STUDY MATERIALS

APPLICATIONS OF VECTOR PRODUCTS

Mathematics Honours
Semester – 2
Paper – C4T   Unit - 4

Dr. Sangita Chakraborty
Associate Professor
Department of Mathematics
Kharagpur College

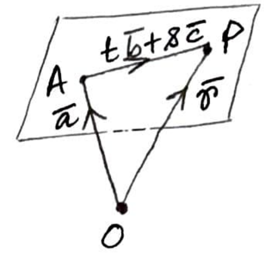
Applications of Vector Products to Space Geometry

· Vector

EQUATION OF A PLANE :—

Parametric form →

(1) A plane which passes through a given point $\bar{a}$ and is parallel to two vectors $\bar{b}$ and $\bar{c}$.



With reference to the origin O,
let the p.v. of the given point A be $\bar{a}$, and
" " " " any point on the plane be $\bar{r}$

Then $\overline{AP}$, $\bar{b}$, $\bar{c}$ are coplanar. $[\because \overline{AP} = \overline{OP} + \overline{OA}]$

i.e., $(\bar{r}-\bar{a})$, $\bar{b}$, $\bar{c}$ " "

$\therefore \bar{r}-\bar{a} = t\bar{b}+s\bar{c}$; t, s are scalars

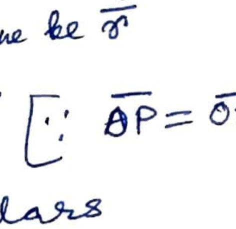
or, $\boxed{\bar{r} = \bar{a}+t\bar{b}+s\bar{c}}$ Parametric form.

$(\bar{r}-\bar{a})\cdot\bar{b}\times\bar{c} = 0$ Using Triple Products.

or, $[\bar{r}\ \bar{b}\ \bar{c}] = [\bar{a}\ \bar{b}\ \bar{c}]$.

Normal Form →

(2) A plane which is perpendicular to the vector $\bar{m}$ and passing through the point $\bar{a}$.



$(\bar{r}-\bar{a})$ lies in the plane
and $\bar{m} \perp$ to the plane.

$\therefore$ $(\bar{r}-\bar{a})\cdot\bar{m} = 0$

or, $\bar{r}\cdot\bar{m} = \bar{a}\cdot\bar{m} = K$

If $\hat{n}$ be the unit vector $\perp$ to the plane, then

$(\bar{r}-\bar{a})\cdot\hat{n} = 0$

$\Rightarrow$ $\bar{r}\cdot\hat{n} = \bar{a}\cdot\hat{n} = p$, the length of the perpendicular from the origin O to the plane $(p = ON)$.

$\boxed{p - \bar{r}\cdot\hat{n} = 0}$ Normal Form. [taking p as positive]

(i) If $\hat{n}$ has (l, m, n) as direction cosines, then eqn of plane reduces in 3D geometry
$$lx + my + nz = p.$$

(ii) If the origin lies on the plane, then $\bar{r}\cdot\hat{n} = 0$.

(iii) Inclination of the two planes $p - \bar{r}\cdot\hat{n} = 0$ and $p' - \bar{r}\cdot\hat{n}' = 0$ is given by $\theta = \cos^{-1}(\hat{n}\cdot\hat{n}')$.

Inclination of two planes $\bar{r}.\bar{m}=p$ and $\bar{r}.\bar{m}'=p'$ where $\bar{m}$ and $\bar{m}'$ are not unit vectors, then

$$\cos\theta = \frac{\bar{m}.\bar{m}'}{|\bar{m}||\bar{m}'|}$$

(iv) Two points $A_1(\bar{a}_1)$, $A_2(\bar{a}_2)$ are on the same or on the different sides of the plane $p-\bar{r}.\hat{n}=0$ according as $p-\bar{a}_1.\hat{n}$ and $p-\bar{a}_2.\hat{n}$ are of the same or of opposite signs.

(v) Distance of a point $A(\bar{a})$ from the plane $p-\bar{r}.\hat{n}=0$ is the required perpendicular distance $p-\bar{a}.\hat{n}$ (is +ve if the point and the origin are in same side
(is −ve if they are in opposite side of the plane.)

(vi) The distance of a point $A(\bar{a})$ to the plane $p-\bar{r}.\hat{n}=0$ measured along a direction whose unit vector $\hat{b}$ is given by

$$d = \frac{p-\bar{a}.\hat{n}}{\hat{b}.\hat{n}}$$

Ex: 9 Find the perpendicular distance from the point $P(1,-2,1)$ to the plane $3x-9y+4z+25=0$.

The given eqn of the plane in normal form:

$$\frac{25}{\sqrt{3^2+9^2+4^2}} - \frac{1}{\sqrt{3^2+9^2+4^2}}\cdot(-3x+9y-4z)=0 \quad \left[p=\frac{25}{\sqrt{106}},\ \hat{n}=\frac{(3,+9,4)}{\sqrt{106}}\right]$$

perpendicular distance from $P(1,-2,1)$ is

$$= \frac{25}{\sqrt{106}} - \frac{1}{\sqrt{106}}\left((-3)\cdot 1+9\cdot(-2)+(-4)\cdot 1\right) = \frac{25}{\sqrt{106}}+\frac{25}{\sqrt{106}}.$$

$$= \frac{50}{\sqrt{106}} = \frac{25\sqrt{106}}{53}.$$

Ex: 8. Show that the points $(1,-1,3)$ and $(3,3,3)$ are equidistant from the plane $5x+2y-7z+9=0$ and are on the opposite sides of it.

