of all integers enumerable(countable) ? -----justify
- Show that the set $(0,1)$ is not enumerable. Hence show that the set R of real numbers is not enumerable.

- Show that the closed interval $[a,b]$ is not enumeable. Hence show that the set R of real numbers is not enumerable.
- Show that the set R of all real numbers is not enumerable(i.e., not countable).
- Define "open set" , "closed set".
- Prove that the set Q of all rational numbers is neither open nor closed.
- Prove that the intersection of finite collection of open sets is open. Show by an example that the intersection of an infinite collection open sets may not be open.
- Prove that the union of finite collection of closed sets is closed. Show by an example that the union of an infinite collection closed sets may not be closed.
- Find the derived set of the set $S = \left\{ \frac{(-1)^m}{m} + \frac{1}{n} : m \in N, n \in N \right\}$.
- Find the derived set of the set $S = \left\{ \frac{1}{m} + \frac{1}{n} : m \in N, n \in N \right\}$.
- Find the derived set of the set $S = \left\{ \frac{2}{m} + \frac{3}{n} : m \in N, n \in N \right\}$.
- Find the derived set of the set $S = \left\{ (-1)^{n-1} \frac{n}{n-1} : n \in N \right\}$.