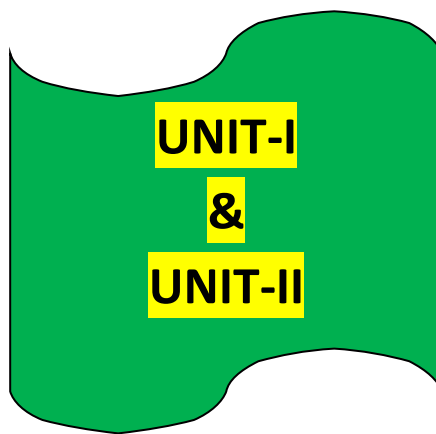


SEMESTER-VI

CORE COURSE C13-T



**BY
DR. PRADIP KUMAR GAIN**

SEMESTER-VI

HONOURS

CORE COURSE---C13-T

UNIT-I (MARKS-07) AND UNIT-II(MARKS-14)

UNIT-I

Dr. Pradip Kumar Gain

Syllabus for Unit-I : Metric spaces: sequences in a metric space, Cauchy sequences, Complete metric spaces, Cantor's theorem.

Sequences in a metric space :

DEFINITION: A **sequence** in a metric space (X, d) is a function defined on the set of natural numbers \mathbb{N} with values in X and is specified by listing its values as $x_1, x_2, x_3, \dots, x_n, \dots$ or as $\{x_n\}_{n=1}^{\infty}$ or as $\{x_n\}$ where x_n is the image of n , $n \in \mathbb{N}$ and is known as the n th term of the sequence.

NOTE : The function stated above is not necessarily one-to-one and therefore, the range set of the sequence may be finite or infinite whereas set of all terms of a sequence is always infinite.

EXAMPLES: The range set of the sequence $\left\{\frac{1}{n}\right\}$ in R is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$ is infinite. The set of all terms is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$ which also infinite. Again the range set the sequence $\{(-1)^n\}$ is $\{-1, 1\}$ which is finite. But the set of all terms of a sequence is infinite.

DEFINITION: Let $\{x_n\}$ be a sequence in the metric space (X, d) . Let $\{n_1, n_2, n_3, \dots, n_k, \dots\}$ be a strictly increasing sequence of natural numbers. Then the sequence $\{x_{n_k}\}$ ie, $\{x_{n_1}, x_{n_2}, x_{n_3}, \dots, x_{n_k}, \dots\}$ is called a **subsequence** of the sequence $\{x_n\}$.

EXAMPLES:

DEFINITION: A sequence $\{x_n\}$ in a metric space (X, d) is said to **converge** to a point $x \in X$, if for given $\varepsilon > 0$, we can find a positive integer m (depending on ε) such that $d(x_n, x) < \varepsilon$, whenever $n \geq m$.

We then write $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ or, $\lim_{n \rightarrow \infty} x_n = x$ or, $x_n \rightarrow x$ as $n \rightarrow \infty$.

OR

Equivalently, a sequence $\{x_n\}$ in a metric space (X, d) is said to converge to a point $x \in X$, if for given $\varepsilon > 0$, we can find a positive integer m (depending on ε) such that $x_n \in S(x, \varepsilon)$ for all $n \geq m$ where, $S(x, \varepsilon)$ is a sphere of radius ε centred at x .

EXAMPLE: The sequence $\left\{\frac{1}{n}\right\}$ converge to 0.

DEFINITION (Cauchy Sequence): A **sequence** $\{x_n\}$ in a metric space (X, d) is said to be a **Cauchy sequence** or **Fundamental sequence** iff for each $\varepsilon > 0$ there exists a positive integer p such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq p$. That is, $d(x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \infty$.

THEOREM 1.1: Every convergent sequence is Cauchy sequence. Converse is not necessarily true.

Proof: Let $\{x_n\}$ be a convergent sequence in the metric space (X, d) and let $x_n \rightarrow x$. Hence for given $\varepsilon > 0$, there exists a positive integer p such that $d(x_n, x) < \frac{\varepsilon}{2}$, $d(x_m, x) < \frac{\varepsilon}{2}$ for all $n, m \geq p$(1)

$$\begin{aligned} \text{Now } d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) \text{ [since } d \text{ is metric . Triangle inequality holds]} \\ &= d(x_n, x) + d(x_m, x) \text{ [since } d \text{ is metric . Symmetric property holds]} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ [by (1)]} \\ &= \varepsilon \quad \forall n, m \geq p \end{aligned}$$

Thus $d(x_n, x_m) < \varepsilon$ for all $n, m \geq p$. Hence $\{x_n\}$ is a Cauchy sequence in the metric space (X, d) .

To show converse is not true let us consider the space $X = (0, 1]$ of the real line with usual metric. Let us consider the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbb{N}$. For a given $\varepsilon > 0$, we choose a

$$\text{positive integer } p (> \frac{2}{\varepsilon}), \quad d(x_n, x_m) = |x_n - x_m| \leq |x_n| + |x_m| = \frac{1}{n} + \frac{1}{m}, \forall n, m \geq p$$

$$\Rightarrow d(x_n, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall n, m \geq p$$

$$\Rightarrow d(x_n, x_m) < \varepsilon \quad \forall n, m \geq p$$

Hence $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbb{N}$ is a Cauchy sequence in $X = (0, 1]$.

But $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbb{N}$, converges to 0 which is not a point of $X = (0, 1]$.

Thus $\{x_n\} = \left\{\frac{1}{n}\right\}$ is a Cauchy sequence in X but does not converge to any point in X .

THEOREM 1.2: Let $\{x_n\}$ be a Cauchy sequence in the metric space (X, d) . If $\{x_n\}$ possesses a convergent subsequence $\{x_{n_k}\}$ converging to x , then the sequence $\{x_n\}$ also converges to x .

Proof: Let $\{x_{n_k}\}$ be a convergent subsequence of the Cauchy sequence $\{x_n\}$ converges to $x \in X$. Then for each $\varepsilon > 0$ there exists a positive integer p such that $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ for all $n_k \geq p$ (1).

Again as $\{x_n\}$ is a Cauchy sequence, for each $\varepsilon > 0$ there exists a positive integer q such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for all $n, m \geq q$ (2).

$$\text{Let } r = \text{Max}(p, q)$$

Then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$ [since d is metric . Triangle inequality holds]

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all } n \geq r .$$

$\Rightarrow d(x_n, x) < \varepsilon$ for all $n \geq r$. Hence the Cauchy sequence converges to $x \in X$.

THEOREM 1.3: A Cauchy sequence $\{x_n\}$ in a metric space (X, d) converges if and only if it has a convergent subsequence $\{x_{n_k}\}$.

Proof: Let $\{x_n\}$ be a cauchy sequence converges to $x \in X$. Hence for given $\varepsilon > 0$, there exists a positive integer p such that $d(x_n, x) < \varepsilon$ for all $n \geq p$ and hence $d(x_{n_k}, x) < \varepsilon$ for all $n_k \geq p$. Therefore, the subsequence $\{x_{n_k}\}$ of the cauchy sequence $\{x_n\}$ converges to $x \in X$. Conversely, let $\{x_{n_k}\}$ be a convergent subsequence of the cauchy sequence $\{x_n\}$ converges to $x \in X$. Then by previous theorem, the cauchy sequence $\{x_n\}$ converges to $x \in X$.

