STUDY MATERIALS

VECTOR FIELDS

Mathematics Honours
Semester – 4
Paper – C9T Unit - 3

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CC-9 / C9T: Multivariate Calculus
Marks: 60 **Credits: 06**

Unit-I: (Functions of several variables) **Marks: 21**
(To be taught by: S. Das)

Functions of several variables, limit and continuity of functions of two or more variables,Partial differentiation, total differentiability and differentiability, sufficient condition for differentiability. Chain rule for one and two independent parameters, directional derivatives, the gradient, maximal and normal property of the gradient, tangent planes, Extrema of functions of two variables, method of Lagrange multipliers, constrained optimization problems

Unit-II: (Multivariable Integration) **Marks: 14**
(To be taught by: S. Das)

Double integration over rectangular region, double integration over non-rectangular region, Double integrals in polar co-ordinates, Triple integrals, triple integral over a parallelepiped and solid regions. Volume by triple integrals, cylindrical and spherical co-ordinates.Change of variables in double integrals and triple integrals.

Unit-III: (Vector Field and Line Integration) **Marks: 16**
(To be taught by: **S. Chakraborty**)

Definition of vector field, divergence and curl.
Line integrals, applications of line integrals: mass and work. Fundamental theorem for line integrals, conservative vector fields, independence of path.

Unit-IV: (Green's, Stoke's and Divergence Theorem) **Marks: 09**
(To be taught by S. Chakraborty)

Green's theorem, surface integrals, integrals over parametrically defined surfaces.Stoke's theorem, The Divergence theorem.

Reference Books

- G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson Education, Delhi, 2005.
- M.J. Strauss, G.L. Bradley and K. J. Smith, Calculus, 3rd Ed., Dorling Kindersley (India) Pvt. Ltd. (Pearson Education), Delhi, 2007.
- E. Marsden, A.J. Tromba and A. Weinstein, Basic Multivariable Calculus, Springer (SIE), Indian reprint, 2005.

Three field operators: $\boxed{\text{Gradient, Divergence, \& Curl.}}$

① The gradient of a scalar field,

② The divergence of a vector field, and

③ The curl of a vector field.

- **The gradient of a scalar field:—**

If $\phi(\bar{r}) = \phi(x, y, z)$ is a scalar field, i.e., a scalar function of position $\bar{r} = (x, y, z)$ in 3D, then its gradient at any point is defined in Cartesian co-ordinates by

$$\text{grad}\,\phi = \bar{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}.$$

where the vector operator $\bar{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ is called "del" or "nabla".

Note: $\bar{\nabla}\phi$ is a vector field. ∴ The gradient of a scalar field tends to point in the direction of greatest change of the field!

- **The Significance of grad:—**

If our current position is $\bar{r}$ in some scalar field ϕ, and we move an infinitesimal distance $d\bar{r}$, we have the change in ϕ is given by—

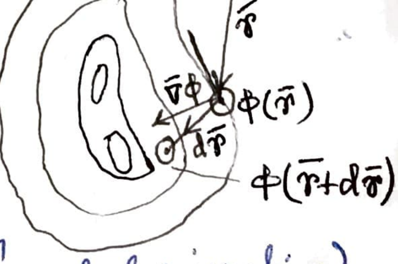
$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz.$$

Also, $d\bar{r} = (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$ and $\bar{\nabla}\phi = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$.

Then $d\phi = \bar{\nabla}\phi \cdot d\bar{r}$; dividing bothsides by ds,

$$\frac{d\phi}{ds} = \bar{\nabla}\phi \cdot \frac{d\bar{r}}{ds}$$; where we have $|d\bar{r}| = ds$, and

$\frac{d\bar{r}}{ds}$ is a unit vector in the direction $d\bar{r}$.



① $\bar{\nabla}\phi$ has the property that the rate of change of ϕ w.r.t. distance in a particular direction $(\hat{a})$ is the projection of $\bar{\nabla}\phi$ onto that direction (or component); which is $\bar{\nabla}\phi \cdot \hat{a}$. (Directional derivative).

