Study Material – Sem. 5. – DSE1T –

– Special Theory of Relativity –

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The theory of relativity starts by examining how measurements of physical quantities are affected by relative motion between an observer and what is being observed. The theory of relativity was first definitely formulated by Einstein. The term 'relativity' is applied to Einstein's theory because the primary quantities in mechanics and astronomy such as space, time and mass are shown by it to be relative, i.e, any one of these factors cannot be reckoned absolutely by itself, but only in relation to others. The new theory is in direct contradiction with the fundamental ideas of classical Newtonian mechanics, where space, time and mass are considered absolute. Einstein's theory rejects this absolute nature of fundamental quantities, space, time and mass postulated by classical mechanics by denying their independence from the position or motion of bodies or observers.

Galilean Transformation : Classical Physics

In classical physics, the motion of a body or any physical system is governed by the Newton's laws of motion. Let us consider the simplest mechanical system : a particle of mass m acted upon by a force $\vec{F}$. Then by second law of Newton,

$$\left.\begin{aligned} m\left(\frac{d^2x}{dt^2}\right) &= F_x \\ m\left(\frac{d^2y}{dt^2}\right) &= F_y \\ m\left(\frac{d^2z}{dt^2}\right) &= F_z \end{aligned}\right] \longrightarrow (1)$$

where F_x, F_y, F_z are the x, y and z components of the vector $\vec{F}$ and $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ are the accelerations of the particle along x, y and z axis respectively.

The above equations are valid only if the frame of reference S defined by the coordinates x, y, z, t is an inertial frame. We define an inertial frame as a frame of reference in which the law of inertia — Newton's first law — holds. In such a system, which we may also describe as an unaccelerated system, a body that is acted on by zero net external force will move with a constant velocity. Newton assumed that a frame of reference fixed with respect to the stars is an inertial system. A rocket ship drifting in outer space, without spinning and with its engines cut off, provides an ideal inertial system. Frames accelerating with respect to such a system are not inertial. In practice, we can often neglect the small (acceleration) effects due to the rotation and orbital motion of the earth and to solar motion. Thus we may regard any set of axes fixed on the earth as forming (approximately) an inertial coordinate system. Likewise, any set of axes moving at uniform velocity w.r.t. the earth, as in a train, ship or airplane, will be (nearly) inertial because motion at uniform velocity does not introduce acceleration. However, a system of axes which accelerates w.r.t the earth, such as one fixed to a spinning merry-go-round or to an accelerating car, is not an inertial system.

Let us now consider another inertial frame $S'(x', y', z', t')$ whose axes are parallel to the corresponding axes of the old frame $S(x, y, z, t)$. The new frame moves to the right with a constant velocity v along x-axis relative to old one. Here t and t' are the times measured in the two frames and defined such that $t = t' = 0$ when the two frames coincide. What is the relation between the sets of numbers (x, y, z, t) and (x', y', z', t')?

Plainly,
$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right] \longrightarrow (2)$$

These equations are known as the Galilean transformation equations.

From the first equation of set (2), we have,
$$\frac{dx'}{dt} = \frac{dx}{dt} - v$$

But, because $t = t'$, the operation $\frac{d}{dt}$ is identical to the operation $\frac{d}{dt'}$, so that

$$\frac{dx'}{dt} = \frac{dx'}{dt'}$$

Therefore, $\frac{dx'}{dt'} = \frac{dx}{dt} - v$

Similarly, $\frac{dy'}{dt'} = \frac{dy}{dt}$

and $\frac{dz'}{dt'} = \frac{dz}{dt}$

Then, $\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2}$

$$\frac{d^2y'}{dt'^2} = \frac{d^2y}{dt^2}$$

$$\frac{d^2z'}{dt'^2} = \frac{d^2z}{dt^2}$$

So, in the reference frame $S'(x', y', z', t')$, Newton's second law can be written as –

$$\left.\begin{aligned} m\frac{d^2x'}{dt'^2} &= F_x \\ m\frac{d^2y'}{dt'^2} &= F_y \\ m\frac{d^2z'}{dt'^2} &= F_z \end{aligned}\right] \longrightarrow (3)$$

The equations have exactly the same mathematical form as equation (1). Newton's laws, which govern the behaviour of the system, do not change when we make Galilean transformation. Newton's laws are therefore said to remain invariant in Galilean transformation. The behaviour of all mechanical system will thus be identical in all inertial frames in uniform translation w.r.t each other. A series of mechanical experiments, whether performed in a car at rest w.r.t the ground or moving smoothly over a straight track at constant speed, gives therefore the same result.