Complete metric spaces

DEFINITION: A *metric space* (X, d) is said to be **complete** if every Cauchy sequence in X converges to some point in X .

The *metric space* (X, d) is called **incomplete** if it is not complete.

EXAMPLES (complete metric spaces):

Ex-1. Any set X with discrete metric forms a complete metric space.

Solution : Let (X, d) be a metric space with discrete metric d such that

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} . \text{ Let } \{x_n\} \text{ be a Cauchy sequence in the discrete metric space } (X, d) .$$

Then $d(x_n, x_m) = \begin{cases} 0 & \text{if } x_n = x_m \\ 1 & \text{if } x_n \neq x_m \end{cases} .$ Since as $\{x_n\}$ is a Cauchy sequence, for each $\varepsilon > 0$ there

exists a positive integer p such that $d(x_n, x_m) < \frac{1}{2}$ for all $n, m \geq p$ [taking $\varepsilon = \frac{1}{2}$]. Then by definition of discrete metric space $d(x_n, x_p) = 0 \forall n \in \mathbb{N}$.

$\Rightarrow x_n \rightarrow x_p$ as $n \rightarrow \infty$ which shows that every Cauchy sequence converges to a point of X which is also a term of the sequence. Hence discrete metric space X is complete.

Ex-2. The real line R is complete.

Solution : Let $\{x_n\}$ be a Cauchy sequence in R . By the definition of Cauchy sequence for each $\varepsilon > 0$ there exists a positive integer p such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq p$. Since R is a metric space with usual metric, we must have $d(x_n, x_m) = |x_n - x_m| < \varepsilon \forall n, m \geq p$. But it follows from the Cauchy's general principle of convergence of a sequence of real numbers that the above situation implies the convergence of a sequence $\{x_n\}$ to some point $x \in R$. Hence R is complete.

Ex-3. Prove that the space $C[0,1]$ of all continuous real valued functions on $[0,1]$ with the metric d , defined by $d(f, g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$, is a complete metric space.

Solution : Clearly, $d(f, g) = \sup\{|f(x) - g(x)| : x \in [0,1]\} \geq 0$

Also $d(f, g) = 0$ iff $\sup\{|f(x) - g(x)| : x \in [0,1]\} = 0$

$$\text{iff } f(x) - g(x) = 0 \quad \forall x \in [0,1]$$

$$\text{iff } f(x) = g(x) \quad \forall x \in [0,1]$$

$$\text{iff } f = g \quad [\text{non-negative property holds}]$$

Also $d(f, g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$

$$= \sup\{|g(x) - f(x)| : x \in [0,1]\}$$

$$= d(g, f) \quad [\text{symmetric property holds}]$$

Also for any three functions f, g, h , we have, $d(f, g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}$

$$= \sup\{|f(x) - h(x) + h(x) - g(x)| : x \in [0,1]\}$$

$$\leq \sup\{|f(x) - h(x)| : x \in [0,1]\} + \sup\{|h(x) - g(x)| : x \in [0,1]\}$$

$$= d(f, h) + d(h, g)$$

$$\text{Thus } d(f, g) \leq d(f, h) + d(h, g) \quad [\text{Triangle inequality holds}]$$

Hence $\{C[0,1], d\}$ is a metric space.

Let $\{f_n\}$ be a Cauchy sequence in $C[0,1]$. Then for each $\varepsilon > 0$ there exists a positive integer p such that $d(f_n, f_m) < \varepsilon$, for all $n, m \geq p$.

$$\Rightarrow \sup\{|f_n(x) - f_m(x)| : x \in [0,1]\} < \varepsilon, \text{ for all } n, m \geq p.$$

$\Rightarrow \{f_n(x) - f_m(x)\} < \varepsilon$, for all $n, m \geq p$ and for all $x \in [0,1]$. Using Cauchy's condition for convergence, we can say that $\{f_n\}$ converges uniformly on $[0,1]$. If $f_n \rightarrow f$ then f is also continuous on $[0,1]$. Therefore, the Cauchy sequence $\{f_n\}$ converges to $f \in C[0,1]$.

Hence $C[0,1]$ is a complete metric space.

EXAMPLES (incomplete metric spaces):

Ex-1. The space $X = (0, 1]$ of the real line with usual metric $d(x, y) = |x - y|$, $\forall x, y \in X$ is not complete.

Solution : Let us consider the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbb{N}$. For a given $\varepsilon > 0$, we choose a

positive integer $p (> \frac{2}{\varepsilon})$, $d(x_n, x_m) = |x_n - x_m| \leq |x_n| + |x_m| = \frac{1}{n} + \frac{1}{m}, \forall n, m \geq p$

$$\Rightarrow d(x_n, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \forall n, m \geq p$$

$$\Rightarrow d(x_n, x_m) < \varepsilon \quad \forall n, m \geq p.$$

Hence $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbb{N}$ is a Cauchy sequence in $X = (0, 1]$.

But $\{x_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbb{N}$, converges to 0 which is not a point of $X = (0, 1]$.

Thus $\{x_n\} = \left\{\frac{1}{n}\right\}$ is a Cauchy sequence in X but does not converge to any point in X .

Hence $X = (0, 1]$ is not complete.

Ex-2. The set Q of all rational numbers with usual metric $d(x, y) = |x - y|$, $\forall x, y \in Q$ is not complete.

Solution : With usual metric $d(x, y) = |x - y|$, $\forall x, y \in Q$ is metric space. Let us consider the

sequence $\{x_n\} = \left\{\frac{1}{3^n}\right\}$. This is a Cauchy sequence in Q . $\{x_n\} = \left\{\frac{1}{3^n}\right\}$ converges to $0 \in Q$.

Again let us consider a sequence $\{x_n\} = \left\{\left[1 + \frac{1}{n}\right]^n\right\}$. This is a Cauchy sequence in Q but this

sequence $\{x_n\} = \left\{\left[1 + \frac{1}{n}\right]^n\right\}$ converge to a point $e \notin Q$. So every Cauchy sequence in Q is

not convergent. Hence (Q, d) is not complete metric space.

DEFINITION : A **sequence** $\{F_n\} = \{F_1, F_2, F_3, \dots\}$ of sets is said to be **nested** if $F_1 \supset F_2 \supset F_3 \supset \dots \supset F_n \supset \dots$. That is, if $F_n \supset F_{n+1} \quad \forall n \in \mathbb{N}$

CANTOR INTERSECTION THEOREM

CANTOR INTERSECTION THEOREM : *If $\{F_n\}$ is a nested sequence of non-empty closed subsets of metric space (X, d) such that $\delta(F_n) \rightarrow 0$ as $n \rightarrow \infty$, then X is complete iff*

$\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point, where $\delta(F_n)$ denotes the diameter of F_n .