The quantity $\frac{d\phi}{ds}$ is called a directional derivative

Note: In general, the directional derivative $\frac{d\phi}{ds}$ has a different value for each direction, and so has no meaning until we specify the direction.

Therefore, we also say that —

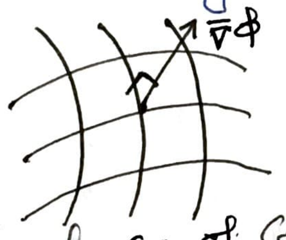
At any point P, $\bar{\nabla}\phi$ (or grad ϕ) points in the direction of greatest change of ϕ at P, and has magnitude equal to the rate of change of ϕ w.r.t. distance in that direction.

② Another property of grad ϕ:

If we consider a surface of constant ϕ, i.e., the locus of (x, y, z) for which $\phi(x,y,z) = \text{Constant}$, then if we move a small amount within that surface, there is no change in ϕ, so $\frac{d\phi}{ds} = 0$.

$\therefore$ for any $\frac{d\bar{r}}{ds}$ in that surface, $\bar{\nabla}\phi \cdot \frac{d\bar{r}}{ds} = 0$.

Since $\frac{d\bar{r}}{ds}$ is a unit tangent vector to the surface, so $\bar{\nabla}\phi$ is everywhere NORMAL to a surface of constant ϕ.



Surface of Constant ϕ : $\phi(x,y,z) = \text{Constant}$.

- The divergence of a vector field

The divergence computes a scalar quantity from a vector field by differentiation.

If $\bar{V}(x, y, z)$ is a vector function of position in 3D, i.e., $\bar{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, then its divergence at any point is defined in cartesian co-ordinates by

$$\text{div}\,\bar{V} = \bar{\nabla}\cdot\bar{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Note: the divergence of a vector field is a scalar field

GRADIENT, DIVERGENCE and CURL

The vector differential operator DEL (∇),

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

The GRADIENT: Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space.

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}.$$

$\nabla\phi$ defines a vector field whereas ϕ defines a differentiable scalar field.

Directional derivative of ϕ in the direction of a unit vector $\bar{a}$ is given by $\bar{\nabla}\phi\cdot\bar{a}$.

Physically, this is the rate of change of ϕ at (x,y,z) in the direction $\bar{a}$.

The DIVERGENCE: Let $\bar{V}(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space. i.e., $\bar{V}$ defines a differentiable vector field.

$$\bar{\nabla}\cdot\bar{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Note that $\bar{\nabla}\cdot\bar{V} \neq \bar{V}\cdot\nabla$

The CURL: If $\bar{V}(x,y,z)$ is a differentiable vector field then

$$\bar{\nabla}\times\bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

FORMULAE: If $\bar{A}$ and $\bar{B}$ are differentiable vector functions and ϕ and ψ are differentiable scalar functions of position (x,y,z), then

1. $\bar{\nabla}(\phi+\psi) = \bar{\nabla}\phi + \nabla\psi$
2. $\bar{\nabla}\cdot(\bar{A}+\bar{B}) = \bar{\nabla}\cdot\bar{A} + \bar{\nabla}\cdot\bar{B}$
3. $\bar{\nabla}\times(\bar{A}+\bar{B}) = \bar{\nabla}\times\bar{A} + \bar{\nabla}\times\bar{B}$
4. $\bar{\nabla}\cdot(\phi\bar{A}) = (\bar{\nabla}\phi)\cdot\bar{A} + \phi(\bar{\nabla}\cdot\bar{A})$
5. $\bar{\nabla}\times(\phi\bar{A}) = (\bar{\nabla}\phi)\times\bar{A} + \phi(\bar{\nabla}\times\bar{A})$

6. $\bar\nabla\cdot(\bar A\times\bar B)=\bar B\cdot(\bar\nabla\times\bar A)-\bar A\cdot(\bar\nabla\times\bar B)$

