STUDY MATERIALS

MATHEMATICS HONOURS [UG]

SEMESTER- 2

PAPER – C 4T

UNIT – 4   [MARKS: 16]

TOPIC: VECTOR CALCULUS

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VECTOR ANALYSIS

Triple Product

Scalar Triple Product $\bar{a}.\bar{b}\times\bar{c}$

Vector Triple Product $\bar{a}\times(\bar{b}\times\bar{c})$

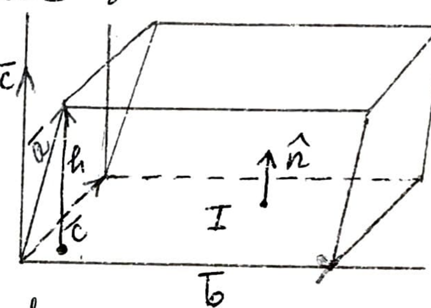
Scalar Triple Product —

If $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\bar{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\bar{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$,

then $\bar{a}\cdot\bar{b}\times\bar{c}=a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)$

$$=\bar{a}\cdot\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ b_1&b_2&b_3\\ c_1&c_2&c_3\end{vmatrix}=\begin{vmatrix}a_1&a_2&a_3\\ b_1&b_2&b_3\\ c_1&c_2&c_3\end{vmatrix}$$

(1) Show that $\bar{a}\cdot\bar{b}\times\bar{c}$ is in absolute value equal to the volume of a parallelopiped having $\bar{a},\bar{b},\bar{c}$ as concurrent edges.



Let $\hat{n}$ be a unit vector perpendicular to the plane of the parallelogram (I), having the direction of $\bar{b}\times\bar{c}$, and let h be the height of the terminal point of $\bar{a}$ above the parallelogram (I).

Volume of the parallelopiped

$=(\text{height } h)\cdot(\text{area of I})=(\bar{a}\cdot\hat{n})(|\bar{b}\times\bar{c}|)$

$=\bar{a}\cdot\{|\bar{b}\times\bar{c}|\hat{n}\}=\bar{a}\cdot(\bar{b}\times\bar{c})$

If $\bar{a},\bar{b},\bar{c}$ do not form a right-handed system, then $\bar{a}\cdot\hat{n}<0$, and the volume $=|\bar{a}\cdot(\bar{b}\times\bar{c})|$.

(*) The above refers to the geometrical significance of $\bar{a}.\bar{b}\times\bar{c}$, and is called Box Product $[\bar{a}\,\bar{b}\,\bar{c}]$.

(*) The scalar tripple product will be
(i) positive, if $\bar{a},\bar{b},\bar{c}$ form a right-handed system.
(ii) negative, " " " " left — " "

(*) It follows that the volume of the parallelopiped

Parallelopiped Law
$$\begin{cases}=\bar{b}\cdot(\bar{c}\times\bar{a})=(\bar{c}\times\bar{a})\cdot\bar{b}=-\bar{b}\cdot(\bar{a}\times\bar{c})=-(\bar{a}\times\bar{c})\cdot\bar{b}\\ =\bar{c}\cdot(\bar{a}\times\bar{b})=(\bar{a}\times\bar{b})\cdot\bar{c}=-\bar{c}\cdot(\bar{b}\times\bar{a})=-(\bar{b}\times\bar{a})\cdot\bar{c}\\ =\bar{a}\cdot(\bar{b}\times\bar{c})=(\bar{b}\times\bar{c})\cdot\bar{a}=-\bar{a}\cdot(\bar{c}\times\bar{b})=-(\bar{c}\times\bar{b})\cdot\bar{a}\end{cases}$$

(1) The sign of the scalar triple product is unchanged as long as the cyclical order of the factors remains unchanged.

(2) For every change of cyclic order a minus sign occurs.

(3) The scalar triple product is independent of the position of dot and cross. i.e., The dot and cross may be interchanged at pleasure.

Prob. (2).

Prove that $\bar{a}.(\bar{b}\times\bar{c}) = \bar{b}.(\bar{c}\times\bar{a}) = \bar{c}.(\bar{a}\times\bar{b})$

we have,

$$\bar{a}.(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \bar{b}.(\bar{c}\times\bar{a})$$

Similarly, $\bar{a}.(\bar{b}\times\bar{c}) = \bar{c}.(\bar{a}\times\bar{b})$, using the theory of determinants.

(3) The that a necessary and sufficient condition for three non zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ to be coplanar is that $\bar{a}.\bar{b}\times\bar{c} = 0$

Let $\bar{a}, \bar{b}, \bar{c}$ be non-zero and coplanar vectors.
Now $\bar{b}\times\bar{c}$ is a vector which is $\perp^r$ to the plane of $\bar{b}$ and $\bar{c}$, and hence so to the vector $\bar{a}$ lying on this plane.

$\therefore\ \bar{a}.\bar{b}\times\bar{c} = 0$.

Conversely,

Let $\bar{a}.\bar{b}\times\bar{c} = 0$

$\Rightarrow \bar{a}\perp(\bar{b}\times\bar{c})$. Again $(\bar{b}\times\bar{c})\perp$ to the plane of $\bar{b}$ & $\bar{c}$

$\Rightarrow \bar{a}$ lies in the plane of $\bar{b}$ and $\bar{c}$

$\Rightarrow \bar{a}, \bar{b}, \bar{c}$ are Coplanar.

$\boxed{\text{OTHERWISE;}}$ Let $\bar{a}, \bar{b}, \bar{c}$ be coplanar

$\Rightarrow$ Volume of the parallelopiped formed by them is zero (height = 0)

$\Rightarrow \bar{a}.\bar{b}\times\bar{c} = 0$

Conversely, let $\bar{a}.\bar{b}\times\bar{c} = 0$

$\Rightarrow$ Volume of the parallelopiped formed by them $= 0$

$\Rightarrow$ they are coplanar.