Proof:

The condition is necessary

Let (X, d) be a complete metric space and let $\{F_n\} = \{F_1, F_2, F_3, \dots\}$ be a nested of non-empty closed subsets of X with $\delta(F_n) \rightarrow 0$ as $n \rightarrow \infty$. We shall show that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Since each $F_n (n \in \mathbb{N}) \neq \emptyset$, we can construct a sequence $\{x_n\}$ by choosing $x_1, x_2, x_3, \dots \in F_n$. That is, $x_n \in F_n, \forall n = 1, 2, 3, \dots$. As $\delta(F_n) \rightarrow 0$ as $n \rightarrow \infty$, for given $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that $\delta(F_n) < \epsilon$ for all $n \geq m$ (1). Since $\{F_n\}$ is nested, $F_n \subset F_m$ for all $n \geq m$. Hence $x_n \in F_m$, for all $n \geq m$.

$\Rightarrow d(x_n, x_m) \leq \delta(F_m)$, for all $n \geq m$.

$\Rightarrow d(x_n, x_m) \leq \epsilon$, for all $n \geq m$. [using (1)].

Hence, $\{x_n\}$ is a Cauchy sequence in F_n . That is, in X . Since, (X, d) is complete metric space (given), the sequence $\{x_n\}$ must be convergent. Let it converges to $x \in X$. That is,

$x_n \rightarrow x$ as $n \rightarrow \infty$. We shall show that $x \in \bigcap_{n=1}^{\infty} F_n$. If possible, let $x \notin \bigcap_{n=1}^{\infty} F_n$. This implies that

x should not lie in some of the sets F_1, F_2, F_3, \dots . Let $x \notin F_k$. Since F_k is closed and $x \notin F_k, d(x, F_k) = \inf\{d(x, y) : y \in F_k\} > 0$. Let $d(x, F_k) \geq r$. Then $d(x, y) \geq r$ for all $y \in F_k$.

Therefore, $S(x, \frac{r}{2})$ and F_k are disjoint. Now $n > k \Rightarrow F_n \subset F_k \Rightarrow \{x_1, x_2, x_3, \dots\} \subset F_k$ [since $x_n \in F_n \forall n = 1, 2, 3, \dots$]

$\Rightarrow x_n \notin S(x, \frac{r}{2})$ which is not possible since $\{x_n\}$ converges to $x \in X$.

Hence $x \in \bigcap_{n=1}^{\infty} F_n$ showing that $\bigcap_{n=1}^{\infty} F_n$ is non-empty.

In order to prove that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point, let us suppose in contrary that

$\bigcap_{n=1}^{\infty} F_n$ contains two points x and y .

Then $\delta(F_1) > d(x, y), \delta(F_2) > d(x, y), \delta(F_3) > d(x, y), \dots$

Since d is metric and so $d(x, y) > 0$, $\delta(F_n)$ does not tend to 0 which contradicts the fact $\delta(F_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

The condition is sufficient

Let us suppose that the given condition is sufficient. We shall show that X is complete. Let $\{x_n\}$ be a Cauchy sequence in X .

Let us consider for each $n \in \mathbb{N}$, $A_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}$.

That is, $A_1 = \{x_1, x_2, x_3, \dots\}$, $A_2 = \{x_2, x_3, x_4, \dots\}$, $A_3 = \{x_3, x_4, x_5, \dots\}, \dots$

Obviously, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ and we have, $\overline{A_1} \supseteq \overline{A_2} \supseteq \overline{A_3} \supseteq \dots$

Since, $\{x_n\}$ is a Cauchy sequence and $\delta(A_n) \rightarrow 0$ as $n \rightarrow \infty$, we have, $\delta(\overline{A_n}) \rightarrow 0$ as $n \rightarrow \infty$.

So $\{\overline{A_n}\} = \{\overline{A_1}, \overline{A_2}, \overline{A_3}, \dots\}$ is a nested sequence of closed and non-empty sets in X ,

where $\delta(\overline{A_n}) \rightarrow 0$ as $n \rightarrow \infty$. So by hypothesis there exists an $x \in X$ such that $x \in \bigcap_{n=1}^{\infty} \overline{A_n}$.

Now $x_n \in A_n \subset \overline{A_n} \Rightarrow x_n \in \overline{A_n}$. Also $x \in \overline{A_n}$. Therefore, $d(x, x_n) < \delta(\overline{A_n})$. Since $\delta(\overline{A_n}) \rightarrow 0$,

$d(x, x_n) \rightarrow 0$ as $n \rightarrow \infty$. That is, $x_n \rightarrow x$ as $n \rightarrow \infty$. Hence the Cauchy sequence $\{x_n\}$

converges to $x \in X$. As $\{x_n\}$ is arbitrary it follows that every Cauchy sequence in (X, d)

converges. Hence (X, d) is complete metric space.

INSTRUCTION FOR STUDENTS :

NOTE: Definition of Cauchy sequence and theorems related to Cauchy sequence, Definition of Complete metric space and examples related to Complete metric spaces and Incomplete metric spaces are important.

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UNIT-II (MARKS-14)

UNIT-II

Dr. Pradip Kumar Gain

Syllabus for Unit-II : Continuity: Continuous mappings, Sequential criterion and other characterizations of continuity, Uniform continuity, Connectedness: Connected subsets of R . Compactness: Sequential compactness, Heine-Borel property, totally bounded spaces, finite intersection property(FIP), continuous functions on compact sets. Homeomorphism. Contraction mappings, Banach fixed point theorem and its applications to ordinary differential equations.

Functions/Mappings

DEFINITION : Let X and Y be two non-empty sets. If there is a rule of correspondence f which corresponds each element $x \in X$ a unique element $y \in Y$, then f is said to be a **function or mapping** or a map from X to Y or f maps X into Y .

In symbol we write $f : X \rightarrow Y$. In such a case the set X is called the Domain of f and the set Y is called Codomain of f . If f relates $x \in X$ with $y \in Y$, we write $y = f(x)$. Here x is called preimage of y under f and y is called image of x under f .

Continuous mappings

DEFINITION : Let (X, d) and (Y, d') be two metric spaces. A **function** $f : (X, d) \rightarrow (Y, d')$ is said to be **continuous** at a point $a \in X$, if and only if for all $\varepsilon > 0$, chosen arbitrarily, there

exists a $\delta(> 0)$ (depending on ε and a) such that $d(x, a) < \delta \Rightarrow d'(f(x), f(a)) < \varepsilon$. That is, $x \in S_X(a, \delta) \Rightarrow f(x) \in S_Y(f(a), \varepsilon)$.

The function f is said to be continuous on (X, d) if and only if it is continuous at each point of X .

REMARKS : It is clear that a function $f : (X, d) \rightarrow (Y, d')$ is continuous at a point $a \in X$, if and only if for all $\varepsilon > 0$, chosen arbitrarily, there exists a $\delta(> 0)$ (depending on ε and a) such that $f(S_X(a, \delta)) \subset S_Y(f(a), \varepsilon)$.