Normal form of the given plane is:

$$\frac{9}{\sqrt{5^2+2^2+7^2}} - \frac{1}{\sqrt{5^2+2^2+7^2}}(-5x-2y+7z)=0. \quad \left[p=\frac{9}{\sqrt{78}},\ \hat{n}=\frac{1}{\sqrt{78}}(-5,-2,7)\right]$$

perp. dist. from $(1,-1,3)$ is $= \frac{9}{\sqrt{78}} - \frac{1}{\sqrt{78}}(-5+2+21) = \frac{9}{\sqrt{78}} - \frac{18}{\sqrt{78}}$

$$= \left|\frac{-9}{\sqrt{78}}\right| = \frac{9}{\sqrt{78}}$$

" " " $(3,3,3)$ is $= \frac{9}{\sqrt{78}} - \frac{1}{\sqrt{78}}(-15-6+21) = \frac{9}{\sqrt{78}}$

$\therefore$ The points are equidistant.

Since the signs are of opposite, they are on the opposite sides of it.

Ex: 1. Find the equation of the plane passing through $(2,-3,1)$ and perp. to the line joining $(3,4,-1)$ and $(2,-1,5)$

$$\{(x,y,z)-(2,-3,1)\}\cdot\{(3-2,4+1,-1-5)\}=0.$$

or, $(x-2,y+3,z-1)\cdot(1,5,-6)=0$

or, $(x-2)+5(y+3)-6(z-1)=0.$

or, $x+5y-6z+19=0.$

Ex. 2: Find the eqn of the plane passing through $(2,3,4)$ and parallel to the plane $5x-6y+7z=8$.

$$\{(x,y,z)-(2,3,4)\}\cdot(5,-6,7)=0.$$

or, $5(x-2)-6(y-3)+7(z-4)=0.$

or, $5x-6y+7z-10+18-28=0.$

or, $5x-6y+7z=20$

Vector Equations of Planes Using Triple Products

① Equation of a plane which is parallel to $\bar{c}$, $\bar{d}$ and passing through $\bar{a}$.

$$(\bar{r}-\bar{a})\cdot\bar{c}\times\bar{d}=0 \Rightarrow [\bar{r}\,\bar{c}\,\bar{d}]=[\bar{a}\,\bar{c}\,\bar{d}]$$

② Plane passing through $\bar{a}, \bar{b}, \bar{c}$

$$(\bar{r}-\bar{a})\cdot(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})=0$$

a, $\bar{r}\cdot(\bar{b}-\bar{a})\times(\bar{c}-\bar{a}) = \bar{a}\cdot\{(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})\}$

$$= \bar{a}\cdot(\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a}+\bar{a}\times\bar{a})$$

$$= \bar{a}\cdot(\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a})$$

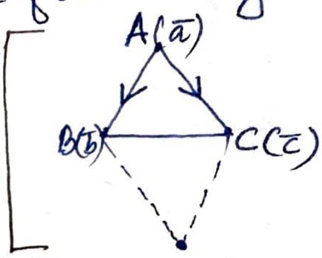
$$= [\bar{a}\,\bar{b}\,\bar{c}].$$

N.B. $|\bar{b}\times\bar{c}+\bar{a}\times\bar{b}+\bar{c}\times\bar{a}|$ = Twice the area of the triangle formed by $\bar{a}, \bar{b}, \bar{c}$

③ Plane containing the line $\bar{r}=\bar{a}+t\bar{b}$ and parallel to $\bar{c}$

$$(\bar{r}-\bar{a})\cdot\bar{b}\times\bar{c}=0$$

a, $[\bar{r}\,\bar{b}\,\bar{c}]=[\bar{a}\,\bar{b}\,\bar{c}]$



④ Plane passing through $\bar{a}$ and $\bar{b}$ and parallel to $\bar{c}$.

$$(\bar{r}-\bar{a})\cdot(\bar{b}-\bar{a})\times\bar{c}=0$$

a, $\bar{r}\cdot(\bar{b}\times\bar{c}+\bar{c}\times\bar{a})=\bar{a}\cdot(\bar{b}\times\bar{c}+\bar{c}\times\bar{a})=[\bar{a}\,\bar{b}\,\bar{c}]$

⑤ Plane containing the line $\bar{r}=\bar{a}+t\bar{b}$ and passing through $\bar{c}$.

$$(\bar{r}-\bar{a})\cdot\{(\bar{a}-\bar{c})\times\bar{b}\}=0$$

a, $\bar{r}\cdot(\bar{a}\times\bar{b}+\bar{b}\times\bar{c})=\bar{a}\cdot(\bar{a}\times\bar{b}+\bar{b}\times\bar{c})=[\bar{a}\,\bar{b}\,\bar{c}]$.

⑥ Plane passing through the line of intersection of two planes $\bar{r}\cdot\bar{n}=p$, $\bar{r}\cdot\bar{n}'=p'$ and contains the point $\bar{b}$.