(4) $\bar{a}.\bar{b}\times\bar{c} = 0 \Rightarrow$ Either one of the three vectors is a zero vector!

Or, two of them are parallel or

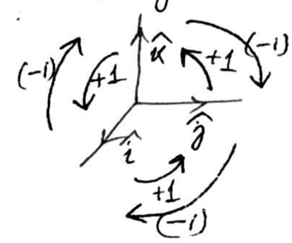
[Say, $\bar{a}\parallel\bar{b} \Rightarrow \bar{a} = k\bar{b}$, k is a scalar;
$\bar{a}.(\bar{b}\times\bar{c}) = k\bar{b}.(\bar{b}\times\bar{c}) = (k\bar{b}\times\bar{b}).\bar{c} = \bar{0}$]

Or, the volume of the parallelopiped being zero, $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

(5) The vectors of the fundamental system ($\hat i$ - $\hat j$ - $\hat k$) satisfy the relations:

$$[\hat i\,\hat j\,\hat k] = [\hat j\,\hat k\,\hat i] = [\hat k\,\hat i\,\hat j] = 1$$

and $$[\hat i\,\hat k\,\hat j] = [k\,\hat j\,\hat i] = [\hat j\,\hat i\,\hat k] = -1$$



(6) If $\bar a$, $\bar b$, $\bar c$ are expressed in terms of three fundamental vectors $\hat i, \hat j, \hat k$, say,

$$\left.\begin{aligned} \bar a &= a_1\hat i + a_2\hat j + a_3\hat k \\ \bar b &= b_1\hat i + b_2\hat j + b_3\hat k \\ \bar c &= c_1\hat i + c_2\hat j + c_3\hat k, \end{aligned}\right\} \quad \text{then } \bar a\cdot\bar b\times\bar c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Instead of the above fundamental system, if $\bar a$, $\bar b$, $\bar c$ are expressed in terms of three non coplanar vectors l, m, n, say,

$$\left.\begin{aligned} \bar a &= a_1\bar l + a_2\bar m + a_3\bar n \\ \bar b &= b_1\bar l + b_2\bar m + b_3\bar n \\ \bar c &= c_1\bar l + c_2\bar m + c_3\bar n \end{aligned}\right\}, \quad \text{then } \bar a\cdot\bar b\times\bar c = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\bar l\,\bar m\,\bar n]$$

Because,

$$\begin{aligned} \bar a\cdot\bar b\times\bar c &= (a_1\bar l + a_2\bar m + a_3\bar n)\cdot\{(b_1c_2-b_2c_1)\bar l\times\bar m + (b_2c_3-b_3c_2)\bar m\times\bar n \\ &\qquad + (b_3c_1-b_1c_3)\bar n\times\bar l\} \\ &= a_3(b_1c_2-b_2c_1)(\bar n\cdot\bar l\times\bar m) + a_1(b_2c_3-b_3c_2)(\bar l\cdot\bar m\times\bar n) \\ &\qquad + a_2(b_3c_1-b_1c_3)(\bar m\cdot\bar n\times\bar l) \end{aligned}$$

(7) Any vector $\bar r$ may be resolved into 3 component vectors in the directions of 3 non-coplanar vectors as
$$\bar r = x\bar a + y\bar b + z\bar c, \text{ where}$$
x, y, z are scalars, determined as follows:

$$\begin{aligned} \bar r\cdot\bar b\times\bar c &= x\bar a\cdot\bar b\times\bar c + y\bar b\cdot\bar b\times\bar c + z\bar c\cdot\bar b\times\bar c \\ &= x[\bar a\,\bar b\,\bar c] + y[\bar b\,\bar b\,\bar c] + z[\bar c\,\bar b\,\bar c] \end{aligned}$$

a, $[\bar r\,\bar b\,\bar c] = x[\bar a\,\bar b\,\bar c] + 0 + 0$

a, $x = \dfrac{[\bar r\,\bar b\,\bar c]}{[\bar a\,\bar b\,\bar c]}$, $[\bar a\,\bar b\,\bar c] \neq 0$, since they are non-coplanar.

Similarly, $$\left.\begin{aligned} y &= \frac{[\bar r\,\bar c\,\bar a]}{[\bar a\,\bar b\,\bar c]} \\ z &= \frac{[\bar r\,\bar a\,\bar b]}{[\bar a\,\bar b\,\bar c]} \end{aligned}\right\} \therefore \bar r = \frac{[\bar r\,\bar b\,\bar c]}{[\bar a\,\bar b\,\bar c]}\bar a + \frac{[\bar r\,\bar c\,\bar a]}{[\bar a\,\bar b\,\bar c]}\bar b + \frac{[\bar r\,\bar a\,\bar b]}{[\bar a\,\bar b\,\bar c]}\bar c$$

VECTOR Triple Product.

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$$

$$(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{c} \cdot \bar{a})\bar{b} - (\bar{c} \cdot \bar{b})\bar{a}$$

Vector Triple Product = (outer . Remote) Adjacent − (Outer . Adjacent) Remote.

Prove: $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

Let $\bar{q} = \bar{a} \times (\bar{b} \times \bar{c})$. So, $\bar{q}$ is perpendicular to both $\bar{a}$ and $\bar{b} \times \bar{c}$

ie., $\bar{q} \cdot \bar{a} = 0$ and $\bar{q} \cdot \bar{b} \times \bar{c} = 0$

$\bar{q} \cdot \bar{b} \times \bar{c} = 0 \Rightarrow \bar{q}$ lies in the plane of $\bar{b}$ and $\bar{c}$.

Therefore, $\bar{q} = x\bar{b} + y\bar{c}$, where x and y are suitable scalars.