Newtonian mechanics thus predicts that all inertial frames in uniform translation relative to each other are equivalent so far as the mechanical phenomena are concerned. A corollary to this is that no mechanical phenomena can distinguish between two inertial frames, so much so that it is impossible by mechanical experiments to prove that one of the frames is in a state of absolute rest while the other moves relative to it. This equivalence of all inertial frames is called Newtonian relativity.

Galilean Transformation & Electromagnetic Theory—

Electromagnetic phenomena are discussed in classical physics in terms of the Maxwell's equations which govern their behaviour. Maxwell's relation comprise a set of differential equations and have the similar role in electromagnetic phenomena as the Newton's laws in mechanical phenomena. Without actually carrying out the Galilean transformation of Maxwell's equations, the calculation being too complicated owing to the nature of the equation, we shall just state the result. It is found that, on Galilean transformation, Maxwell's equations do change their mathematical form, in sharp contrast to Newton's equations. This is a very important point in the development of the theory of relativity.

Maxwell's equations predict the existence of electromagnetic waves that propagate with a velocity not dependent on the frequency of wave motion. With the mechanistic outlook of nineteenth century, physicists felt quite sure that the electromagnetic waves require the existence of a propagation medium. The medium was given the name ether. It appeared from the very start that ether must be attributed some strange properties in order to be in agreement with certain known facts. For instance, as these waves can travel through a vacuum, ether must be massless. But they must at the same time be highly elastic to sustain the vibrations. Despite the difficulties, the ether concept was more attractive than the alternative that electromagne-tic waves require no medium for propagation. Now, once the ether is assumed, electromagnetic waves would expectedly propagate with a fixed velocity with respect to it, as sound waves do with respect to air.

It was assumed the Maxwell's electro-magnetic equations were valid for the frame of reference which was at rest with respect to ether - the so-called ether frame. A solution of these equations gives the propagation velocity of the electromagnetic waves : $c\ (= 3 \times 10^8$

m/sec.) and this is in agreement with experimental observation. But in a new frame in uniform translation with respect to ether, Maxwell's equations, we have seen, do change their form and when the propagation velocity of electromagnetic waves as seen from the new frame was evaluated using the changed form of Maxwell's equations, it was found to have a value different from c.

If $\bar{c}$ be the fixed velocity of light relative to ether, $\bar{c}_0$ the velocity of light relative to the new frame and $\bar{v}$ the velocity of the new frame relative to ether, then $\bar{c}_0 = \bar{c} - \bar{v}$. The velocity $-\bar{v}$ sometimes called the ether wind velocity. The above vector addition of velocities accepts tacitly the validity of the Galilean transformations.

To summarise: Towards the end of the nineteenth century, physics was based upon the following three fundamental assumptions:

(i) The validity of Newton's laws,

(ii) the validity of Maxwell's equation, and

(iii) The validity of Galilean transformation

Everything that was derivable from the above assumptions, was in good experimental agreement. These assumptions predicted that —

(a) all inertial frames in uniform translation with respect to each other were equivalent in so far as the mechanical phenomena were concerned.

(b) but in regard to electromagnetic phenomena, they were not equivalent; there was only one frame of reference, the ether frame, in which the velocity of light was c.

Einstein's Postulates

It seemed imperative to conclude that there is no special frame of reference, the ether frame, with the unique property that

The velocity of light in this frame is equal to c. All frames in uniform translation relative to each other seem equivalent in that the light velocity is the same, measured in any direction, and is c.

Einstein based his special theory of relativity on the following two postulates:

Postulate 1: The laws of physics are the same for all inertial frames of reference having uniform translational velocity with respect to one another.

Postulate 2: The velocity of light in any given frame of reference is independent of the motion of its source.

The consequence of postulate 1 is that all inertial frames are completely equivalent. This may however appear similar to Newtonian relativity. There is a fundamental difference. But Newton assumed the existence of an absolute frame in which his laws of motion are valid and as a result they were also valid in all other frames in uniform translation relative to it. But Einstein does not bring any absolute frame in the picture but considers only the frames in uniform transla-tion with respect to one another. Further, Newton's equivalence of the inertial frames was in respect of mechanical phenomena only. Einstein extends it to all the laws of physics, including the electromagnetic phenomena. Because of this postulate, an observer cannot detect the motion of the system by performing any expriment — mechanical, optical, electrical etc. - conducted in the system. This postulate required Einstein modify either Maxwell's equations or the Galilean transformation, for the two together imply the contrary of the postula-te. Einstein decided to retain the Maxwell's equations unchanged and this was indicated by the second postulate.