Any plane passing through the line of intersection of two planes $\bar{r}\cdot\bar{n}=p$, $\bar{r}\cdot\bar{n}'=p'$ is given by $\bar{r}\cdot(\bar{n}-\lambda\bar{n}')=p-\lambda p'$ (λ is a parameter). This plane passes through $\bar{b}$, then $\bar{b}\cdot(\bar{n}-\lambda\bar{n}')=p-\lambda p' \Rightarrow \lambda=\dfrac{\bar{b}\cdot\bar{n}-p}{\bar{b}\cdot\bar{n}'-p'}$

∴ The req. plane is given by

$$\bar{r}\cdot(\bar{n}-\lambda\bar{n}')=p-\lambda p' \text{ where } \lambda=\frac{\bar{b}\cdot\bar{n}-p}{\bar{b}\cdot\bar{n}'-p'}.$$

(7) Find the eqn of the plane through two given lines $\bar{r} = \bar{a} + t\bar{b}$, $\bar{r} = \bar{c} + s\bar{d}$ (t, s are scalars).

$$(\bar{r}-\bar{a}) \cdot \bar{b} \times \bar{d} = 0$$

$$\Rightarrow [\bar{r}\,\bar{b}\,\bar{d}] = [\bar{a}\,\bar{b}\,\bar{d}]$$

Condition of coplanarity of two non-parallel lines is obtained by considering that the plane also passes through $\bar{c}$.

i.e., $[\bar{c}\,\bar{b}\,\bar{d}] = [\bar{a}\,\bar{b}\,\bar{d}]$.

(8) Show that the 4 points $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar iff $[\bar{b}\,\bar{c}\,\bar{d}] + [c\,a\,d] + [a\,b\,d] = [a\,b\,c]$.

Four points $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar iff $\bar{d}$ lies on the plane through the 3 points $\bar{a}, \bar{b}, \bar{c}$, ie, if $\bar{d}$ satisfies the equation

$$(\bar{r}-\bar{a}) \cdot (\bar{b}-\bar{a}) \times (\bar{c}-\bar{a}) = 0$$

a, $(\bar{d}-\bar{a}) \cdot (\bar{b}-\bar{a}) \times (\bar{c}-\bar{a}) = 0$

$$\Rightarrow (\bar{d}-\bar{a}) \cdot (\bar{b}\times\bar{c} + \bar{c}\times\bar{a} + \bar{a}\times\bar{b}) = 0$$

$$\Rightarrow [\bar{d}\,\bar{b}\,\bar{c}] + [\bar{d}\,\bar{c}\,\bar{a}] + [\bar{d}\,\bar{a}\,\bar{b}] = [\bar{a}\,\bar{b}\,\bar{c}]$$ proved.

Otherwise. We know that the necessary and sufficient condition for four points $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ to be coplanar is that there exists four scalars x, y, z, t, not all zero, such that $x\bar{a} + y\bar{b} + z\bar{c} + t\bar{d} = \bar{0}$, $x+y+z+t=0$.

Eliminating t, the condition becomes:

$$x(\bar{a}-\bar{d}) + y(\bar{b}-\bar{d}) + z(\bar{c}-\bar{d}) = \bar{0}.$$

Now x, y, z cannot be all zero, for then $t=0$ and all 4 scalars will become zero.

$x(\bar{a}-\bar{d}) \cdot (\bar{b}-\bar{d}) \times (\bar{c}-\bar{d}) = 0$

Suppose, $x \neq 0$, then $(\bar{a}-\bar{d}) \cdot (\bar{b}-\bar{d}) \times (\bar{c}-\bar{d}) = 0$ ($\because x \neq 0$)

$\Rightarrow (\bar{a}-\bar{d}) \cdot (\bar{b}-\bar{d}) \times (\bar{c}-\bar{d})$ [sic]. Expanding this, we get the required result.

Vector Equation of a Straight Line

Parametric Form:

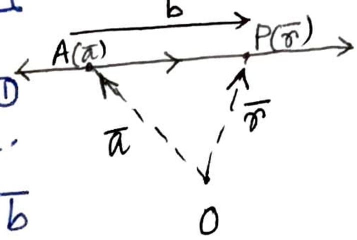
Find the equation of a line passing through $\bar{a}$ and parallel to $\bar{b}$ is given by

$$\overline{OP} = \overline{OA} + \overline{AP}$$

Here, $\bar{r} - \bar{a} = t\bar{b}$

$$\bar{r} = \bar{a} + t\bar{b} \longrightarrow (1)$$

where t is a scalar parameter.



Non-parametric Form: $(\bar{r} - \bar{a}) \parallel \bar{b}$

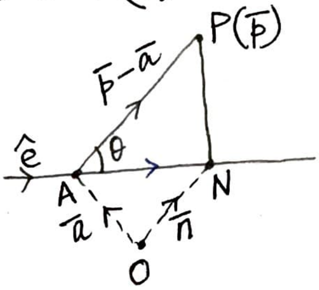
$$\Rightarrow (\bar{r} - \bar{a}) \times \bar{b} = \bar{0} \longrightarrow (2)$$

(2) can be derived from (1) as follows: $(\bar{r} - \bar{a}) \times \bar{b} = t\,\bar{b} \times \bar{b} = \bar{0}$.

If $\hat{e}$ be the unit vector along $\bar{b}$, then

$$(\bar{r} - \bar{a}) \times \hat{e} = \bar{0}$$

Perpendicular distance from a point $P(\bar{p})$ on a line $(\bar{r} - \bar{a}) \times \hat{e} = \bar{0}$



$$PN = AP \sin\theta = |(\bar{p} - \bar{a}) \times \hat{e}|$$

$$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \bar{a} + \{(\bar{p} - \bar{a}) \cdot \hat{e}\}\hat{e}$$

Note:

(i) $\bar{r} \cdot \bar{n} = 0$ is the equation of a plane passing through the origin and perpendicular to $\bar{n}$

(ii) $\bar{r} \times \bar{n} = \bar{0}$ is the equation of a line passing through the origin and parallel to $\bar{n}$.