Sequential criterion of continuity

THEOREM 2.1: Let (X, d) and (Y, d') be two metric spaces. A function $f : (X, d) \rightarrow (Y, d')$ is said to be continuous at a point $x \in X$, if and only if for all sequences $\{x_n\}$ of elements of X converging to the point x in (X, d) , the sequences $\{f(x_n)\}$ of elements of Y converge to $f(x)$ in (Y, d') .

Proof :

The condition is necessary

Let us suppose that the function $f : (X, d) \rightarrow (Y, d')$ is continuous at a point $x \in X$. We shall show that $x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$. Let $\varepsilon > 0$, be arbitrarily chosen. Since f is continuous at the point x , there exists a $\delta(> 0)$ such that $d(x_n, x) < \delta \Rightarrow d'(f(x_n), f(x)) < \varepsilon$. Since $x_n \rightarrow x$ as $n \rightarrow \infty$ in (X, d) , corresponding to $\delta(> 0)$ there exists a natural number m depending on δ such that $n > m \Rightarrow d(x_n, x) < \delta$. Combining the two results above we conclude that $n > m \Rightarrow d'(f(x_n), f(x)) < \varepsilon$, where m is a natural number depending on δ and hence dependent on $\varepsilon > 0$. This implies $\{f(x_n)\}$ converges to $f(x)$ in (Y, d') .

The condition is sufficient

We shall show that if for all sequences $\{x_n\}$ converging to the point x in (X, d) the corresponding sequences $\{f(x_n)\}$ converge to $f(x)$ in (Y, d') , then f is continuous at the point x . If possible let f is not continuous at the point x . Then there exists atleast one $\varepsilon > 0$ such that for all $\delta(> 0)$ $d(x', x) < \delta$ but $d'(f(x'), f(x)) \geq \varepsilon$ for at least one $x' \in X$.

Let us consider a sequence of δ 's given by $\delta = \frac{1}{n}$ for all $n \in \mathbb{N}$. So, corresponding to each natural number n , there exists $x_n \in X$ such that $d(x_n, x) < \frac{1}{n}$ but $d'(f(x_n), f(x)) \geq \varepsilon$. This implies $f(x_n)$ does not tend to $f(x)$ in (Y, d') although $x_n \rightarrow x$ as $n \rightarrow \infty$ in (X, d) , which is a contradiction to our hypothesis. Hence f must be continuous at the point x .

REMARKS : The above theorem shows that convergence of sequence of points remains preserved under a continuous map.

From the above theorem the following theorem follows:

THEOREM 2.2__: Let (X, d) and (Y, d') be two metric spaces. A function $f : (X, d) \rightarrow (Y, d')$ is continuous if and only if for any $x \in X$ and for all sequences $\{x_n\}$ in X converging to x in (X, d) , the sequences $\{f(x_n)\}$ converge to $f(x)$ in (Y, d') .

Other characterizations of continuity :

THEOREM 2.3: Let (X, d) and (Y, d') be two metric spaces. A function $f : (X, d) \rightarrow (Y, d')$ is continuous if and only if for any open set G in (Y, d') , $f^{-1}(G)$ is open in (X, d) .

Proof : Let us assume $f : (X, d) \rightarrow (Y, d')$ is continuous and a set G is open in (Y, d') . We shall show that its inverse image $f^{-1}(G)$ is open in (X, d) . If $f(X) \cap G = \phi$, then $f^{-1}(G) = \phi$ and remains nothing to prove. Let $f(X) \cap G \neq \phi$, then $f^{-1}(G) \neq \phi$. So, there exists atleast one $x \in f^{-1}(G)$. This implies $f(x) \in G$. Since G is open $f(x)$ is an interior point of the set G . So, we can find an $\varepsilon > 0$ such that $S_Y(f(x), \varepsilon) \subset G$. Since f is continuous at the point x , there exists a $\delta (> 0)$ such that $d(x, x') < \delta \Rightarrow d'(f(x), f(x')) < \varepsilon$. That is, $x' \in S_X(x, \delta) \Rightarrow f(x') \in S_Y(f(x), \varepsilon)$. That is, $f(S_X(x, \delta)) \subset S_Y(f(x), \varepsilon) \subset G$. That is, $x \in S_X(x, \delta) \subset f^{-1}(G)$. Thus x is an interior point of the set $f^{-1}(G)$ in (X, d) . Since $x \in f^{-1}(G)$ is arbitrarily chosen it follows that $f^{-1}(G)$ is open in (X, d) .

Conversely, we assume that the inverse image of every open set G in (Y, d') is open in (X, d) . We shall show that f is continuous. We choose any $x \in X$, then $f(x)$ is uniquely determined. For $\varepsilon > 0$ chosen arbitrarily, $S_Y(f(x), \varepsilon)$ is an open set in (Y, d') . By proposition $f^{-1}(S_Y(f(x), \varepsilon))$ is open in (X, d) . Now, $x \in f^{-1}(S_Y(f(x), \varepsilon))$. So, there exists a $\delta (> 0)$ such that $x \in S_X(x, \delta) \subset f^{-1}(S_Y(f(x), \varepsilon))$. This implies, $f(S_X(x, \delta)) \subset (S_Y(f(x), \varepsilon))$. That is, $d(x', x) < \delta \Rightarrow d'(f(x'), f(x)) < \varepsilon$ where $\delta (> 0)$ depends on $\varepsilon > 0$. Consequently, f is continuous at x in (X, d) . Since x is chosen arbitrarily, f is continuous.

THEOREM 2.4: Let (X, d) and (Y, d') be two metric spaces. A function $f : (X, d) \rightarrow (Y, d')$ is continuous if and only if for any closed set F in (Y, d') , $f^{-1}(F)$ is closed in (X, d) .

Proof : Let us assume $f : (X, d) \rightarrow (Y, d')$ is continuous and a set F is closed in (Y, d') . So, $Y \setminus F$ is open in (Y, d') and therefore, $f^{-1}(Y \setminus F)$ is open in (X, d) (since f is continuous). Now, $f^{-1}(F) = X \setminus f^{-1}(Y \setminus F)$

$\Rightarrow X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$. So, $\Rightarrow X \setminus f^{-1}(F)$ is open in (X, d) . Consequently, $f^{-1}(F)$ is closed in (X, d) .

Conversely, we assume that for all sets F closed in (Y, d') , $f^{-1}(F)$ is closed in (X, d) . We shall show that f is continuous. Let G be any open set in (Y, d') . $Y \setminus G$ is closed in (Y, d') hence $f^{-1}(Y \setminus G)$ is closed in (X, d) . Since $f^{-1}(Y \setminus G) = X \setminus f^{-1}(G)$, it follows that $X \setminus f^{-1}(G)$ is closed in (X, d) . That is, $f^{-1}(G)$ is open in (X, d) . Therefore, f is continuous.

THEOREM 2.5: Let (X, d) and (Y, d') be two metric spaces. A function $f : (X, d) \rightarrow (Y, d')$ is continuous if and only if for any set $A \subset X$, $f(ClA) \subset Cl(f(A))$.