7. $\bar\nabla\times(\bar A\times\bar B)=(\bar B\cdot\bar\nabla)\bar A-\bar B(\bar\nabla\cdot\bar A)-(\bar A\cdot\bar\nabla)\bar B+\bar A(\bar\nabla\cdot\bar B)$

8. $\bar\nabla(\bar A\cdot\bar B)=(\bar B\cdot\bar\nabla)\bar A+(\bar A\cdot\bar\nabla)\bar B+\bar B\times(\bar\nabla\times\bar A)+\bar A\times(\bar\nabla\times\bar B)$

9. $\bar\nabla\cdot(\bar\nabla\phi)=\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$

where $\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is called the Laplacian Operator

10. $\bar\nabla\times(\bar\nabla\phi)=\bar 0$

11. $\bar\nabla\cdot(\bar\nabla\times\bar A)=0$

12. $\bar\nabla\times(\bar\nabla\times\bar A)=\bar\nabla(\bar\nabla\cdot\bar A)-\nabla^2 A$

Prove that $\bar\nabla(\bar A\cdot\bar B)=(\bar B\cdot\bar\nabla)\bar A+(\bar A\cdot\bar\nabla)\bar B+\bar B\times(\bar\nabla\times\bar A)+\bar A\times(\bar\nabla\times\bar B)$

Now, $\bar\nabla(\bar A\cdot\bar B)=\sum\hat i\frac{\partial}{\partial x}(\bar A\cdot\bar B)=\sum\hat i\left\{\bar A\cdot\frac{\partial\bar B}{\partial x}+\bar B\cdot\frac{\partial\bar A}{\partial x}\right\}$

$=\sum\hat i\,\bar A\cdot\frac{\partial\bar B}{\partial x}+\sum\hat i\,\bar B\cdot\frac{\partial\bar A}{\partial x}$

Now, $\bar A\times\left(\hat i\times\frac{\partial\bar B}{\partial x}\right)=\left(\bar A\cdot\frac{\partial\bar B}{\partial x}\right)\hat i-(\bar A\cdot\hat i)\frac{\partial\bar B}{\partial x}$

a, $\hat i\left(\bar A\cdot\frac{\partial\bar B}{\partial x}\right)=\bar A\times\left(\hat i\times\frac{\partial\bar B}{\partial x}\right)+(\bar A\cdot\hat i)\frac{\partial\bar B}{\partial x}$.

Similarly, $\bar B\times\left(\hat i\times\frac{\partial\bar A}{\partial x}\right)=\left(\bar B\cdot\frac{\partial\bar A}{\partial x}\right)\hat i-(\bar B\cdot\hat i)\frac{\partial\bar A}{\partial x}$

a, $\hat i\left(\bar B\cdot\frac{\partial A}{\partial x}\right)=\bar B\times\left(\hat i\times\frac{\partial\bar A}{\partial x}\right)+(\bar B\cdot\hat i)\frac{\partial\bar A}{\partial x}$.

Hence $\bar\nabla(\bar A\cdot\bar B)=\sum\bar A\times\left(\hat i\times\frac{\partial\bar B}{\partial x}\right)+\sum\bar B\times\left(\hat i\times\frac{\partial\bar A}{\partial x}\right)+\sum(\bar A\cdot\hat i)\frac{\partial\bar B}{\partial x}$

$+\sum(\bar B\cdot\hat i)\frac{\partial\bar A}{\partial x}$.

$=\bar A\times\sum\hat i\times\frac{\partial\bar B}{\partial x}+\bar B\times\sum\hat i\times\frac{\partial\bar A}{\partial x}+\sum(\bar A\cdot\hat i)\frac{\partial\bar B}{\partial x}+\sum(\bar B\cdot\hat i)\frac{\partial\bar A}{\partial x}$

$=\bar A\times(\bar\nabla\times\bar B)+\bar B\times(\bar\nabla\times\bar A)+(\bar A\cdot\bar\nabla)\bar B+(\bar B\cdot\bar\nabla)\bar A$ proved.