Vector Equations of Lines Using vector products.

(1) A line passes through $\bar{a}$ and perpendicular to $\bar{b}$ and $\bar{c}$

$$(\bar{r} - \bar{a}) \times (\bar{b} \times \bar{c}) = \bar{0}$$

(2) A line passes through $\bar{a}$ and parallel to the line of intersection of the planes $\bar{r} \cdot \bar{n}_1 = p_1$ and $\bar{r} \cdot \bar{n}_2 = p_2$.

$$(\bar{r} - \bar{a}) \times (\bar{n}_1 \times \bar{n}_2) = \bar{0}$$

(3) A line passes through $\bar{a}$ and parallel to the plane $\bar{r} \cdot \bar{n} = p$ and perpendicular to the line $\bar{r} = \bar{c} + t\bar{d}$

$$(\bar{r} - \bar{a}) \times (\bar{d} \times \bar{n}) = \bar{0}.$$

(4) A line passes through $\bar{c}$, parallel to the plane $\bar{r} \cdot \bar{n} = p$ and intersecting the line $\bar{r} = \bar{a} + t\bar{b}$

$$(\bar{r} - \bar{c}) \times [\{(\bar{a} - \bar{c}) \times \bar{b}\} \times \bar{n}] = \bar{0}$$

The result

(5) A line passes through the point $\bar{c}$, intersecting both the lines $\bar{r} = \bar{a} + s\bar{b}$ and $\bar{r} = \bar{a}' + t\bar{b}'$.

$$(\bar{r} - \bar{c}) \times \{(\bar{a} - \bar{c}) \times \bar{b}\} \times \{(\bar{a}' - \bar{c}) \times \bar{b}'\} = \bar{0}$$

(6) A line of intersection of two planes $\bar{r}\cdot\bar{n}_1 = p_1$ and $\bar{r}\cdot\bar{n}_2 = p_2$.

To find, at first, one point through which the line passes. Let O be the origin and $\overline{ON}$ be the perpendicular drawn from O on the line.

$\therefore$ $\overrightarrow{ON}$, $\bar{n}_1$, $\bar{n}_2$ are coplanar.

$\therefore$ $\overrightarrow{ON} = t\bar{n}_1 + s\bar{n}_2$, where t, s are scalars.

Again, N is a common point of two given planes

$\therefore$ $(t\bar{n}_1 + s\bar{n}_2)\cdot\bar{n}_1 = p_1$, $(t\bar{n}_1 + s\bar{n}_2)\cdot\bar{n}_2 = p_2 \longrightarrow (1)$

Thus the required line passes through N and parallel to $\bar{n}_1 \times \bar{n}_2$.

The eqn of line will be then

$$\{\bar{r} - (t\bar{n}_1 + s\bar{n}_2)\} \times (\bar{n}_1 \times \bar{n}_2) = \bar{0} \longrightarrow (2)$$; t, s can be obtained from (1).

<u>Note</u>: Interpret the vector equn: $(\bar{\rho} - \bar{\alpha}) \times (\bar{\beta} \times \bar{\gamma}) = \bar{0}$, where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ and $\bar{\rho}$ be the p.vs of four points A, B, C and P respectively of which P is a variable point.

The given vector equation is a equation of a line passing through the point A and parallel to $\bar{\beta} \times \bar{\gamma}$, i.e., parallel to the normal to the plane containing $\overline{OB}$ and $\overline{OC}$, O being the origin.

(7) Find the ~~condition~~ point of intersection of the three planes $\bar{r}\cdot\bar{n}_1 = p_1$, $\bar{r}\cdot\bar{n}_2 = p_2$, $\bar{r}\cdot\bar{n}_3 = p_3$; $[\bar{n}_1\bar{n}_2\bar{n}_3] \neq 0$

$$[\bar{n}_2 \times \bar{n}_3, \bar{n}_3 \times \bar{n}_1, \bar{n}_1 \times \bar{n}_2] = (\bar{n}_2 \times \bar{n}_3)\cdot\{(\bar{n}_3 \times \bar{n}_1) \times (\bar{n}_1 \times \bar{n}_2)\}$$

$$= (n_2 \times n_3)\cdot$$

$$= (n_2 \times n_3)\cdot\{[n_3\, n_1\, n_2]n_1 - [n_3 n_1 n_1]n_2\}$$

$$= [n_1\, n_2\, n_3]^2 \neq 0.$$

$\bar{r} =$

Ex. Reduce the expression $(\bar{b}+\bar{c})\cdot\{(\bar{c}+\bar{a})\times(\bar{a}+\bar{b})\}$ in its simplest form and show that it vanishes when $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

Solution: $(\bar{b}+\bar{c})\cdot\{(\bar{c}+\bar{a})\times(\bar{a}+\bar{b})\}$

$= (\bar{b}+\bar{c})\cdot\{\bar{c}\times\bar{a}+\bar{c}\times\bar{b}+\bar{a}\times\bar{a}+\bar{a}\times\bar{b}\}$

$= (\bar{b}+\bar{c})\cdot(\bar{c}\times\bar{a}-\bar{b}\times\bar{c}+\bar{a}\times\bar{b})$

$= \bar{b}\cdot(\bar{c}\times\bar{a}) - \bar{b}\cdot(\bar{b}\times\bar{c}) + \bar{b}\cdot(\bar{a}\times\bar{b}) + \bar{c}\cdot(\bar{c}\times\bar{a}) - \bar{c}\cdot(\bar{b}\times\bar{c})$
$+ \bar{c}\cdot(\bar{a}\times\bar{b}) = [\bar{b}\,\bar{c}\,\bar{a}] - 0 + 0 + 0 - 0 + [\bar{c}\,\bar{a}\,\bar{b}]$