THEOREM 2.6: Let (X, d) and (Y, d') be two metric spaces. A function $f : (X, d) \rightarrow (Y, d')$ is continuous if and only if for any set $A \subset X$, $Cl\{f^{-1}(B)\} \subset f^{-1}(ClB)$.

Uniform continuity

DEFINITION : Let (X, d) and (Y, d') be two metric spaces. A function $f : (X, d) \rightarrow (Y, d')$ is said to be **uniformly continuous** on (X, d) if and only if corresponding to $\epsilon > 0$, chosen arbitrarily, there exists a $\delta(> 0)$ (depending on ϵ alone) such that $d(x_1, x_2) < \delta \Rightarrow d'(f(x_1), f(x_2)) < \epsilon \forall x_1, x_2 \in X$

THEOREM 2.7: Let (X, d) and (Y, d') be two metric spaces and a function $f : (X, d) \rightarrow (Y, d')$ is a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in (X, d) then $\{f(x_n)\}$ is a Cauchy sequence in (Y, d') .

Proof : Let $\epsilon > 0$, be arbitrarily chosen. Since f is uniformly continuous in (X, d) , there exists a $\delta(> 0)$ (depending on ϵ alone) such that $d(x_1, x_2) < \delta \Rightarrow d'(f(x_1), f(x_2)) < \epsilon \forall x_1, x_2 \in X$ (1). Since $\{x_n\}$ is a Cauchy sequence in (X, d) , corresponding to $\delta(> 0)$ there exists a positive integer $m = m(\delta)$ such that $n > m \Rightarrow d(x_n, x_{n+p}) < \delta$, for all $p \in \mathbb{N}$ (2). Combining (1) & (2) we get, $n > m \Rightarrow$ for all $p \in \mathbb{N}$, $d(x_n, x_{n+p}) < \delta \Rightarrow d'(f(x_n), f(x_{n+p})) < \epsilon$. This implies that $\{f(x_n)\}$ is a Cauchy sequence in (Y, d') .

Examples of continuous functions:

EXAMPLES 2.1: Show that the function $f(x) = \frac{1}{x}$ mapping the real line into itself is continuous everywhere on the real line except at the origin.

EXAMPLES 2.2: Show that the function $f(x) = \frac{1}{x}$ mapping the real line into itself given by

$$f(x) = \begin{cases} x, & x \in \mathcal{Q} \\ 1-x, & \text{otherwise} \end{cases} \text{ continuous only at the point } \frac{1}{2}.$$

EXAMPLES 2.3: Let (X, d) be a metric space and A and B are two non-empty disjoint closed sets in X . Prove that there exists a continuous function $f: X \rightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} 1, & x \in A \\ -1, & x \in B \end{cases}.$$

CONNECTEDNESS

DEFINITION : Let (X, d) be a metric space and A and B are two subsets of X . The sets A and B are said to be **separated** in if and only if neither has a point in common with the closure of another. That is, $A \cap Cl(B) = \phi$, $Cl(A) \cap B = \phi$.

These two conditions can be expressed by $\{A \cap Cl(B)\} \cup \{Cl(A) \cap B\} = \phi$. This is known as "**Hausdorff-Lennes condition**".

NOTE : Two sets A and B are may be separated in one metric space but not in other.

For example, let us consider the set \mathbb{R} of all real numbers along usual metric. Then let us consider two sets $\{0\}$ and $(0,1)$. Now, $\{0\} \cap Cl((0,1)) = \{0\} \cap [0,1] = \{0\} \neq \phi$.

Again, let us consider the set \mathbb{R} of all real numbers along with discrete metric. Let us consider two sets $\{0\}$ and $(0,1)$. The open sphere $S\left(0, \frac{1}{2}\right)$ with centre 0 and radius

$\frac{1}{2}$ contains no point of the set $(0,1)$. Therefore, in this metric space $Cl(\{0\}) \cap (0,1) = \{0\} \cap (0,1) = \phi$, $\{0\} \cap Cl((0,1)) = \{0\} \cap (0,1) = \phi$ and hence the sets $\{0\}$ and $(0,1)$ are separated in this metric space.

THEOREM 2.8: Let a set G is open in a metric space (X, d) . If G is expressed as the union of two non-empty separated sets A and B . Then both the sets A and B are open in (X, d) .

Disconnected Spaces and Disconnected Sets:

DEFINITION : A *metric space* (X, d) is said to be **disconnected** if and only if it can be expressed as the union of two non-empty separated sets. That is, $X = A \cup B$ where $A \neq \phi, B \neq \phi$ and $A \cap Cl(B) = \phi, Cl(A) \cap B = \phi$.

By theorem 2.8, both the sets A and B are open in (X, d) .

RESULT : A metric space (X, d) is disconnected if and only if it can be expressed as the union of two non-empty disjoint open sets.

RESULT : A metric space (X, d) is disconnected if and only if it can be expressed as the union of two non-empty disjoint closed sets.

DEFINITION : A *non-empty subset* A of a metric space (X, d) is **disconnected** if and only if it can be expressed as the union of two non-empty separated sets. That is, $A = A_1 \cup A_2$ where $A_1 \neq \phi, A_2 \neq \phi$ and $A_1 \cap Cl(A_2) = \phi, Cl(A_1) \cap A_2 = \phi$.

THEOREM 2.8: Let (X, d) be a metric space. Then the following conditions are equivalent :

(i) (X, d) is disconnected.

(ii) X can be expressed as the union of two non-empty disjoint closed sets in (X, d) .

(iii) X can be expressed as the union of two non-empty disjoint open sets in (X, d) .

(iv) there exists a non-empty proper subset of X , which is both open and closed in the metric space (X, d) .

Connected Spaces and Connected Sets:

DEFINITION : A *metric space* (X, d) is said to be **connected** if and only if X is not expressible as the union of two non-empty separated sets in (X, d) . In other words, (X, d) is connected if and only if X is not disconnected.

DEFINITION : A *non-empty subset* A of a metric space (X, d) is **connected** if and only if it cannot be expressed as the union of two non-empty separated sets.

THEOREM 2.9: A metric space (X, d) is connected if and only if X and ϕ are the only sets which are both open and closed in (X, d) .

Proof : Let X and ϕ are the only sets which are both open and closed in (X, d) . That is, X is the only non-empty set which is both open and closed in (X, d) . We shall prove that (X, d)

is connected. If possible, let X is disconnected. Then there exists a disconnection (A, B) of (X, d) . Obviously, both the sets A and B are non-empty. Since X is open, we find both the sets A and B are open. Similarly, both the sets A and B are closed. Thus there exists a non-empty proper subset A of X which is both open and closed in (X, d) . This is a contradiction to our hypothesis that X is the only non-empty set which is both open and closed in (X, d) . Therefore, (X, d) is connected.

