**Example 1:**

Prove that $\frac{3^n\Gamma\left(n+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = 2.5.8 \ldots.(3n-1)$

Ans: We know that $\Gamma(n+1) = n\Gamma(n)$

$$\therefore \quad \Gamma\left(n+\frac{2}{3}\right) = \left(n-\frac{1}{3}\right)\Gamma\left(n-\frac{1}{3}\right)$$

$$= \left(n-\frac{1}{3}\right)\left(n-\frac{4}{3}\right)\Gamma\left(n-\frac{4}{3}\right)$$

$$= \left(n-\frac{1}{3}\right)\left(n-\frac{4}{3}\right)\left(n-\frac{7}{3}\right)\ldots\frac{8}{3}\,\frac{5}{3}\,\frac{2}{3}\,\Gamma\left(\frac{2}{3}\right)$$

$$= \frac{(3n-1)(3n-4)(3n-7)\ldots\ldots 8.5.2\ \Gamma\left(\frac{2}{3}\right)}{3^n}$$

Therefore, $\frac{3^n\Gamma\left(n+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = 2.5.8 \ldots.(3n-1).$

**Other forms of Gamma function:**

Gamma function can be expressed in different forms as follow,

1) $\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-ky} y^{n-1}\, dy$

Proof: we know that $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx, \qquad for\ n > 0$

Substituting *x* by *ky* so that $dx = kdy$

Then, $\Gamma(n) = \int_0^\infty e^{-ky}\,(ky)^{n-1} k\, dy = k^n \int_0^\infty e^{-ky}\, y^{n-1}\, dy$

Therefore, $\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-ky} y^{n-1}\, dy.$

2) $\frac{\Gamma(n/2)}{2k^n} = \int_0^\infty e^{-k^2x^2} x^{n-1}\, dx$

Proof: we know that $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx, \qquad for\ n > 0$

Putting $y = k^2x^2 \qquad or, x = \frac{\sqrt{y}}{k}$ so that $dx = \frac{dy}{2k\sqrt{y}}$

Then, $\int_0^\infty e^{-k^2x^2} x^{n-1}\, dx = \int_0^\infty e^{-y}\left(\frac{\sqrt{y}}{k}\right)^{n-1} \frac{dy}{2k\sqrt{y}} = \frac{1}{2k^n}\int_0^\infty e^{-y} y^{\frac{n-1}{2}} y^{-\frac{1}{2}}\, dy$

$$= \frac{1}{2k^n}\int_0^\infty e^{-y} y^{\frac{n-2}{2}}\, dy = \frac{1}{2k^n}\int_0^\infty e^{-y} y^{\frac{n}{2}-1}\, dy = \frac{\Gamma(n/2)}{2k^n}.$$

Therefore, $\frac{\Gamma(n/2)}{2k^n} = \int_0^\infty e^{-k^2x^2} x^{n-1}\, dx$

3) $\Gamma(n) = \int_0^1 \left(log\,\frac{1}{y}\right)^{n-1} dy$

Proof: we know that $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx, \qquad for\ n > 0$

Substituting $e^{-x} = y \quad or, -x = \log y \quad or, x = log\frac{1}{y}$

And $\quad -e^{-x}dx = dy \quad or, dx = -\frac{dy}{e^{-x}} = -\frac{dy}{y}$

Then, $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}dx = -\int_1^0 \left(log\frac{1}{y}\right)^{n-1} y\ \frac{1}{y}dy$

$$= \int_0^1 \left(log\frac{1}{y}\right)^{n-1} dy$$

4) $\frac{\Gamma(m)}{n^m} = \int_0^1 \left[log\left(\frac{1}{x}\right)\right]^{m-1} x^{n-1}\, dx$

<u>Proof:</u> Putting $log\left(\frac{1}{x}\right) = y \quad or, x = e^{-y} \quad and\ dx = -e^{-y}dy$ we get

$$\int_0^1 \left[log\left(\frac{1}{x}\right)\right]^{m-1} x^{n-1}\, dx = -\int_\infty^0 (e^{-y})^{n-1}\, y^{m-1}e^{-y}\, dy$$

$$= \int_0^\infty e^{-ny}\, y^{m-1}\, dy$$

Again, put $ny = t \quad or, y = \frac{t}{n} \quad and\ dy = \frac{1}{n}dt$

$\therefore \quad \int_0^\infty e^{-ny}\, y^{m-1}\, dy = \int_0^\infty e^{-t}\left(\frac{t}{n}\right)^{m-1}\frac{1}{n}dt$

$$= \frac{1}{n^m}\int_0^\infty e^{-t}\, t^{m-1}dt$$

$$= \frac{\Gamma(m)}{n^m}$$

5) $\frac{1}{(loga)^{a+1}}\Gamma(a+1) = \int_0^\infty \frac{x^a}{a^x}dx$

Proof: Substituting $a^x = e^y \quad or, x(\log a) = y \quad or, x = \frac{y}{\log a}$ and $dx = \frac{1}{\log a}dy$ we get $\int_0^\infty \frac{x^a}{a^x}dx = \int_0^\infty \left(\frac{y}{\log a}\right)^a e^{-y}\frac{1}{\log a}dy$

$$= \frac{1}{(loga)^{a+1}}\int_0^\infty e^{-y}\, y^a dy$$

$$= \frac{1}{(loga)^{a+1}}\Gamma(a+1)$$

**Beta function (Euler's integral of 1st kind):**

The Beta function having two indices $m, n$ written by $\beta(m,n)$ is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx \text{ ----------------- (1)}$$

Which is valid for real positive values of $m$ and $n$ ($i.e.\ m > 0, n > 0$), because it is for just these values of $m$ and $n$ the integral is convergent. We shall not however prove the statement.

