Some Special Integrals

L6: Error Function (Probability Integral):

The error function erf (x), also known as the probability integral, is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, du$$

The integral arises in the solution of certain differential equation and is of importance in the theory of probability.

Note: 1) Remember the Gaussian distribution function is written as

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

and it depicts a bell-shaped curve in $-\infty \; to \; +\infty$.

Further, $\mu = 0 \; and \; \sigma = 1$ for standard normal distribution change the above function as

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

Now to compute the total probability we can integrate the above function in the interval $-\infty \; to \; +\infty$ by putting $\frac{1}{2}y^2 = u^2$ as

$$\int_{-\infty}^{\infty} f(y)\, dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, du$$

This must be equal to unity, as this integration represents total probability in $-\infty \; to \; +\infty$.

Therefore, $\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$

Since e^{-u^2} is an even function we can write $\int_0^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$

That's why, the normalizing factor $\frac{2}{\sqrt{\pi}}$ arise in the definition of error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, du$$

Again, the error function (part from the factor $\frac{2}{\sqrt{\pi}}$) gives the area under the curve e^{-u^2} from $u = 0$ to $u = x$, and hence the name **probability integral**.

2) We shall see later that the Gamma function of *n* is defined as

$$\Gamma(n) = \int_0^{\infty} e^{-y}\, y^{n-1} dy, \qquad for\; n > 0$$

For *n*=1/2, we get $\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-y}\, y^{-\frac{1}{2}}\, dy$

Putting $y = u^2 \; and \; dy = 2u\, du$ we get $\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-u^2}\, du$, same integral as we got earlier.

Since the limit of the integral for error function is from $u = 0$ to $u = x$ and that for Gamma function is $u = 0$ to $u = \infty$, the error function is also known as **incomplete Gamma function**.

The following properties of error function immediately follow:

Property 1: $\mathbf{erf}(-x) = -\mathbf{erf}\,(x)$

Proof: We have $\quad \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$

$\therefore \quad \text{erf}(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-u^2}\,du$

Putting $u = -y\,,\ du = -dy$. For $u = 0, y = 0$ $and\ for\ u = -x, y = x$

Therefore, $\text{erf}(-x) = -\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy = -\text{erf}\,(x)$.

Property 2: $\mathbf{erf}(\mathbf{0}) = \mathbf{0}$

Proof: By definition $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$

$\therefore \quad \text{erf}(0) = \frac{2}{\sqrt{\pi}}\int_0^0 e^{-u^2}\,du = 0$

Alternatively, from property 1 we can write

$\text{erf}(0) = -\text{erf}\,(0)$

$2\,\text{erf}(0) = 0$

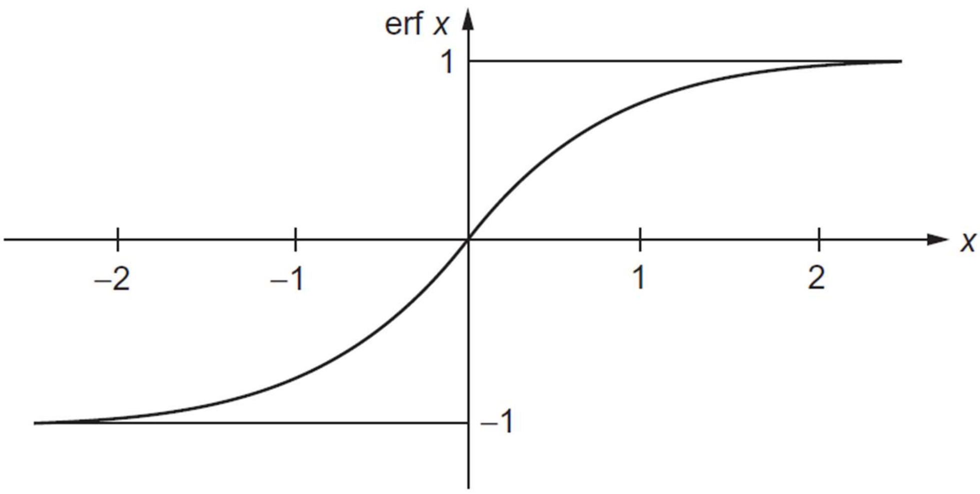
$\therefore \quad \text{erf}(0) = 0$

Property 3: $\boldsymbol{erf}\,(\infty) = \mathbf{1}$

Proof: By definition $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$

$\therefore \quad er\,f(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\,du = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1,\ \ \text{since}\ \int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$

This property shows that the total area under the normal or Gaussian distribution is unity.



Property 4: $\boldsymbol{erf(x) + erf_C(x) = 1}$

Where $erf_C(x)$ is known as complementary error function and is defined as

$$erf_C(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\ du.$$

Since the limit of the integral for error function is from $u = 0$ to $u = x$ and that for complementary error function is $u = x$ to $u = \infty$. Hence, the name 'complementary'.