$= 2[\bar{a}\,\bar{b}\,\bar{c}]$, it vanishes when $[\bar{a}\,\bar{b}\,\bar{c}] = 0$,
i.e., when $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

Ex. Prove that $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [\bar{a}\,\bar{b}\,\bar{d}]\bar{c} - [\bar{a}\,\bar{b}\,\bar{c}]\bar{d} = [\bar{a}\,\bar{c}\,\bar{d}]\bar{b} - [\bar{b}\,\bar{c}\,\bar{d}]\bar{a}$
Hence show that $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are l.d.

Let $(\bar{a}\times\bar{b}) = \bar{p}$, $\therefore (\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = \bar{p}\times(\bar{c}\times\bar{d})$
$= (\bar{p}\cdot\bar{d})\bar{c} - (\bar{p}\cdot\bar{c})\bar{d} = [\bar{a}\,\bar{b}\,\bar{d}]\bar{c} - [\bar{a}\,\bar{b}\,\bar{c}]\bar{d}$

Let $(c\times\bar{d}) = \bar{q}$, $\therefore (\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = (\bar{a}\times\bar{b})\times\bar{q}$
$= (\bar{a}\cdot\bar{q})\bar{b} - (\bar{b}\cdot\bar{q})\bar{a} = [\bar{a}\,\bar{c}\,\bar{d}]\bar{b} - [\bar{b}\,\bar{c}\,\bar{d}]\bar{a}$

Now $[\bar{a}\,\bar{b}\,\bar{d}]\bar{c} - [\bar{a}\,\bar{b}\,\bar{c}]\bar{d} = [\bar{a}\,\bar{c}\,\bar{d}]\bar{b} - [\bar{b}\,\bar{c}\,\bar{d}]\bar{a}$

or, $\bar{d} = \dfrac{[\bar{b}\,\bar{c}\,\bar{d}]}{[\bar{a}\,\bar{b}\,\bar{c}]}\bar{a} - \dfrac{[\bar{a}\,\bar{c}\,\bar{d}]}{[\bar{a}\,\bar{b}\,\bar{c}]}\bar{b} + \dfrac{[\bar{a}\,\bar{b}\,\bar{d}]}{[\bar{a}\,\bar{b}\,\bar{c}]}\bar{c}$

$\bar{d} = \dfrac{[\bar{d}\,\bar{b}\,\bar{c}]}{[\bar{a}\,\bar{b}\,\bar{c}]}\bar{a} + \dfrac{[\bar{d}\,\bar{c}\,\bar{a}]}{[\bar{a}\,\bar{b}\,\bar{c}]}\bar{b} + \dfrac{[\bar{d}\,\bar{a}\,\bar{b}]}{[\bar{a}\,\bar{b}\,\bar{c}]}\bar{c}$

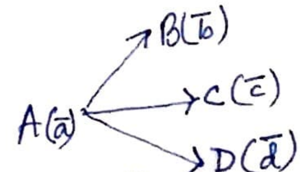
When $\bar{a}, \bar{b}, \bar{c}$ are non-coplanar so that $[\bar{a}\,\bar{b}\,\bar{c}] \neq 0$, $\bar{d}$ can be expressed as a l.c. of $\bar{a}, \bar{b}, \bar{c}$, such that $\bar{d} = x\bar{a} + y\bar{b} + z\bar{c}$; then $\bar{d} - x\bar{a} - y\bar{b} - z\bar{c} = \bar{0}$
Here coeff. of $\bar{d}$ is $1 (\neq 0)$

$x = \dfrac{[\bar{d}\,\bar{b}\,\bar{c}]}{[\bar{a}\,\bar{b}\,\bar{c}]}$, $y = \dfrac{[\bar{d}\,\bar{c}\,\bar{a}]}{[\bar{a}\,\bar{b}\,\bar{c}]}$, $z = \dfrac{[\bar{d}\,\bar{a}\,\bar{b}]}{[\bar{a}\,\bar{b}\,\bar{c}]}$.

$\therefore$ The four vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are l.d.

Ex. Show that the four points with p.v.s $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar iff $[b\,c\,d] + [c\,a\,d] + [a\,b\,d] = [a\,b\,c]$

Four points $A(\bar{a}), B(\bar{b}), C(\bar{c}), D(\bar{d})$ will be coplanar iff $\overline{AB}(\bar{b}-\bar{a})$, $\overline{AC}(\bar{c}-\bar{a})$, $\overline{AD}(\bar{d}-\bar{a})$ will be coplanar iff $[\bar{b}-\bar{a}, \bar{c}-\bar{a}, \bar{d}-\bar{a}] = 0$

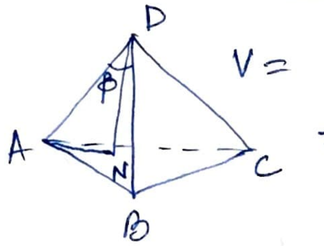


i.e., $(\bar{b}-\bar{a})\cdot\{(\bar{c}-\bar{a})\times(\bar{d}-\bar{a})\} = 0$. This gives the result

① Find the S.D. between two lines, one joining the points $A(-1,2,-3)$, $B(-16,6,4)$ and other " " $C(1,-1,3)$, $D(4,9,7)$. [Ans; 7]

② Find the S.D. between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$. Are they coplanar? [0; Coplanar]

Volume of a Tetrahedron:



$V=\frac{1}{3}$ Area of $\triangle ABC \times$ height DN

$=\frac{1}{3}\left\{\frac{1}{2}AB.AC\sin BAC\right\} AD\cos\phi$

$=\frac{1}{6}\,\overline{\overline{AB}\times\overline{AC}\cdot\overline{AD}}$.