This integral is also called the Euler's integral of the first kind. From this definition it immediately follows that, for $m = n = 1$

$$\beta(1,1) = \int_0^1 x^{1-1}\,(1-x)^{1-1}dx = \int_0^1 dx = 1$$

Which is rather an important result.

An important property of Beta function is its symmetry. It can be easily seen that

$\beta(m,n) = \beta(n,m)$ i.e. Beta function is symmetric with respect to $m$ and $n$.

Substituting $x = 1 - y$ and $dx = -dy$ in equation (1) we get

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx = -\int_1^0 (1-y)^{m-1}\,y^{n-1}dy$$

$$= \int_0^1 y^{n-1}\,(1-y)^{m-1}dy = \beta(n,m)$$

$$\therefore \quad \beta(m,n) = \beta(n,m)$$

**Other forms of Beta function:**

Beta function can be expressed in different forms as follow,

1) $\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$ --------------- (2)

Proof: we know that $\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$

Putting $x = \frac{1}{1+y}$, so that $dx = -\frac{1}{(1+y)^2}dy$ and $1 - x = \frac{y}{1+y}$

We get $\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left[-\frac{1}{(1+y)^2}\right]dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$

Using symmetry property of Beta function, we can write

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy$$

2) $\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}dx$ ------------ (3)

Proof: similar as previous one (Try this)

3) $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$ ----------------------- (4)

Proof: From equation (2) we have $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$\therefore \quad \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Consider the second integral $\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Putting $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2} dy$ we get

$$\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = -\int_1^{0} \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \frac{1}{y^2}\, dy = \int_0^1 \frac{\left(\frac{1}{y}\right)^{m-1} \frac{1}{y^2}\, dy}{\left(\frac{1}{y}\right)^{m+n} (y+1)^{m+n}}$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$\therefore \quad \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$

Therefore, $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$

4) $\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \; cos^{2n-1}\theta \; d\theta$

Proof: we know that $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Substituting $x = sin^2\theta$ so that $dx = 2\sin\theta\cos\theta \; d\theta$ we get

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} (sin^2\theta)^{m-1} (cos^2\theta)^{n-1} 2\sin\theta\cos\theta \; d\theta$$

$$\therefore \quad \beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \; cos^{2n-1}\theta \; d\theta$$

**Relation between *β* and *Γ* function:**

We know that $\Gamma(m) = \int_0^{\infty} e^{-x} x^{m-1} dx$

And $\frac{\Gamma(m)}{k^m} = \int_0^{\infty} e^{-kx} x^{m-1} dx$

$\therefore \quad \Gamma(m) = \int_0^{\infty} k^m e^{-kx} x^{m-1} dx$

Multiplying both sides of the above equation by $e^{-k}k^{n-1}$, we get

$$\Gamma(m) e^{-k} k^{n-1} = \int_0^{\infty} k^{m+n-1} e^{-k(1+x)} x^{m-1} dx$$

Integrating both sides with respect to *k* we have

$$\int_0^\infty \Gamma(m)e^{-k}k^{n-1}\,dk = \int_0^\infty \int_0^\infty e^{-k(1+x)}\,k^{m+n-1}x^{m-1}\,dx\,dk$$

$$\Gamma(m)\int_0^\infty e^{-k}k^{n-1}\,dk = \int_0^\infty x^{m-1}\,dx \int_0^\infty e^{-k(1+x)}\,k^{m+n-1}\,dk$$

$$\Gamma(m)\Gamma(n) = \int_0^\infty x^{m-1}\,dx\,\frac{\Gamma(m+n)}{(1+x)^{m+n}}$$

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$$

$$\therefore \qquad \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

**Example 2**

Show that $\int_0^{\pi/2} sin^p\theta\, cos^q\theta\;\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

Ans: we know that $\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$

Substituting $x = sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\;\, d\theta$ we get

$$\beta(m,n) = \int_0^{\frac{\pi}{2}}(sin^2\theta)^{m-1}\,(cos^2\theta)^{n-1}2\sin\theta\cos\theta\;\, d\theta$$

$$\therefore \qquad \beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta\;\, cos^{2n-1}\theta\;\, d\theta$$

$$\therefore \qquad \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} sin^{2m-1}\theta\;\, cos^{2n-1}\theta\;\, d\theta$$

Putting $2m-1 = p,\quad i.e.\, m = \frac{p+1}{2}$ and $2n-1 = q,\; i,e.\; n = \frac{q+1}{2}$

$$\therefore \qquad \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\pi/2} sin^p\theta\;\, cos^q\theta\;\, d\theta$$

**Example 3**

Show that $\int_0^{\pi/2} \sqrt{\cot\theta}\;\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

In previous example we obtain, $\int_0^{\pi/2} sin^p\theta\, cos^q\theta\;\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

$$\therefore \qquad \int_0^{\pi/2} \sqrt{\cot\theta}\;\, d\theta = \int_0^{\pi/2} sin^{-\frac{1}{2}}\theta\, cos^{\frac{1}{2}}\theta\;\, d\theta = \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$(since\;\; \Gamma(1) = 1)$