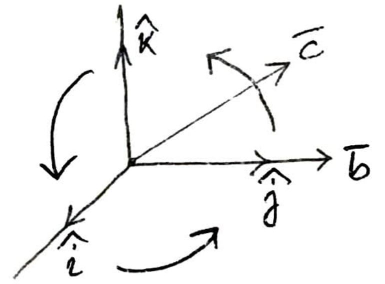
Let us consider a unit vector $\hat{j}$ along $\bar{b}$ and another unit vector $\hat{k}$ perpendicular to $\hat{j}$ in the plane of $\bar{b}$ and $\bar{c}$.

Now we choose the unit vector $\hat{i}$ so that $\hat{i}, \hat{j}, \hat{k}$ form a right-handed vector triad.

$$\bar{b} = 0\hat{i} + b_2\hat{j} + 0\hat{k} = b_2\hat{j}$$

$$\bar{c} = 0\hat{i} + c_2\hat{j} + c_3\hat{k} = c_2\hat{j} + c_3\hat{k}$$

$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$



$$\bar{a} \cdot \bar{b} = a_2 b_2$$

$$\bar{a} \cdot \bar{c} = a_2c_2 + a_3c_3$$

$$\bar{b} \times \bar{c} = b_2c_2(\hat{j} \times \hat{j}) + b_2c_3(\hat{j} \times \hat{k}) = \vec{0} + b_2c_3\hat{i} = b_2c_3\hat{i}.$$

Hence

$$\bar{a} \times (\bar{b} \times \bar{c}) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_2c_3\hat{i})$$

$$= (a_1b_2c_3)(\hat{i} \times \hat{i}) + (a_2b_2c_3)(\hat{j} \times \hat{i}) + (a_3b_2c_3)(\hat{k} \times \hat{i})$$

$$= -a_2b_2c_3\hat{k} + a_3b_2c_3\hat{j}.$$

$$= a_3c_3b_2\hat{j} + a_2c_2b_2\hat{j} - a_2c_2b_2\hat{j} - a_2b_2c_3\hat{k}$$

$$= (a_2c_2 + a_3c_3)b_2\hat{j} - a_2b_2(c_2\hat{j} + c_3\hat{k})$$

$$= (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}. \quad \text{(proved)}.$$

Reciprocal System of Vectors

Definition $\rightarrow$ Let $\bar{a}, \bar{b}, \bar{c}$ be 3 non-coplanar vectors, then all vectors can be expressed linearly in terms of $\bar{a}, \bar{b}, \bar{c}$, as $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$, where x, y, z are suitable scalars.

Then the set $\{\bar{a}, \bar{b}, \bar{c}\}$ form a basis for 3D-space.

Now, if $\{\bar{a}, \bar{b}, \bar{c}\}$ form a basis, then $[\bar{a}\,\bar{b}\,\bar{c}] \neq 0$.

Because, any vector $\bar{r} = \dfrac{[\bar{r}\,\bar{b}\,\bar{c}]}{[\bar{a}\,\bar{b}\,\bar{c}]}\bar{a} + \dfrac{[\bar{r}\,\bar{c}\,\bar{a}]}{[\bar{a}\,\bar{b}\,\bar{c}]}\bar{b} + \dfrac{[\bar{r}\,\bar{a}\,\bar{b}]}{[\bar{a}\,\bar{b}\,\bar{c}]}\bar{c}$,

where $[\bar{a}\,\bar{b}\,\bar{c}] \neq 0$.

Let us consider another basis $\{\bar{a}', \bar{b}', \bar{c}'\}$ such that

$$\text{①}\quad \left.\begin{array}{lll} \bar{a}.\bar{a}' = 1, & \bar{a}.\bar{b}' = 0, & \bar{a}.\bar{c}' = 0 \\ \bar{b}.\bar{a}' = 0, & \bar{b}.\bar{b}' = 1, & \bar{b}.\bar{c}' = 0 \\ \bar{c}.\bar{a}' = 0, & \bar{c}.\bar{b}' = 0, & \bar{c}.\bar{c}' = 1 \end{array}\right\}$$

Then, $\{\bar{a}, \bar{b}, \bar{c}\}$ and $\{\bar{a}', \bar{b}', \bar{c}'\}$ are reciprocal sets to each other.

Relations between two sets:

From the first column of ①, we see that

$\bar{a}' \perp \bar{b}$ and, $\bar{a}' \perp \bar{c}$ $\Rightarrow \bar{a}' \parallel (\bar{b}\times\bar{c}) \Rightarrow \bar{a}' = K\bar{b}\times\bar{c}$, K being a scalar

Further, $\bar{a}.\bar{a}' = 1 \Rightarrow \bar{a}.K\bar{b}\times\bar{c} = 1 \Rightarrow K = 1/[\bar{a}\,\bar{b}\,\bar{c}]$.

$\therefore\ \bar{a}' = \dfrac{\bar{b}\times\bar{c}}{[\bar{a}\,\bar{b}\,\bar{c}]}$. Similarly, $\bar{b}' = \dfrac{\bar{c}\times\bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]}$, $\bar{c}' = \dfrac{\bar{a}\times\bar{b}}{[\bar{a}\,\bar{b}\,\bar{c}]}$. $\longrightarrow$ ②

The symmetry of the relations in ① gives:

$$\bar{a} = \frac{\bar{b}'\times\bar{c}'}{[\bar{a}'\,\bar{b}'\,\bar{c}']},\quad \bar{b} = \frac{\bar{c}'\times\bar{a}'}{[\bar{a}'\,\bar{b}'\,\bar{c}']},\quad \bar{c} = \frac{\bar{a}'\times\bar{b}'}{[\bar{a}'\,\bar{b}'\,\bar{c}']} \longleftrightarrow ③$$

We can derive relations ① from relations ② as follows:

$$\bar{a}.\bar{a}' = a.\frac{\bar{b}\times\bar{c}}{[\bar{a}\,\bar{b}\,\bar{c}]} = \frac{[\bar{a}\,\bar{b}\,\bar{c}]}{[\bar{a}\,\bar{b}\,\bar{c}]} = 1.$$

$$\bar{a}.\bar{b}' = a.\frac{\bar{c}\times\bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]} = \frac{0}{[\bar{a}\,\bar{b}\,\bar{c}]} = 0,$$ and so on.