Regular Tetrahedron: whose edges are all equal.

$\therefore$ The faces are all equilateral triangles and hence angle between any two of its concurrent edges is 60°.

① Angle θ between any two plane faces is given by $\cos\theta=\frac{1}{3}$.

Taking A as origin and p.v.s of B, C, D are $\bar{b},\bar{c},\bar{d}$.

$\therefore\ |\bar{b}|=|\bar{c}|=|\bar{d}|=a$ (say)

$\cos\theta=\frac{(\bar{b}\times\bar{c})\cdot(\bar{b}\times\bar{d})}{|\bar{b}\times\bar{c}|\,|\bar{b}\times\bar{d}|}=\frac{\frac{1}{4}a^4}{a^4\times\frac{3}{4}}=\frac{1}{3}$.

Now, $(\bar{b}\times\bar{c})\cdot(\bar{b}\times\bar{d})=\begin{vmatrix}\bar{b}\cdot\bar{b} & \bar{b}\cdot\bar{d}\\ \bar{c}\cdot\bar{b} & \bar{c}\cdot\bar{d}\end{vmatrix}=\begin{vmatrix}a^2 & \frac{1}{2}a^2\\ \frac{1}{2}a^2 & \frac{1}{2}a^2\end{vmatrix}=\frac{1}{4}a^4$.

and $|\bar{b}\times\bar{c}|=|\bar{b}||\bar{c}|\sin60^\circ=a^2\frac{\sqrt{3}}{2}$, $|\bar{b}\times\bar{d}|=a^2\frac{\sqrt{3}}{2}$

Ex. If the volume of a tetrahedron is 2 and 3 of its vertices have position vectors $A(1,1,0)$, $B(1,0,1)$, $C(2,-1,1)$. Find the locus of its fourth vertex D.

$V=\left|\frac{1}{6}(\overline{AB}\times\overline{AC})\cdot\overline{AD}\right|=2$

Ex. Find the volume of the tetrahedron formed by planes whose eqns are $y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=1$.

Let us obtain the point of intersection of first 3 planes: $(0,0,0)$, which is one of the vertices of the tetrahedron.

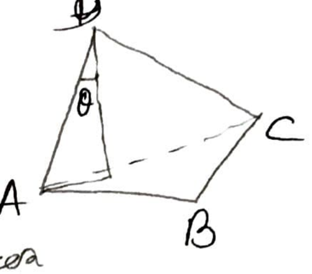
Similarly we obtain the other 3 vertices: $(-1,1,1)$, $(1,1,-1)$, $(1,-1,1)$

Volume $=2/3$.

Pr

Volume of a Tetrahedron.

① If the volume of the tetrahedron is 2 and 3 of its vertices have p.v.s $(1,1,0),(1,0,1),(2,-1,1)$, find the locus of the 4th vertex.



Let ABCD be a tetrahedron.
Let the p.v. of 4th vertex D be (α,β,γ)

The volume of the tetrahedron

$$V=\frac{1}{6}(\overline{AB}\times\overline{AC})\cdot\overline{AD}=\frac{1}{6}\begin{vmatrix}0 & -1 & 1\\ 1 & -2 & 1\\ \alpha-1 & \beta-1 & \gamma\end{vmatrix}=\pm 2$$

a, $\alpha+\beta+\gamma=\pm 12+2 \Rightarrow \alpha+\beta+\gamma=14$, $\alpha+\beta+\gamma=-10$

② S.T. the angle between any two plane faces of a regular tetrahedron is $\cos^{-1}\frac{1}{3}$.

Soln; Let ABCO be a tetrahedron and O be chosen as the origin.
Let $\overline{OA}=\bar{a}$, $\overline{OB}=\bar{b}$, $\overline{OC}=\bar{c}$
Now the angle (θ) between any two planes is the angle between their normals.
Normals on OAB and OAC are $\bar{a}\times\bar{b}$ and $\bar{c}\times\bar{a}$,

$$\therefore \cos\theta=\frac{(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{a})}{|\bar{a}\times\bar{b}|\,|\bar{c}\times\bar{a}|}=\frac{(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{a})-(\bar{a}\cdot\bar{a})(\bar{b}\cdot\bar{c})}{(ab\sin\pi/3)(ac\sin\pi/3)}$$

$$=\frac{a^2bc\cos^2\pi/3-a^2bc\cos\pi/3}{a^2bc\sin^2\pi/3}=\frac{1}{3}.$$

$\therefore\ \theta=\cos^{-1}\frac{1}{3}$

Note; The necessary and sufficient condition for the coplanarity of the four points with p.v.s $\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta}$ is that volume of the tetrahedron is zero, i.e.,

$$[\beta\gamma\delta]+[\gamma\alpha\delta]+[\delta\alpha\beta]=[\alpha\beta\gamma]$$

Resultant of Concurrent forces F_1, F_2; acting at a point O, is given by

$\bar{F} = \sum_{i=1}^{n} \bar{F}_i$; If $\bar{F} = \bar{0}$, then the forces are in equilibrium

Lami's theorem;

If three coplanar forces acting at a point be in equilibrium, then magnitude of each force is proportional to the sine of the angle between the other two.