The second postulate follows directly from the null result of the Michelson's experiment. By this postulate Einstein was forced to modify the galilean transformation. This was a very bold move. First, the intuitive belief in the validity of Galilean Transformation was too strong among the contemporary physicists. Secondly, any modification in Galilean transformation would at once require some compensatory change in Newton's equations in order that the first postulate is satisfied for mechanics.

Insufficiency of Galilean transformations

Let us consider two frames of reference S and S', S' moving w.r.t. S with uniform velocity v along the x-direction. Let the origins O and O' coincides at a particular instant $t = t' = 0$. Let at that instant a light wave originate at O and O'. In S-system, the wavefront would lie on a sphere, given by —

$$x^2 + y^2 + z^2 = c^2t^2 \longrightarrow (1)$$

where c is the velocity of light.
The eq. (1) may be rewritten, using Galilean transformation equations, as —

$$(x' - vt)^2 + y'^2 + z'^2 = c^2t'^2 \longrightarrow (2)$$

in S'-system. Clearly eq. (2) does not represent a sphere and thus is not in accord with the postulates of relativity. The relativity principle demands that the wavefront in S'-system should also be a sphere given by —

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \longrightarrow (3)$$

This is clearly an impossibility by Galilean transformation equations. A new set of transformation formulae is therefore required for the purpose.

Lorentz Transformation

H.A. Lorentz in 1904 discovered a transformation that does not alter the form of Maxwell's equations upon suitable changes in the field components. Let us consider two inertial coordinate systems, namely, the unprimed system x, y, z, t (called frame S) and the primed system x', y', z', t' (called frame S'). Suppose that the S'-frame moves relative to S-frame in the x-direction with a uniform velocity v. The Lorentz transformation equations are —

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \longrightarrow (1)$$

$$y' = y \longrightarrow (2)$$

$$z' = z \longrightarrow (3)$$

$$t' = \frac{t - (vx/c^2)}{\sqrt{1 - v^2/c^2}} \longrightarrow (4)$$

It can be shown that the form of the Maxwell's wave equations is invariant to Lorentz tranformation, i.e, with suitable changes in the field components, the form of wave equation is the same in frames S and S'.

Derivation of Lorentz Transformation Equations

Here, we shall derive the Lorentz transformation equations on the basis of the special theory of relativity. Let us consider two reference frames S and S', the corresponding coordinate axes of which are parallel. Let the S'-frame moves along the x-axis with a uniform velocity v relative to the S-frame. If an event occurs at the point (x, y, z) at time t to an observer in the S-frame, the same event will occur at the point (x', y', z') at time t' to an observer in the S'-frame. Let us assume that when the origins of the two frames coincide, the two observers in the two frames regulate their clocks to show zero time, i.e, t = t' = 0. Exactly at this instant, a signal of light is sent out from a source at the common origin of S and S'. According to the second postulate of the special Theory of relativity, the velocity of light must be the same in both the frames S and S'. The observers in both the frames will see a spherical light wave propagated outward from the origin of the coordinate systems with the velocity c. For the S-frame observer, the coordinates of a point on the wavefront at time t must satisfy the equation —

$$x^2 + y^2 + z^2 = c^2 t^2 \longrightarrow (1)$$

For an S'-frame observer, the corresponding equation will be —

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \longrightarrow (2)$$

We assume that the transformation relations between the primed and the unprimed coordinates are —

$$x' = k(x - vt) \longrightarrow (3)$$

$$y' = y \longrightarrow (4)$$

$$z' = z \longrightarrow (5)$$

$$t' = at + bx \longrightarrow (6)$$

where k, a and b are the constants to be determined. The transformation relations are linear, i.e, They contain only The first power of The variables. Because oTherwise, a given length in S would have different values in S' depending on its position in S; or a given time interval in S would have different values in S' depending on The numerical settings of The hands of The clock in S. This would violate The homogeneity requirement.