Note: Relation ④ also implies that $[\bar{a}\,\bar{b}\,\bar{c}]$ and $[\bar{a}'\,\bar{b}'\,\bar{c}']$ must have the same sign, i.e., two reciprocal sets are either both right handed or both left handed.

To show $[\bar{a}\,\bar{b}\,\bar{c}][\bar{a}'\,\bar{b}'\,\bar{c}'] = 1 \Rightarrow [\bar{a}\,\bar{b}\,\bar{c}]$ is reciprocal to $[\bar{a}'\,\bar{b}'\,\bar{c}']$

$$\bar{b}'\times\bar{c}' = \frac{(\bar{c}\times\bar{a})\times(\bar{a}\times\bar{b})}{([\bar{a}\,\bar{b}\,\bar{c}])^2} = \frac{\{(\bar{c}\times\bar{a}).\bar{b}\}\bar{a} - \{(\bar{c}\times\bar{a}).\bar{a}\}\bar{b}}{([\bar{a}\,\bar{b}\,\bar{c}])^2}$$

$$= \frac{[\bar{c}\,\bar{a}\,\bar{b}]\,\bar{a}}{([\bar{a}\,\bar{b}\,\bar{c}])^2} = \frac{\bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]}.$$

$$\therefore \boxed{\bar{a} = \frac{\bar{b}'\times\bar{c}'}{[\bar{a}'\,\bar{b}'\,\bar{c}']}}$$

$$\bar{a}'.\bar{b}'\times\bar{c}' = \frac{b\times\bar{c}}{[\bar{a}\,\bar{b}\,\bar{c}]}.\frac{\bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]} = \frac{1}{[\bar{a}\,\bar{b}\,\bar{c}]} \Rightarrow [\bar{a}'\,\bar{b}'\,\bar{c}'][\bar{a}\,\bar{b}\,\bar{c}] = 1 \longrightarrow ④$$

Problems.

Ex. (1) Show that the vectors $\bar{a}\times(\bar{b}\times\bar{c})$, $\bar{b}\times(\bar{c}\times\bar{a})$, $\bar{c}\times(\bar{a}\times\bar{b})$ are coplanar.

Solution: The sum of given 3 vectors is zero.
$\therefore$ Any of these 3 vectors can be expressed in terms of the other two and so they are coplanar.

(2) Decompose a vector $\bar{r}$ as a l.c. of a vector $\bar{a}$ and another vector $\perp$ to $\bar{a}$ and coplanar with $\bar{r}$ and $\bar{a}$.

Solution: Any vector $\perp$ to $\bar{a}$ and coplanar with $\bar{r}$ and $\bar{a}$ is given by $(\bar{a}\times(\bar{r})\times\bar{a})$.

Let $\bar{r} = x\bar{a} + y\{\bar{a}\times(\bar{r}\times\bar{a})\}$ (Let us take dot product with $\bar{a}$ on bothsides)

$$\bar{r}\cdot\bar{a} = x(\bar{a}\cdot\bar{a}) + y\{\bar{a}\times(\bar{r}\times\bar{a})\}\cdot\bar{a}$$
$$= x\bar{a}^2 + y[\bar{a}\,(\bar{r}\times\bar{a})\,\bar{a}] = x\bar{a}^2 + 0$$

or, $\boxed{x = \frac{\bar{r}\cdot\bar{a}}{\bar{a}^2}}$

Let us take cross product with $\bar{a}$ on bothsides,

$$\bar{r}\times\bar{a} = x\,\bar{a}\times\bar{a} + y\{\bar{a}\times(\bar{r}\times\bar{a})\}\times\bar{a}$$
$$= \bar{0} + y[(\bar{a}\cdot\bar{a})(\bar{r}\times\bar{a}) - \{\bar{a}\cdot(\bar{r}\times\bar{a})\}\bar{a}]$$
$$= y\{\bar{a}^2(\bar{r}\times\bar{a}) - 0\bar{a}\}$$

or, $(\bar{r}\times\bar{a})(1 - y\bar{a}^2) = \bar{0} \Rightarrow 1 - y\bar{a}^2 = 0$ $[\because \bar{r}\times\bar{a} \neq \bar{0}]$

$\therefore$ $\boxed{y = \frac{1}{\bar{a}^2}}$. Since, $\bar{r}$ is not parallel to $\bar{a}$, otherwise $\bar{r}$ can not be decompose as a l.c. of the vector $\bar{a}$ & a vector $\perp$ to $\bar{a}$ and coplanar with $\bar{r}$ and $\bar{a}$

(3) Find the set of vectors reciprocal to $\bar{a} = (1,0,0)$, $\bar{b} = (1,1,0)$, $\bar{c} = (1,1,1)$

Let $\{\bar{a}', \bar{b}', \bar{c}'\}$ be the required reciprocal set to $\{\bar{a}, \bar{b}, \bar{c}\}$

$$[\bar{a}\,\bar{b}\,\bar{c}] = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1$$

Then $\bar{a}' = \frac{\bar{b}\times\bar{c}}{[\bar{a}\,\bar{b}\,\bar{c}]} = \frac{\hat{i}-\hat{j}}{1} = \hat{i}-\hat{j} = (1,-1,0)$

$\bar{b}' = \frac{\bar{c}\times\bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]} = \frac{\hat{j}-\hat{k}}{1} = \hat{j}-\hat{k} = (0,1,-1)$

$\bar{c}' = \frac{\bar{a}\times\bar{b}}{[\bar{a}\,\bar{b}\,\bar{c}]} = \frac{\hat{k}}{1} = \hat{k} = (0,0,1)$