Conversely, let (X, d) is connected. We shall show that X is the only non-empty set which is both open and closed in (X, d) . If possible, let there exists a non-empty proper subset A of X which is both open and closed in (X, d) . Then its complement $A^c = X \setminus A$ is non-empty. Since A is both open and closed A^c is both closed and open. Therefore, (X, d) is disconnected with a disconnection (A, A^c) , which contradicts our assumption. Therefore, X is the only non-empty set which is both open and closed in (X, d) .

THEOREM 2.10: *If two connected sets are not separated, their union is connected.*

THEOREM 2.11: *In a metric space the union of two non-disjoint connected sets is connected.*

THEOREM 2.12: *If every two points of a set A in a metric space (X, d) are contained in some connected subset of A , then A is connected set.*

DEFINITION : Let (X, d) be a **metric space**. If corresponding to every pair a, b of distinct points of X , there exist separated sets A and B in (X, d) with $a \in A$ and $b \in B$, then the space (X, d) is said to be **totally disconnected**.

CONNECTED SETS IN THE REAL LINE :

It is clear that like other spaces, the null set ϕ and singleton sets are connected in the real line.

THEOREM 2.13: **A set $A \subset R$ with at least two points is connected in the real line if and only if A is an interval.**

Proof : Let us assume that A is an interval. We shall show that A is connected. Let us assume, if possible, A is disconnected. Then there exist two non-empty sets B and C both open and closed in the subspace A such that $A = B \cup C$. Since B and C are non-empty disjoint sets we choose any $b \in B$ and $c \in C$. Since the sets B and C are disjoint, the points b and c are distinct. That is, $b \neq c$. Let $b < c$. Since A is an interval and $b, c \in A$ it follows that $b < x < c \Rightarrow x \in A$. So, $[b, c] \subset A = B \cup C$. Also, $y \in [b, c] \Rightarrow$ either $y \in B$ or $y \in C$ but not both. Let $E = [b, c] \cap B$. Now $b \in E$. Since E is non-empty and bounded above E has a finite supremum. Let $u = \sup E$. Then $b \leq u \leq c$. Since $u = \sup E$, no real number less than

u can be an upper bound of the set E . Consequently, corresponding to each $\varepsilon(>0)$, however small, there exists a $v \in E$ such that $u - \varepsilon < v \leq u$. Thus every neighbourhood $S(u, \varepsilon)$ of u in the real line contains a point of E . Since $E \subset B$, we conclude that every neighbourhood of u contains a point of B different from u . So u is a point of accumulation of the set B . Since B is closed, we must have $u \in B$. Also $u \notin C$ (Since the sets B and C are disjoint). Hence $u \neq c$. As $b \leq u \leq c$, it follows that $u < c$. Again for each $\varepsilon(>0)$, however small, $u + \varepsilon \in C$, if $u + \varepsilon \leq c$. This implies every neighbourhood $S(u, \varepsilon)$ of the point u in the real line contains some point of C different from u . Therefore, u is a point of accumulation of the set C in the real line. Since the set C is closed, we also have $u \in C$. Thus $u \in B \cap C$, which contradicts that the sets B and C are disjoint. Therefore, A must be connected.

Conversely, if possible, let A is a connected subset of R containing at least two points but A is not an interval. Then there exist three points x, y, z such that $x, z \in A$, $y \notin A$ where $x < y < z$. Now, the sets $A_1 = (-\infty, y)$ and $A_2 = (y, \infty)$ re separated open sets in Euclidean line. Let $B_1 = A_1 \cap A$, $B_2 = A_2 \cap A$. Then $B_1 \subset A_1$, $B_2 \subset A_2$ and consequently B_1 and B_2 are separated. As $x \in B_1$, $z \in B_2$ both B_1 and B_2 are non-empty. Also $A = B_1 \cup B_2$ Hence A has a disconnection (B_1, B_2) . This is a contradiction to the fact that A is a connected set. Thus A is an interval.

EXAMPLE : Show that the set R of all real numbers is connected in the real line.

SOLUTION : If possible, let the set R is disconnected in the real line and (A, B) is a disconnection of R . Then A, B are non-empty separated sets in the real line which are both open and closed. Since A, B are non-empty, there exists at least one $a_1 \in A$ and $b_1 \in B$. Since A and B are disjoint, $a_1 \neq b_1$. So either $a_1 < b_1$ or $a_1 > b_1$. Without loss of generality, let $a_1 < b_1$. Let $I_1 = [a_1, b_1]$. Then $|I_1| = b_1 - a_1 > 0$. Now, $\frac{a_1 + b_1}{2} \in R$. Since

$R = A \cup B$, $\frac{a_1 + b_1}{2}$ belongs either to A or to B or belong to both. Also since, $A \cap B = \phi$,

$\frac{a_1 + b_1}{2}$ can't belong to both the sets A and B . If $\frac{a_1 + b_1}{2} \in A$, we shall consider the interval $\left[\frac{a_1 + b_1}{2}, b_1\right]$. Let $[a_2, b_2] = \left[\frac{a_1 + b_1}{2}, b_1\right]$. That is, $a_2 = \frac{a_1 + b_1}{2}$ and $b_2 = b_1$.

If $\frac{a_1 + b_1}{2} \in B$, we shall consider the interval $\left[a_1, \frac{a_1 + b_1}{2}\right]$. Let $[a_2, b_2] = \left[a_1, \frac{a_1 + b_1}{2}\right]$. That

is, $a_2 = a_1$ and $b_2 = \frac{a_1 + b_1}{2}$. Let $I_2 = [a_2, b_2]$. Then clearly, $I_2 \subset I_1$ and

$|I_2| = b_2 - a_2 = \frac{(b_1 - a_1)}{2}$. If we repeat this process, we must find intervals I_3, I_4, I_5, \dots

and so on. In every case, we select the end points a_n and b_n such that $a_n \in A$ and $b_n \in B$. Thus we get a sequence $\{I_n\}$ of bounded closed intervals, where $I_n = [a_n, b_n]$. Also $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. And, $|I_n| = (b_n - a_n) = \frac{1}{2^{n-1}}(b_1 - a_1) \rightarrow 0$ as $n \rightarrow \infty$. Thus $\{I_n\}$ forms a nest of closed intervals with diameter tending towards zero in the real line. Nested Interval Theorem says that there exists one and only one point $c \in \bigcap \{I_n : n \in \mathbb{N}\}$. It can easily be seen that both the sequences $\{a_n\}$ and $\{b_n\}$ are convergent in the real line and both converges to c . Since $\{a_n\} \subset A$, c is a point of accumulation of the set A in the real line. As A is closed, $c \in A$. Similarly, $c \in B$. Thus $A \cap B \neq \emptyset$, which contradicts the disconnection of the real line. Therefore, the set R of all real numbers is connected in the real line.

EXAMPLE : Let (X, d) be a connected metric space and $f : (X, d) \rightarrow (Y, d')$. Prove that $f(X)$ is a connected subset of Y .

SOLUTION : If possible, let the set $f(X)$ is not connected in the metric space (Y, d') . Then we can find a non-empty proper subset H of $f(X)$ which is both open and closed in the subspace $f(X)$. Since H is open and f is continuous, $f^{-1}(H)$ is open in (X, d) . Again since H is closed and f is continuous, $f^{-1}(H)$ is closed in (X, d) .