Proof: Let P, Q, R be the magnitudes of the forces and $\bar{a}, \bar{b}, \bar{c}$ be the unit vectors along them.

Since the forces are in equilibrium

$\therefore P\bar{a} + Q\bar{b} + R\bar{c} = \bar{0}$ [Resultant force $= \bar{0}$]

$P(\bar{a}\times\bar{a}) + Q(\bar{a}\times\bar{b}) + R(\bar{a}\times\bar{c}) = \bar{0}$

$$\frac{Q}{|\bar{c}\times\bar{a}|} = \frac{R}{|\bar{a}\times\bar{b}|}, \text{ and } \frac{P}{|\bar{b}\times\bar{c}|} = \frac{R}{|\bar{a}\times\bar{b}|}$$

~~and~~

$$\therefore \frac{P}{|\bar{b}\times\bar{c}|} = \frac{Q}{|\bar{c}\times\bar{a}|} = \frac{R}{|\bar{a}\times\bar{b}|}.$$

$$\frac{P}{\sin \widehat{bc}} = \frac{Q}{\sin \widehat{ca}} = \frac{R}{\sin \widehat{ab}}.$$

Ex: A force $\bar{P} = 4\hat{i} - 3\hat{k}$ passes through the point A whose p.v. is $2i - 2j + 5k$. Find the moment of P about the point B whose p.v. is $i - 3j + k$

$\overline{BA} \times \bar{P}$

Ex: Find the torque about the point A(1, 2, −1) of a force (3, 0, 1) acting the point (2, −1, 3)

$\overline{AB} \times \bar{F}$

Problems on S.D.

① ⓐ Find the S.D. between two lines, one joining the points $A(-1,2,-3)$ and $B(-16,6,4)$ and the other joining the points $C(1,-1,3)$ and $D(4,9,7)$.

Two lines are given by:

$\overleftrightarrow{AB}$: $\quad \bar{r} = (-1,2,-3) + t\{-16+1,\ 6-2,\ 4+3\} = (-1,2,-3)+t(-15,4,7)$

$\overleftrightarrow{CD}$: $\quad \bar{r} = (1,-1,3) + s(4-1,\ 9+1,\ 7-3) = (1,-1,3)+s(3,10,4)$.

$$\text{S.D.}\ (p) = \frac{\overrightarrow{CA}\cdot\{(-15,4,7)\times(3,10,4)\}}{|(-15,4,7)\times(3,10,4)|} = 7 \ \text{(Ans.)}$$

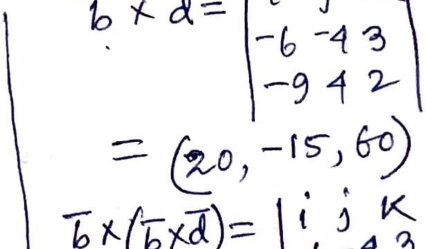
② Find the S.D. between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$. Are they coplanar?

$$\text{S.D.}\ (p) = (\bar{a}-\bar{c})\cdot\frac{\bar{b}\times\bar{d}}{|\bar{b}\times\bar{d}|} \quad ; \quad \bar{a}=(1,2,3),\ \bar{c}=(2,3,4),\ \bar{b}=(2,3,4),\ \bar{d}=(3,4,5)$$

$= 0$.

Hence they are coplanar.

③ Find where the S.D. between the lines $\frac{x-23}{-6}=\frac{y-19}{-4}=\frac{z-25}{3}$ and $\frac{x-12}{-9}=\frac{y-1}{4}=\frac{z-5}{2}$ meets them.



Here $\bar{b}=(-6,-4,3)$, $\bar{d}=(-9,4,2)$.

The eqn of the plane APN is given by;

$$(x-23, y-19, z-25)\cdot\bar{b}\times(\bar{b}\times\bar{d})=0$$

$\therefore\ (x-23, y-19, z-25)\cdot(-195,420,170)=0$.

a, $-39(x-23)+84(y-19)+34(z-25)=0$.

The line CN has the eqn;

$$\frac{x-12}{-9}=\frac{y-1}{4}=\frac{z-5}{2}=t.$$

The line CN meets the plane APN at N.

$\therefore\ -39(12-9t-23)+84(1+4t-19)+34(5+2t-25)=0$

a, $(39\times9+84\times4+34\times2)t = -11\times39+18\times84+20\times34$

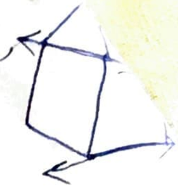
a, $\cancel{1808\times t = 1763}$ a, $755\times t = 1763$

a,

$$\bar{b}\times\bar{d} = \begin{vmatrix} i & j & k \\ -6 & -4 & 3 \\ -9 & 4 & 2 \end{vmatrix} = (20,-15,60)$$

$$\bar{b}\times(\bar{b}\times\bar{d}) = \begin{vmatrix} i & j & k \\ -6 & -4 & 3 \\ 20 & -15 & 60 \end{vmatrix} = (-195,420,170) = 5(-39,84,34)$$

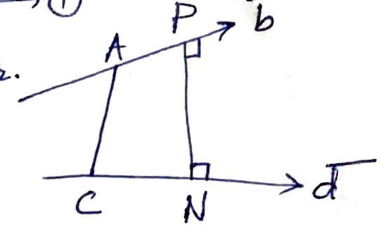
Shortest Distance (S.D.) between two skew lines



Skew lines: Lines that lie in different planes, not parallel and do not intersect.