Proof:

$$erf(x) + erf_C(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\ du + \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\ du = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\ du = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1$$

Series expansion of error function:

When $|x|$ is small, the error function may be expanded with help of the following series

$e^{-u^2} = 1 - \frac{u^2}{1!} + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots$ as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\ du$$

$$= \frac{2}{\sqrt{\pi}}\int_0^x \left(1 - \frac{u^2}{1!} + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots\right) du$$

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{1! \times 3} + \frac{x^5}{2! \times 5} - \frac{x^7}{3! \times 7} + \cdots\right]$$

$$\mathrm{erf}\ (x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!\,(2n+1)}$$

Gamma function (Euler integral of 2nd kind):

The Gamma function of n, written as $\Gamma(n)$,is defined as

$$\Gamma(n) = \int_0^{\infty} e^{-x}\, x^{n-1} dx, \qquad for\; n > 0$$

It follows that for n=1, $\Gamma(1) = \int_0^{\infty} e^{-x}\, dx = |-e^{-x}|_0^{\infty} = 1$

Recurrence formula of Gamma function:

As $\Gamma(n) = \int_0^{\infty} e^{-x}\, x^{n-1} dx$

$$\therefore \quad \Gamma(n+1) = \int_0^{\infty} e^{-x}\, x^{n+1-1} dx = \int_0^{\infty} e^{-x}\, x^{n} dx = |-e^{-x} x^n|_0^{\infty} + \int_0^{\infty} e^{-x}\, n x^{n-1} dx$$

$$= 0 + n \int_0^{\infty} e^{-x}\, x^{n-1} dx = n\Gamma(n)$$

Therefore, $\Gamma(n+1) = n\Gamma(n)$ -------------------- (1)

This is a very important property, also known as **reduction formula** for $\Gamma(n)$.

It follows that $\Gamma(n) = (n-1)\Gamma(n-1)$

Again equation (1) gives $\Gamma(n) = \frac{1}{n}\Gamma(n+1)$, which tends to ∞ as $n \to 0$.

As equation (1) gives $\Gamma(n+1) = n\Gamma(n)$, we can write

$$\Gamma(n+2) = (n+1)\Gamma(n+1) = (n+1)n\Gamma(n)$$

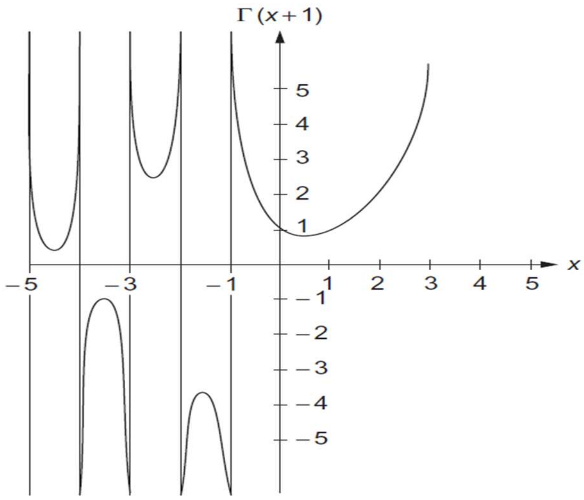
$$\Gamma(n+3) = (n+2)\Gamma(n+2) = (n+2)(n+1)\Gamma(n+1) = (n+2)(n+1)n\Gamma(n)$$

and so on.

In general, $\Gamma(n+m+1) = (n+m)(n+m-1) \ldots \ldots (n+2)(n+1)n\Gamma(n)$

Therefore, $\Gamma(n) = \frac{\Gamma(n+m+1)}{n(n+1)(n+2)\ldots\ldots(n+m-1)(n+m)}$

Clearly, at $n = 0, -1, -2, \ldots\ldots$, $\Gamma(n)$ has singularity (simple poles), i.e. $\Gamma(n)$ is defined at all the real values of n except at zero and negative integers.



As $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$

$$= n(n-1)(n-2)\Gamma(n-2) \text{ and so on}$$

$$= n(n-1)(n-2)(n-3)\ldots.3.2.1\,\Gamma(1) = n!\,\Gamma(1) = n! \quad as\,\Gamma(1) = 1$$

Therefore, $\Gamma(n+1) = n!$, provided n is positive integer.

Similarly, $\Gamma(n) = (n-1)!$

Value of $\Gamma(1/2)$:

The value of $\Gamma\left(\frac{1}{2}\right)$ is often required and may be obtained directly from the definition of Gamma function as $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}\,x^{-\frac{1}{2}}\,dx$

Putting $x = y^2$ and $dx = 2y\,dy$ we get $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\,dy = 2\int_0^\infty e^{-x^2}\,dx$

Therefore, $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\int_0^\infty e^{-y^2}\,dy \times 2\int_0^\infty e^{-x^2}\,dx = 4\int_0^\infty . \int_0^\infty e^{-(x^2+y^2)}\,dxdy$

Using polar coordinate (r,θ), so that $x = r\cos\theta\,, y = r\sin\theta$ and $dxdy = rdrd\theta,$ we obtain

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\,r\,drd\theta = 4\times\frac{\pi}{2}\int_{r=0}^{\infty} e^{-r^2} r\,dr = 2\pi\left|-\frac{1}{2}e^{-r^2}\right|_0^\infty = \pi$$

Therefore, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Further, using the relation $\Gamma(n+1) = n\Gamma(n)$, we get

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\times\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$$

Otherwise, $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$, using $\Gamma(n+1) = n\Gamma(n)$

Therefore, $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\times\frac{3}{2}\times\frac{\sqrt{\pi}}{2} = \frac{15\sqrt{\pi}}{8}$.

Gamma function for negative value of n:

As $\Gamma(n) = \frac{1}{n}\Gamma(n+1)$, therefore, $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi}$.

Similarly, $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{4}{3}\sqrt{\pi}$

and $\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = \frac{\Gamma\left(-\frac{3}{2}\right)}{-\frac{5}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = -\frac{8}{15}\sqrt{\pi}$

and $\Gamma\left(-\frac{7}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{7}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{16}{105}\sqrt{\pi}$