H is proper subset of $f(X) \Rightarrow f(X) \setminus H \neq \emptyset \Rightarrow f^{-1}(f(X) \setminus H) \neq \emptyset \Rightarrow f^{-1}(H) \neq \emptyset$. Thus $f^{-1}(H)$ is a non-empty proper subset of X which is both open and closed in (X, d) . So (X, d) not connected, a contradiction. Hence $f(X)$ must be connected in the metric space (Y, d') .

NOTE : In case $f : (X, d) \rightarrow (Y, d')$ is an onto continuous map and X is connected, then $Y = f(X)$ is connected.

COMPACTNESS

DEFINITION : Let X be a non-empty set. A **family** $A = \{A_\alpha : \alpha \in \Lambda\}$ of subsets of X is said to be a **cover** of X if and only if $X = \bigcup \{A_\alpha : \alpha \in \Lambda\}$, where Λ is an index set.

* In such a case, we say that the family $A = \{A_\alpha : \alpha \in \Lambda\}$ covers X .

DEFINITION : Let Y be a non-empty subset of the set X . A family $B = \{B_\alpha : \alpha \in \Lambda\}$ of subsets of X is said to be a cover of Y if and only if $Y \subset \bigcup \{B_\alpha : \alpha \in \Lambda\}$, where Λ is an index set.

* In such a case, we say that the family $B = \{B_\alpha : \alpha \in \Lambda\}$ covers Y .

* If there exists a subfamily B' of B which also covers Y , we say that B' is a subcover of B .

*A cover is said to be a finite cover(respt. Countable) if it contains finite (respt. Countable) number of sets.

* If set in the family $A = \{A_\alpha : \alpha \in \Lambda\}$ are all open sets in a metric space (X, d) , A is said to be an open cover of X in the metric space (X, d) .

NOTE : When a family of subsets of X in a metric space (X, d) covers X , the metric d plays no role. But in order to be an open cover of X for a family of subsets of X , d must have role because, openness of a set depends on the underlying metric.

EXAMPLE : Show that the family $A = \{A_n = (-n, n) : n \in \mathbb{N}\}$, of bounded open intervals, is an open cover of \mathbb{R} .

EXAMPLE : Show that each one of the following families is an open cover of the real line:

(i) $A_1 = \{(-|x|, |x|) : x \in \mathbb{R}\}$

(ii) $A_2 = \{(x, x+1) : x \in \mathbb{N}\}$

(iii) $A_3 = \{(n-1, n+1) : n \in \mathbb{Z}\}$

DEFINITION : A **metric space** (X, d) is said to be a **Lindelöf** space if and only if every open cover of X in the metric space (X, d) admits of a countable subcover.

THEOREM 2.14: (Lindelöf Covering Theorem)

In the real line every open cover of a set has a countable subcover.

DEFINITION(Compact Space) : A **metric space** (X, d) is said to be a **compact metric** space if and only if every open cover of X in the metric space (X, d) admits of a finite subcover.

DEFINITION(Heine-Borel Property) : A **metric space** (X, d) is said to satisfy **Heine-Borel Property** if and only if every open cover of X in the metric space (X, d) admits of a finite subcover.

DEFINITION(Compact Set) : Let Y be a **non-empty set** in a metric space (X, d) . Then Y is said to be a **compact set** if and only if every open cover of Y in the metric space (X, d) has a finite subcover.

NOTE : It is to be noted that by means of open sets, we consider those sets which are open in the metric space (X, d) .

PROPERTIES OF COMPACT SPACES AND COMPACT SETS :

THEOREM 2.15: Every closed subset of a compact metric space is compact

NOTE : If in any metric space we can find at least one closed set which is not compact, we can assert that the space is not compact.

EXAMPLE : Show that the real line is not compact.

SOLUTION : Let us consider the set Z of integers. We know that in the real line Z is closed. The family $A = \{(-n, n) : n \in \mathbb{N}\}$ is an open cover of Z in the real line since $Z \subset \mathbb{R} = \cup \{(-n, n) : \alpha \in \Lambda\}$. Since $A = \{(-n, n) : n \in \mathbb{N}\}$ (cover of Z) has no finite subcover, the set Z is not compact. So Z is closed but not compact in the real line. Consequently, the real line is not compact.

THEOREM 2.16: *In any metric space (X, d) every compact set is closed .*

Combining the theorem 2.15 & theorem 2.16, we get the following theorem:

THEOREM 2.17: *A subset F of a compact metric space (X, d) is compact if and only if it is closed.*

THEOREM 2.18: *Every compact subset of a metric space is bounded.*

HEINE BOREL THEOREM

THEOREM 2.19: (Heine Borel Theorem) *Every closed and bounded set in the real line is compact.*

Converse of Heine Borel Theorem : *Every compact set in the real line is both closed and bounded.*

FINITE INTERSECTION PROPERTY

A family $A = \{A_\alpha : \alpha \in \Lambda\}$ of non-empty sets is said to possess finite intersection property if and only if every finite subfamily of $A = \{A_\alpha : \alpha \in \Lambda\}$ has non-empty intersection.

That is, for any arbitrary finite collection $\{A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3}, \dots, A_{\alpha_n}\}$ of members of $A = \{A_\alpha : \alpha \in \Lambda\}$ we have, $\cap \{A_{\alpha_i} : i = 1, 2, 3, \dots, n\} \neq \emptyset$

EXAMPLE : The collection $A = \{(-n, n) : n \in \mathbb{N}\}$ of open intervals satisfy finite intersection property. If we consider any finite collection $\{(-n_1, n_1), (-n_2, n_2), (-n_3, n_3), \dots, (-n_p, n_p)\}$ of open intervals in \mathbb{R} then $\cap \{(-n_r, n_r) : r = 1, 2, 3, \dots, p\} = (-n_\alpha, n_\alpha) \neq \emptyset$ where $n_\alpha = \min\{n_1, n_2, \dots, n_p\}$.

EXAMPLE : The collection $B = \{(n-1, n+1) : n \in \mathbb{Z}\}$ of open intervals does not satisfy finite intersection property. If we consider the finite collection $\{(1, 3), (3, 5)\}$ of B and we find $(1, 3) \cap (3, 5) = \emptyset$

THEOREM 2.19: *A metric space (X, d) is compact if and only if every infinite family of non-empty closed sets in (X, d) with finite intersection property has non-empty intersection.*

Proof :

CONTINUITY AND COMPACTNESS :

THEOREM 2.20: Continuous image of a compact metric space is compact.