The arguments behind The forms of The assumed transformation relations run as follows. Since S' moves along The common x-axis, y and z coordinate are not affected; justifying equations 4 and 5. As space is isotropic, t' should not involve y and z. Otherwise t' at (x, y, z, t) and at $(x, -y, -z, t)$ would differ. Thus t' must have The form of eq. 6. Again, a point with $x' = 0$ moves in The x-direction with The speed v, so That $x = vt$ must always give $x' = 0$. This expectation leads to eq. 3 for The x'-equation.

Then, equation 2 becomes—

$$k^2(x-vt)^2+y^2+z^2=c^2(at+bx)^2$$

or, $k^2(x^2-2xvt+v^2t^2)+y^2+z^2=c^2(a^2t^2+b^2x^2+2abtx)$

or, $k^2x^2-2k^2xvt+k^2v^2t^2+y^2+z^2=c^2a^2t^2+c^2b^2x^2+2abtc^2x$

or, $x^2(k^2-b^2c^2)+y^2+z^2-2k^2vtx-2abtc^2x=c^2a^2t^2-k^2v^2t^2$

or, $x^2(k^2-b^2c^2)+y^2+z^2-2xt(vk^2+abc^2)=t^2(a^2c^2-k^2v^2)$ $\longrightarrow$ (7)

A comparison of the coefficients of the different terms in eq. (7) with those of the terms in eq. (1) yields —

$k^2-b^2c^2=1$ $\longrightarrow$ (8)

$vk^2+abc^2=0$ $\longrightarrow$ (9)

$a^2c^2-k^2v^2=c^2$ $\longrightarrow$ (10)

From eq. (8) we get,

$1+b^2c^2=k^2$ or, $b^2c^2=k^2-1$ or, $b^2=\frac{k^2-1}{c^2}$ $\longrightarrow$ (11)

From eq. (10) we get,

$a^2c^2=c^2+k^2v^2$ or, $a^2=\frac{k^2v^2+c^2}{c^2}$ $\longrightarrow$ (12)

From eq. (9), we get,

$vk^2=-abc^2$

squaring we get, $v^2k^4=a^2b^2c^4$

or, $k^4v^2=\left(\frac{k^2v^2+c^2}{c^2}\right)\left(\frac{k^2-1}{c^2}\right)c^4=(k^2v^2+c^2)(k^2-1)$

or, $k^4v^2=k^4v^2-k^2v^2+k^2c^2-c^2$

or, $-k^2v^2+k^2c^2-c^2=0$ or, $k^2(c^2-v^2)=c^2$ or, $k^2=\frac{c^2}{c^2-v^2}$

or, $k^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - (v^2/c^2)} = \frac{1}{1 - \beta^2} \longrightarrow$ (13)

where, $\beta = \frac{v}{c}$

$\therefore a^2 = \frac{k^2 v^2 + c^2}{c^2} = 1 + k^2\left(\frac{v^2}{c^2}\right) = 1 + \frac{\beta^2}{1 - \beta^2}$

$= \frac{1 - \beta^2 + \beta^2}{1 - \beta^2} = \frac{1}{1 - \beta^2} = k^2 \longrightarrow$ (14)

$\therefore k = a = \frac{1}{\sqrt{1 - \beta^2}} \longrightarrow$ (15)

from eq. (9), we get,

$abc^2 = -vk^2$

or, $b = -\frac{vk^2}{ac^2} = -\frac{vk}{c^2} = -\frac{v}{c^2\sqrt{1-\beta^2}}$

$= -\frac{\beta}{c\sqrt{1-\beta^2}}$

Substituting k, a & b into eq. (3) & (6), we get

$x' = \frac{x - vt}{\sqrt{1 - \beta^2}}$

$y' = y$

$z' = z$

$t' = \frac{t}{\sqrt{1-\beta^2}} - \frac{\beta x}{c\sqrt{1-\beta^2}} = \frac{t}{\sqrt{1-\beta^2}} - \frac{\frac{v}{c^2}x}{\sqrt{1-\beta^2}}$

$= \frac{t - vx/c^2}{\sqrt{1-\beta^2}}$

These are the well-known Lorentz transformation equations from S-frame to S$'$-frame. For transformation from S$'$ to S, the relations are—

$$x = \frac{x' + vt}{\sqrt{1-\beta^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + vx/c^2}{\sqrt{1-\beta^2}}$$

Two important aspects of Lorentz transformation are: (i) measurements of both time and position depend on the frame of reference of the observer, (ii) Lorentz equations reduce to those of Galileo when v is small compared to c.