Find the components of $\bar{r} = (2,3,4)$ relative to the bases $\bar{a}, \bar{b}, \bar{c}$ and $\bar{a}', \bar{b}', \bar{c}'$

Relative to the bases $\bar{a}, \bar{b}, \bar{c}$, the components of $\bar{r}$ are $\bar{r}.\bar{a}', \bar{r}.\bar{b}', \bar{r}.\bar{c}'$, as

$$\bar{r} = (\bar{r}.\bar{a}')\bar{a} + (\bar{r}.\bar{b}')\bar{b} + (\bar{r}.\bar{c}')\bar{c},$$

Also $\bar{r} = (\bar{r}.\bar{a})\bar{a}' + (\bar{r}.\bar{b})\bar{b}' + (\bar{r}.\bar{c})\bar{c}'$.

$$\bar{r}.\bar{a}' = (2,3,4).(1,-1,0) = -1,$$
$$\bar{r}.\bar{b}' = (2,3,4).(0,1,-1) = -1.$$
$$\bar{r}.\bar{c}' = (2,3,4).(0,0,1) = 4.$$

(4) Find the set reciprocal to $\bar{a}, \bar{b}, \bar{a}\times\bar{b}$

The set reciprocal to given set be $\{\bar{x}, \bar{y}, \bar{z}\}$, say.

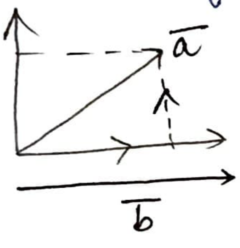
$$[\bar{a}\,\bar{b}\,(\bar{a}\times\bar{b})] = \bar{a}.\{\bar{b}\times(\bar{a}\times\bar{b})\}$$
$$= \bar{a}.\{(\bar{b}.\bar{b})\bar{a} - (\bar{b}.\bar{a})\bar{b}\}$$
$$= \bar{a}^2\bar{b}^2 - (\bar{a}.\bar{b})^2$$
$$= \bar{a}^2\bar{b}^2(1-\cos^2\theta) = \bar{a}^2\bar{b}^2\sin^2\theta$$
$$= |\bar{a}\times\bar{b}|^2$$

$$\bar{x} = \frac{\bar{b}\times(\bar{a}\times\bar{b})}{[\bar{a}\,\bar{b}\,(\bar{a}\times\bar{b})]} = \frac{\bar{b}\times(\bar{a}\times\bar{b})}{|\bar{a}\times\bar{b}|^2}$$

$$\bar{y} = \frac{(\bar{a}\times\bar{b})\times\bar{a}}{|\bar{a}\times\bar{b}|^2}, \quad \bar{z} = \frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|^2}.$$

(10) Express a vector $\bar{a}$ as the sum of two component vectors, one $\parallel$ and the other $\perp$ to $\bar{b}$ in the form:

$$\bar{a} = \frac{1}{b^2}\{(\bar{a}.\bar{b})\bar{b} + \bar{b}\times(\bar{a}\times\bar{b})\}.$$



Solution: Component vector of $\bar{a}$ along

$$\bar{b} = (\bar{a}.\hat{b})\hat{b} = \frac{(\bar{a}.\bar{b})}{|\bar{b}|}\frac{\bar{b}}{|\bar{b}|} = \frac{(\bar{a}.\bar{b})\bar{b}}{|\bar{b}|^2}$$

Component vector of $\bar{a}$ $\perp$ to $\bar{b}$ is given by

$$\bar{a} - \frac{(\bar{a}.\bar{b})\bar{b}}{\bar{b}^2} = \frac{\bar{a}(\bar{b}.\bar{b}) - (\bar{a}.\bar{b})\bar{b}}{\bar{b}^2} = \frac{\bar{b}\times(\bar{a}\times\bar{b})}{\bar{b}^2}$$

$$\therefore \bar{a} = \frac{1}{b^2}\{(\bar{a}.\bar{b})\bar{b} + \bar{b}\times(\bar{a}\times\bar{b})\}.$$

(11) Let $\bar{a}$ and $\bar{c}$ be two given vectors and K be a given scalar. Determine all vectors $\bar{b}$ such that

$$\bar{a}\times\bar{b} = \bar{c} \text{ and } \bar{a}.\bar{b} = K.$$

$$(\bar{a}\times\bar{b})\times\bar{a} = \bar{c}\times\bar{a} \Rightarrow (\bar{a}.\bar{a})\bar{b} - (\bar{a}.\bar{b})\bar{a} = \bar{c}\times\bar{a}$$

$$\Rightarrow a^2\bar{b} - K\bar{a} = \bar{c}\times\bar{a} \Rightarrow \bar{b} = \frac{(\bar{c}\times\bar{a}) + K\bar{a}}{a^2} \quad (\bar{a}\neq\bar{0})$$

[Note: $\bar{a}.\bar{c} = 0$, if $\bar{a} = \bar{0}$ then $K = 0$, $\bar{c} = \bar{0} \Rightarrow \bar{b}$ can be any vector.]

(Ex). If $\bar{a}, \bar{b}, \bar{c}$ be any 3 non-coplanar vectors, then $\bar{b}\times\bar{c}$, $\bar{c}\times\bar{a}$, $\bar{a}\times\bar{b}$ are also non-coplanar. Express $\bar{a}, \bar{b}, \bar{c}$ in terms of $\bar{b}\times\bar{c}, \bar{c}\times\bar{a}, \bar{a}\times\bar{b}$, and conversely.

Soln: Since $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar, $[\bar{a}\bar{b}\bar{c}] \neq 0$.