Let the eqns of two straight lines be

$$\bar{r} = \bar{a} + t\bar{b}, \qquad \bar{r} = \bar{c} + s\bar{d} \longrightarrow (1)$$



Let PN be their common perpendicular. $\bar{b} \times \bar{d}$ is perpendicular to both and hence $\parallel$ PN.

S.D. (p) is the length of projection of CA on $\hat{n}$, the unit vector in the direction $\bar{b} \times \bar{d}$.

Hence $$p = \vec{CA} \cdot \hat{n} = (\bar{a} - \bar{c}) \cdot \frac{\bar{b} \times \bar{d}}{|\bar{b} \times \bar{d}|} \longrightarrow (2)$$

To find the equation of the line PN:

The line PN is the line of intersection of the two planes drawn through the given lines and PN.

The eqn of the plane APN containing the first line and PN is given by

$$(\bar{r} - \bar{a}) \cdot \bar{b} \times (\bar{b} \times \bar{d}) = 0 \longrightarrow (3)$$

Now the point N is that in which the second line meets this plane.

Similary, the eqn of the plane CNP containing the second line and PN is given by

$$(\bar{r} - \bar{c}) \cdot \bar{d} \times (\bar{b} \times \bar{d}) = 0 \longrightarrow (4)$$

The two planes (3) & (4) determine the line PN.

To find the condition of intersection of two lines:

The reqd. condition is given by

S.D. (p) = 0. Hence from (2), we obtain:

$$(\bar{a} - \bar{c}) \cdot \bar{b} \times \bar{d} = 0$$

or, $[\bar{b}, \bar{d}, \bar{a} - \bar{c}] = 0$. Condition of coplanarity of two lines.

Ex: find the eqn of the plane passing through the line of intersection of the planes $\bar{r}\cdot(i+2j-k)=3$ and $r\cdot(3i+2j+k)=5$ and $\perp$ to the plane $r\cdot(i+j+k)=4$.

$r\cdot\{(i+2j-k)+t(3i+2j+k)\}=3+5t$, t is a parameter.

or, $r\cdot\{(1+3t)i+2(1+t)j-(1-t)k\}=3+5t$.

$\{(1+3t)i+2(1+t)j-(1-t)k\}\cdot(i+j+k)=0$

or, $1+3t+2(1+t)-(1-t)=0$

or, $2+6t=0$ or, $t=-\frac{1}{3}$.

$\therefore$ $r\cdot\{0i+\frac{4}{3}j-\frac{4}{3}k\}=\frac{4}{3}$

$\Rightarrow$ $\vec{r}\cdot(\hat{j}-\hat{k})=1$

Ex: PQRS is a tetrahedron with $(-5,-4,8),(2,3,1),(4,1,2),(6,3,7)$ as the coordinates of P, Q, R, S resp. find the distance of the point P from the plane determined by Q, R and S.

Eqn of the plane through Q, R, S is given by

~~$(\bar{r}-\bar{q})\cdot(\bar{r}-\bar{q})\times($~~

$(\bar{r}-\bar{Q})\cdot(\bar{R}-\bar{Q})\times(\bar{S}-\bar{Q})=0,$

or, $\{\bar{r}-(2,3,1)\}\cdot\{(2,-2,1)\times(4,0,6)\}=0$.

or, $\{\bar{r}-(2,3,1)\}\cdot(-12,-8,8)=0$

or, $\bar{r}\cdot(-12,-8,8)=(2,3,1)\cdot(-12,-8,8)=-24-24+8=-40$.

or, $40+\bar{r}\cdot(-12,-8,8)=0$ $\quad$ $\bar{r}\cdot(3,2,-2)=10$

or, $40-\bar{r}\cdot(12,8,-8)=0$ $\Rightarrow$ $\frac{10}{\sqrt{17}}-(-5,-4,8)\cdot\frac{1}{\sqrt{17}}(3,2,-2)$

Distance of P $=$ ~~$|40-(-5,-4,8)\cdot(12,8,-8)|$~~

$=\frac{49}{\sqrt{17}}$. $\quad$ $=$ ~~$|40-(-60-32-64)|$~~

$=$ ~~$|196|$~~

Ex: Show that the lines L_1 & L_2 given by
$\bar{r} = 3i + j + t(2j + k)$ and $\bar{r} = 4k + s(i + j - k)$
intersect and find the eqn of the plane containing them.

Two lines intersect means they are coplanar.
The condition of coplanarity of two lines of the form $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ is

$$[\bar{a} - \bar{c}, \bar{b}, \bar{d}] = 0.$$

Here $\bar{a} = (3,1,0)$, $\bar{b} = (0,2,1)$

$\bar{c} = (0,0,4)$, $\bar{d} = (1,1,-1)$

$$\therefore [\bar{a} - \bar{c}, \bar{b}, \bar{d}] = (3,1,-4) \cdot (0,2,1) \times (1,1,-1)$$
$$= (3,1,-4) \cdot (-3,1,-2)$$
$$= -9 + 1 + 8 = 0.$$

Hence L_1 & L_2 intersect.

The plane containing L_1 & L_2 is given by

$$[\bar{r} - \bar{a}, \bar{b}, \bar{d}] = 0.$$

a, $[\bar{r}\ \bar{b}\ \bar{d}] = [\bar{a}\ \bar{b}\ \bar{d}]$

a, $\bar{r} \cdot (-3i + j - 2k) = -8.$