Proof : Let (X, d) be a compact metric space and f is a continuous mapping from (X, d) into another metric space (Y, d') . If $Y' = f(X) \subset Y$, then we are to prove that the set Y' is a compact subset of (Y, d') . Let $A = \{A_\alpha : \alpha \in \Lambda\}$ be any open cover of Y' in (Y, d') . We are to show that it has a finite subcover. By proposition $Y' = \cup\{A_\alpha : \alpha \in \Lambda\}$. Hence we get,

$$X = f^{-1}(Y') = f^{-1}[\cup\{A_\alpha : \alpha \in \Lambda\}] = \cup\{f^{-1}(A_\alpha) : \alpha \in \Lambda\} \dots\dots\dots(1)$$

Since for all $\alpha \in \Lambda$, A_α is open in (Y, d') and f is continuous, it follows that $f^{-1}(A_\alpha)$ is open in (X, d) , for all $\alpha \in \Lambda$. Then from (1) it follows that $\{f^{-1}(A_\alpha) : \alpha \in \Lambda\}$ is an open cover of (X, d) . Since (X, d) is compact, it has a finite subcover, say, $\{f^{-1}(A_{\alpha_i}) : i = 1, 2, 3, \dots, n : \alpha_i \in \Lambda\}$. We show shall that $\{A_{\alpha_i} : i = 1, 2, 3, \dots, n : \alpha_i \in \Lambda\}$ is an open cover Y' in (Y, d') . Let $y \in Y'$ be arbitrarily chosen. Then there exists at least one $x \in X$ such that $f(x) = y$. Since $\{f^{-1}(A_{\alpha_i}) : i = 1, 2, 3, \dots, n : \alpha_i \in \Lambda\}$ is an open cover of (X, d) , for some integer $i' (1 \leq i' \leq n)$, $x \in f^{-1}(A_{\alpha_{i'}})$. Hence $y = f(x) \in A_{\alpha_{i'}}$. Therefore, $Y' = \cup\{f^{-1}(A_{\alpha_i}) : i = 1, 2, 3, \dots, n : \alpha_i \in \Lambda\}$. Consequently, Y' is a compact set in (Y, d') . Therefore, continuous image of a compact metric space is compact.

NOTE : In case $f : (X, d) \rightarrow (Y, d')$ is an onto continuous map and (X, d) is compact, then (Y, d') is also compact.

NOTE : If $f : (X, d) \rightarrow (Y, d')$ is continuous map and $A \subset X$ is a compact set in (X, d) , then $f(A) \subset Y$ is also compact in (Y, d') .

SEQUENTIALLY COMPACT SPACE

DEFINITION : A metric space (X, d) is said to be **sequentially compact** if and only if every sequence in X has a convergent subsequence.

DEFINITION : A non-empty set $A \subset X$ is said to be **sequentially compact** if and only if every sequence in A has a convergent subsequence.

EXAMPLE : In the real line the set R of all real numbers is not sequentially compact.

SOLUTION : Let us consider the sequence $\{x_n\}$ in R defined by $x_n = n$ for all $n \in \mathbb{N}$. Clearly, $\{x_n\}$ has no convergent subsequence. Hence R is not sequentially compact.

EXAMPLE : In the metric space R with usual metric, the set $(0, 1) \subset R$ is not sequentially compact.

SOLUTION : Let us consider the sequence $\{x_n\}$ in R defined by $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. $\{x_n\}$ has no subsequence which converges to any point in $(0,1)$. Hence $(0,1)$ is not sequentially compact set.

NOTE : However, in the real line the closed interval $[0,1]$ is sequentially compact set.

PROPERTIES OF SEQUENTIALLY COMPACT SETS

THEOREM 2.21: *In a metric space a sequentially compact set is both bounded and closed.*

THEOREM 2.22: *A sequentially compact metric space is complete.*

Proof : Let (X, d) be a sequentially compact metric space. In order to prove the theorem it is sufficient to show that any Cauchy sequence $\{x_n\}$ in (X, d) converges in X . Since $\{x_n\}$ is a Cauchy sequence in (X, d) , corresponding to $\varepsilon (> 0)$, chosen arbitrarily, there exists a positive integer N_1 , depending on $\varepsilon (> 0)$, such that $d(x_{n+p}, x_n) < \varepsilon$, for all $p \in \mathbb{N}$ and $n > N_1$. Since the metric space (X, d) is sequentially compact, the sequence $\{x_n\}$ must have a subsequence which converges in X . Let the subsequence be $\{x_{n_k}\}$ and $x_{n_k} \rightarrow x \in X$ as $n_k \rightarrow \infty$. So, there exists a positive integer N_2 , depending on $\varepsilon (> 0)$, such that $d(x_{n_k}, x) < \varepsilon$, whenever $n_k > N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $n_k > n > N$ we get, $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon + \varepsilon = 2\varepsilon$. Therefore, the sequence $\{x_n\}$ converges to $x \in X$. Consequently, the metric space (X, d) is complete.

THEOREM 2.23: *Every compact metric space is sequentially compact.*

COMPACTNESS AND TOTAL BOUNDEDNESS

DEFINITION : Let (X, d) be a metric space and ε be an arbitrarily chosen positive quantity. A non-empty subset A of X is said to be an ε -net of (X, d) , if the set of all open spheres of radius ε with centres in A covers X .

This implies for any $x \in X$, we can find at least one $a \in A$, such that $x \in S(a, \varepsilon)$, that is, $d(a, x) < \varepsilon$.

EXAMPLE : In the real line the set Z of all integers is an 1-net but not a $\frac{1}{2}$ -net.

DEFINITION : Let (X, d) be a metric space. A non-empty subset A of X is said to be totally bounded if and only if for every $\varepsilon (> 0)$, the set A has a finite ε -net.

This implies for any $\varepsilon (> 0)$, a finite collection of open spheres of radius ε covers A .

THEOREM 2.24: *A metric space (X, d) is totally bounded if and only if every sequence in X has a Cauchy subsequence.*

THEOREM 2.25: *A metric space (X, d) is sequentially compact if and only if it is complete and totally bounded.*

Proof : Let the metric space (X, d) is complete and totally bounded. Since (X, d) is totally bounded, every sequence $\{x_n\}$ in (X, d) has a Cauchy subsequence $\{y_n\}$. Since (X, d) is complete, any Cauchy sequence $\{y_n\}$ is convergent. So every sequence $\{x_n\}$ in (X, d) has a convergent subsequence. Therefore, the metric space (X, d) is sequentially compact.

Conversely, let the metric space (X, d) is sequentially compact. Then every sequence $\{x_n\}$ in (X, d) has a convergent subsequence $\{y_n\}$. Since the sequence $\{y_n\}$ satisfies Cauchy property, by theorem 2.24, it follows that (X, d) is totally bounded.

Moreover, as the metric space (X, d) is sequentially compact, every sequence $\{x_n\}$ in (X, d) has a convergent subsequence. Specifically, every Cauchy sequence $\{x_n\}$ in (X, d) has a convergent subsequence. We know that a Cauchy sequence is convergent if and only if it has a convergent subsequence. Thus every Cauchy sequence in (X, d) is convergent. Hence the metric space (X, d) is complete.

COROLLARY : Let (X, d) be a complete metric space. Then a non-empty subset A of X is compact if and only if A is totally bounded in (X, d) .

THEOREM 2.26: *Every sequentially compact metric space is compact.*