Now $[\bar{b}\times\bar{c}, \bar{c}\times\bar{a}, \bar{a}\times\bar{b}] = (\bar{b}\times\bar{c})\cdot(\bar{c}\times\bar{a})\times(\bar{a}\times\bar{b})$

Let $\bar{c}\times\bar{a} = \bar{q}$, and $\bar{b}\times\bar{c} = \bar{p}$, then $(\bar{c}\times\bar{a})\times(\bar{a}\times\bar{b}) = \bar{q}\times(\bar{a}\times\bar{b})$

$= (\bar{q}\cdot\bar{b})\bar{a} - (\bar{q}\cdot\bar{a})\bar{b}$

Again $(\bar{b}\times\bar{c})\cdot(\bar{c}\times\bar{a})\times(\bar{a}\times\bar{b}) = \bar{p}\cdot\{(\bar{q}\cdot\bar{b})\bar{a} - (\bar{q}\cdot\bar{a})\bar{b}\}$

$= (\bar{p}\cdot\bar{a})(\bar{q}\cdot\bar{b}) - (\bar{p}\cdot\bar{b})(\bar{q}\cdot\bar{a})$

$= (\bar{b}\times\bar{c}\cdot\bar{a})(\bar{c}\times\bar{a}\cdot\bar{b}) - (\bar{b}\times\bar{c}\cdot\bar{b})(\bar{c}\times\bar{a}\cdot\bar{a})$

$= [\bar{a}\bar{b}\bar{c}][\bar{a}\bar{b}\bar{c}] - 0\times0 = ([\bar{a}\bar{b}\bar{c}])^2 \neq 0$

$\therefore$ The given 3 vectors also are non-coplanar.

Any vector $\bar{a}$ can be expressed as a l.c. of 3 non-coplanar vectors $\bar{b}\times\bar{c}, \bar{c}\times\bar{a}, \bar{a}\times\bar{b}$.

Let $\bar{a} = l\,\bar{b}\times\bar{c} + m\,\bar{c}\times\bar{a} + n\,\bar{a}\times\bar{b}$; l, m, n are scalars.

$\bar{a}\cdot\bar{a} = l(\bar{b}\times\bar{c}\cdot\bar{a}) + m(\bar{c}\times\bar{a}\cdot\bar{a}) + n(\bar{a}\times\bar{b}\cdot\bar{a})$

a, $l = \dfrac{\bar{a}\cdot\bar{a}}{[\bar{a}\bar{b}\bar{c}]}$, Similarly, $m = \dfrac{\bar{a}\cdot\bar{b}}{[\bar{a}\bar{b}\bar{c}]}$, $n = \dfrac{\bar{a}\cdot\bar{c}}{[\bar{a}\bar{b}\bar{c}]}$

$\therefore \quad \bar{a} = \dfrac{\bar{a}\cdot\bar{a}}{[\bar{a}\bar{b}\bar{c}]}\,\bar{b}\times\bar{c} + \dfrac{\bar{a}\cdot\bar{b}}{[\bar{a}\bar{b}\bar{c}]}\,\bar{c}\times\bar{a} + \dfrac{\bar{a}\cdot\bar{c}}{[\bar{a}\bar{b}\bar{c}]}\,\bar{a}\times\bar{b}$

Similarly, we can express $\bar{b}$ and $\bar{c}$ in terms of these 3 non-coplanar vectors.

Conversely, let $\bar{b}\times\bar{c} = l\bar{a} + m\bar{b} + n\bar{c}$

$(\bar{b}\times\bar{c})\cdot(\bar{b}\times\bar{c}) = l\bar{a}\cdot(\bar{b}\times\bar{c}) + 0 + 0$

a, $l = \dfrac{(\bar{b}\times\bar{c})\cdot(\bar{b}\times\bar{c})}{[\bar{a}\bar{b}\bar{c}]}$

Similarly, $m =$

$n =$

Similarly, $\bar{c}\times\bar{a}$, $\bar{a}\times\bar{b}$ can be expressed in terms of $\bar{a}, \bar{b}, \bar{c}$.

(4) A necessary and sufficient condition that the vector equation $\bar{a}\times\bar{r}=\bar{b}$ possesses a solution is that $\bar{a}.\bar{b}=0$. $(\bar{a}\neq\bar{0})$

Necessary $\rightarrow$ $\bar{a}\cdot(\bar{a}\times\bar{r})=\bar{a}\cdot\bar{b} \Rightarrow 0=\bar{a}.\bar{b}$

Sufficient $\rightarrow$ Let us consider 3 non-coplanar vectors $\bar{a},\bar{b},\bar{b}\times\bar{a}$. $\quad \bar{a}.\bar{b}=0$

Any vector $\bar{r}=x\bar{a}+y\bar{b}+z\,\bar{b}\times\bar{a}$.

from $\bar{a}\times\bar{r}=\bar{b}$, we obtain,

$$\bar{a}\times\bar{r}=\bar{b}=x\,\bar{a}\times\bar{a}+y\,\bar{a}\times\bar{b}+z\,\bar{a}\times(\bar{b}\times\bar{a})$$

$$=\bar{0}+y(\bar{a}\times\bar{b})+z\{(\bar{a}.\bar{a})\bar{b}-(\bar{a}.\bar{b})\bar{a}\}$$

$$\Rightarrow y(\bar{a}\times\bar{b})=(za^2-1)\bar{b}=\bar{0}$$

Since $\bar{a}\times\bar{b}$ and $\bar{a}$ are non-collinear,

then $y=0,\ z=1/a^2$.

$\therefore\ \bar{r}=x\bar{a}+\frac{1}{a^2}\bar{b}\times\bar{a}$, It is a general solution of $\bar{r}$, x being the parameter.

(5) Determine all $\bar{r}$ such that $\bar{a}\times\bar{r}=\bar{c}$ and $\bar{a}\cdot\bar{r}=K$

Case I. If $\bar{a}.\bar{c}\neq 0$, there is no solution, by the above problem. Because, $\bar{a}\cdot\bar{a}\times\bar{r}=0$, hence $\bar{a}\times\bar{r}=\bar{c}$ becomes inconsistent then.

Case II. If $\bar{a}.\bar{c}=0$, and $\bar{a}\neq\bar{0}$, $(\bar{a}\times\bar{r})\times\bar{a}=\bar{c}\times\bar{a}$

$$\Rightarrow (\bar{a}.\bar{a})\bar{r}-(\bar{r}.\bar{a})\bar{a}=\bar{c}\times\bar{a}$$

$$\Rightarrow a^2\bar{r}-K\bar{a}=\bar{c}\times\bar{a}$$

$$\Rightarrow \bar{r}=\frac{\bar{c}\times\bar{a}+K\bar{a}}{a^2}$$ which is only one solution

Case III. If $\bar{a}=\bar{0}$, there is no solution unless $\bar{c}=\bar{0}$ and $K=0$.

(6) Solve the vector equation:

$$t\bar{r}+\bar{r}\times\bar{a}=\bar{b} \longrightarrow (1)$$

Now, $(t\bar{r}+\bar{r}\times\bar{a})\cdot\bar{a}=\bar{b}\cdot\bar{a}$

$\Rightarrow t(\bar{r}.\bar{a})=\bar{b}\cdot\bar{a} \longrightarrow (2)$

Again, $(t\bar{r}+\bar{r}\times\bar{a})\times\bar{a}=\bar{b}\times\bar{a}$

$$\Rightarrow t\,\bar{r}\times\bar{a}+(\bar{r}.\bar{a})\bar{a}-(\bar{a}.\bar{a})\bar{r}=\bar{b}\times\bar{a}$$

$$\Rightarrow t(\bar{b}-t\bar{r})+\frac{1}{t}(\bar{b}.\bar{a})\bar{a}-\bar{a}^2\bar{r}=\bar{b}\times\bar{a} \quad [\text{using } (1) \,\&\, (2)]$$

$$\Rightarrow \bar{r}(t^2+\bar{a}^2)=t\bar{b}+\frac{1}{t}(\bar{b}\cdot\bar{a})\bar{a}-(\bar{b}\times\bar{a})$$

$$\Rightarrow \bar{r}=\frac{\bar{a}\times\bar{b}+\frac{1}{t}(\bar{a}.\bar{b})\bar{a}+t\bar{b}}{t^2+\bar{a}^2} \longrightarrow (3)$$

Theorem of Rankine

Ex. If four forces $\bar{P}_1, \bar{P}_2, \bar{P}_3, \bar{P}_4$ acting at a point in equilibrium, then each force is proportional to the volume of the parallelopiped determined by the unit vectors in the directions of other three.

Let $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ be the unit vectors in the directions of forces P_1, P_2, P_3, P_4 so that

$$P_1\hat{a} = \bar{P}_1, \quad P_2\hat{b} = \bar{P}_2, \quad P_3\hat{c} = \bar{P}_3, \quad P_4\hat{d} = \bar{P}_4$$

Since the forces are in equilibrium, their vector sum is zero. $\Rightarrow P_1\hat{a} + P_2\hat{b} + P_3\hat{c} + P_4\hat{d} = \bar{0}. \longrightarrow (1).$

$$(P_1\hat{a} + P_2\hat{b} + P_3\hat{c} + P_4\hat{d}) \cdot (\hat{c}\times\hat{d}) = \bar{0}\cdot(\hat{c}\times\hat{d})$$

$$\Rightarrow P_1[\hat{a}\,\hat{c}\,\hat{d}] + P_2[b\,c\,d] = 0 \longrightarrow (2)$$

$$P_2[b\,a\,c] + P_4[d\,a\,c] = 0 \longrightarrow (3)$$

$$P_3[c\,a\,b] + P_4[d\,a\,b] = 0 \longrightarrow (4).$$

From (2), (3) & (4), we deduce

$$\frac{P_1}{[b\,c\,d]} = \frac{P_2}{[\bar{c}\,\bar{a}\,\bar{d}]} = \frac{P_3}{[a\,b\,d]} = \frac{P_4}{[b\,a\,c]}$$

Thus each force is proportional to the scalar triple product of unit vectors in the directions of the other three forces and hence to the volume of the parallelopiped determined by them.

Special Case: Lami's Theorem:

If three forces acting at a point are in equilibrium, then each force is proportional to the sine of the angle between the other two.

For this case, $P_1\hat{a} + P_2\hat{b} + P_3\hat{c} = \bar{0} \longrightarrow (1)$

Taking successively the vector product of (1) with $\hat{a}, \hat{b}, \hat{c}$, we obtain

$$P_2(\hat{b}\times\hat{a}) + P_3(\hat{c}\times\hat{a}) = \bar{0} \longrightarrow (2)$$

$$P_1(\hat{a}\times\hat{b}) + P_3(\hat{c}\times\hat{b}) = \bar{0} \longrightarrow (3)$$

$$P_1(\hat{a}\times\hat{c}) + P_2(\hat{b}\times\hat{c}) = \bar{0} \longrightarrow (4)$$

From 2, 3 & 4 we obtain

$$\frac{P_1}{|\hat{b}\times\hat{c}|} = \frac{P_2}{|\hat{c}\times\hat{a}|} = \frac{P_3}{|\hat{a}\times\hat{b}|}$$

$$\Rightarrow \frac{P_1}{\sin(\widehat{\vec{P}_2,\vec{P}_3})} = \frac{P_2}{\sin(\widehat{\vec{P}_1,\vec{P}_3})} = \frac{P_3}{\sin(\widehat{\vec{P}_1,\vec{P}_2})}.$$

VECTOR EQUATIONS

A vector equation with unknown scalars and known vector coefficients can be solved by the use of scalar and vector products.

① Find scalars y and z s.t. $\bar{a}y + \bar{b}z = \bar{c}$.
($\bar{a}, \bar{b}, \bar{c}$ are given vectors)

Solution $(\bar{a}y + \bar{b}z) \times \bar{b} = \bar{c} \times \bar{b} \Rightarrow (\bar{a}\times\bar{b})y + (\bar{b}\times\bar{b})z = \bar{c}\times\bar{b}$

$$\Rightarrow y(\bar{a}\times\bar{b}) + \bar{0}z = \bar{c}\times\bar{b}$$

$$\Rightarrow (\bar{a}\times\bar{b})^2 y = (\bar{c}\times\bar{b})\cdot(\bar{a}\times\bar{b})$$

$$\Rightarrow y = \frac{(\bar{c}\times\bar{b})\cdot(\bar{a}\times\bar{b})}{(\bar{a}\times\bar{b})^2}$$

Similarly,

$$(\bar{a}y + \bar{b}z) \times \bar{a} = \bar{c}\times\bar{a}$$

$$\Rightarrow (\bar{b}\times\bar{a})z = \bar{c}\times\bar{a}$$

$$\Rightarrow (\bar{b}\times\bar{a})^2 z = (\bar{c}\times\bar{a})\cdot(\bar{b}\times\bar{a})$$

$$\Rightarrow z = \frac{(\bar{c}\times\bar{a})\cdot(\bar{b}\times\bar{a})}{(\bar{b}\times\bar{a})^2}.$$

② Solve for a vector $\bar{r}$ where $\bar{r}\cdot\bar{a} = t$, $\bar{r}\times\bar{b} = \bar{c}$, and $\bar{a}\cdot\bar{b} \neq 0$ (t is a given scalar; $\bar{a}, \bar{b}, \bar{c}$ are given)

Solution:

$$\bar{a}\times(\bar{r}\times\bar{b}) = \bar{a}\times\bar{c}$$

$$\Rightarrow (\bar{a}\cdot\bar{b})\bar{r} - (\bar{a}\cdot\bar{r})\bar{b} = \bar{a}\times\bar{c}$$

$$\Rightarrow (\bar{a}\cdot\bar{b})\bar{r} - t\bar{b} = \bar{a}\times\bar{c}$$

$$\Rightarrow \bar{r} = \frac{\bar{a}\times\bar{c} + t\bar{b}}{\bar{a}\cdot\bar{b}}$$

; This is uniquely determined.

③ A vector is uniquely determined when its scalar and vector products with two known non-perpendicular vectors are given.

③ A vector can be determined when its scalar products with 3 non-coplanar vectors are given as follows:

$$\bar{r}\cdot\bar{a} = x, \quad \bar{r}\cdot\bar{b} = y, \quad \bar{r}\cdot\bar{c} = z$$

Let us obtain a reciprocal set $(\bar{a}', \bar{b}', \bar{c}')$ to the set $(\bar{a}, \bar{b}, \bar{c})$ by $\bar{a}' = \dfrac{\bar{b}\times\bar{c}}{[\bar{a}\,\bar{b}\,\bar{c}]}$, and so on....

Any vector $\bar{r}$ can be written as

$$\bar{r} = (\bar{r}\cdot\bar{a})\bar{a}' + (\bar{r}\cdot\bar{b})\bar{b}' + (\bar{r}\cdot\bar{c})\bar{c}' = x\bar{a}' + y\bar{b}' + z\bar{c}'.$$

Linear Combination of a Set of Vectors

Definition → Any vector $\bar{r}$ is said to be a linear combination of the finite set of vectors $\bar{a}, \bar{b}, \bar{c}, \cdots$, if $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c} + \cdots$; x, y, z are scalar constants.

(1) For any two vectors $\bar{a}, \bar{b}$, if $\bar{a}$ and $\bar{b}$ are collinear, then $\bar{b} = x\bar{a}$ (x is a scalar). $\bar{b}$ is a l.c. of $\bar{a}$.

(2) Any vector $\bar{c}$, coplanar with $\bar{a}$ and $\bar{b}$, we have $\bar{c} = x\bar{a} + y\bar{b}$. $\bar{c}$ is a l.c. of $\bar{a}$ & $\bar{b}$.

(3) In 3D, any vector $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$, where $\bar{a}, \bar{b}, \bar{c}$ are any 3 non-coplanar vectors. Here $\bar{r}$ is a l.c. of $\bar{a}, \bar{b}$ & $\bar{c}$.

Linearly Dependent and Independent Vectors:

If the scalars $x, y, z, \cdots$ (NOT all zero) exist in the relation $x\bar{a} + y\bar{b} + z\bar{c} + \cdots = \bar{0}$, then the set $\{\bar{a}, \bar{b}, \bar{c}, \cdots\}$ forms a l.d. set of vectors. Otherwise, the set forms a l.i.d. set of vectors, i.e., for a l.i.d. set of vectors,

$x\bar{a} + y\bar{b} + z\bar{c} + \cdots = \bar{0} \Rightarrow x = 0, y = 0, z = 0, \cdots$

Properties:

1. A superset of a l.d. set of vectors is l.d.
2. A single vector is l.d. if it is a zero vector.
   " " " l.i.d. " " " NOT " " .
3. A set of vectors that includes zero vector is l.d.
4. Two non-zero vectors be l.d. iff they are collinear.
5. Three non-zero vectors be l.d. iff they are coplanar.
6. For any 4 vectors, we have, $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$, or, $\bar{r} - x\bar{a} - y\bar{b} - z\bar{c} = \bar{0}$
   ∴ Four vectors or more vectors are always l.d.
7. Two non-collinear vectors $\bar{a}, \bar{b}$ are l.i.d.
8. Three non-coplanar vectors $\bar{a}, \bar{b}, \bar{c}$ are l.i.d.
   Because, $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0} \Rightarrow \bar{a} = \left(-\frac{y}{x}\right)\bar{b} + \left(-\frac{z}{x}\right)\bar{c}$ $(x \neq 0)$
   Which gives $\bar{a}$ is coplanar with $\bar{b}$ and $\bar{c}$, which is a contradiction. Similarly, $y \neq 0, z \neq 0$.