Prove that $\bar\nabla\cdot(\bar A\times\bar B)=\bar B\cdot(\bar\nabla\times\bar A)-\bar A\times(\bar\nabla\times\bar B)$

$\bar\nabla\cdot(\bar A\times\bar B)=\sum\hat i\cdot\frac{\partial}{\partial x}(\bar A\times\bar B)=\sum\hat i\cdot\left\{\frac{\partial\bar A}{\partial x}\times\bar B+\bar A\times\frac{\partial\bar B}{\partial x}\right\}$

$=\sum\hat i\cdot\frac{\partial\bar A}{\partial x}\times\bar B+\sum\hat i\cdot\bar A\times\frac{\partial\bar B}{\partial x}=\sum\left(\hat i\times\frac{\partial\bar A}{\partial x}\right)\cdot\bar B-\sum\left(\hat i\times\frac{\partial\bar B}{\partial x}\right)\cdot\bar A$

$=(\bar\nabla\times\bar A)\cdot\bar B-(\bar\nabla\times\bar B)\cdot\bar A=\bar B\cdot(\bar\nabla\times\bar A)-\bar A\cdot(\bar\nabla\times\bar B)$ proved.

List of some very useful results.

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$; constant vector.
f, g are scalar point functions, both continuously differentiable. $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$.

① $\bar{\nabla} f(u) = f'(u)\,\bar{\nabla} u$. $\left[\bar{\nabla} f(u) = \sum \hat{i}\frac{\partial}{\partial x} f(u) = \sum \hat{i} f'(u)\frac{\partial u}{\partial x} = f'(u)\sum \hat{i}\frac{\partial u}{\partial x} = f'(u)\bar{\nabla} u\right]$

② $\bar{\nabla}(r) = \frac{\bar{r}}{r}$. $\left[\bar{\nabla}(r) = \sum \hat{i}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \sum \hat{i}\frac{1}{2r}\frac{\partial}{\partial x}(x^2+y^2+z^2) = \frac{1}{r}\sum \hat{i}x = \frac{\bar{r}}{r}\right]$

③ $\bar{\nabla} r^n = n r^{n-2}\bar{r}$. $\left[\bar{\nabla} r^n = \frac{d}{dr}(r^n)\,\bar{\nabla} r = n r^{n-1}\frac{\bar{r}}{r} = n r^{n-2}\bar{r}\right]$

④ $\bar{\nabla}\cdot\bar{r} = 3$. $\left[\bar{\nabla}\cdot\bar{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = 1+1+1 = 3\right]$

⑤ $\bar{\nabla}\cdot\left\{\frac{f(r)}{r}\bar{r}\right\} = \frac{1}{r^2}\frac{d}{dr}(r^2 f)$.

$$\bar{\nabla}\cdot\left\{\frac{f(r)}{r}\bar{r}\right\} = \sum \frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\} = \sum\left\{\frac{f(r)}{r} - \frac{1}{r^2}\frac{\partial r}{\partial x}f(r)x + f'(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r}\right\}$$

$$= \frac{3f(r)}{r} - \sum \frac{x}{r^2}f(r)\cdot\frac{x}{r} + \sum f'(r)\frac{x}{r}\cdot\frac{x}{r}$$

$$= \frac{3f(r)}{r} - \frac{1}{r^3}f(r)(x^2+y^2+z^2) + \frac{f'(r)}{r^2}(x^2+y^2+z^2)$$

$$= \frac{3f(r)}{r} - \frac{1}{r^3}f(r)\cdot r^2 + \frac{f'(r)}{r^2}\cdot r^2 = \frac{2f(r)}{r} + f'(r)$$

$$= \frac{1}{r^2}\frac{d}{dr}\{r^2 f(r)\}.$$

⑥ $\bar{\nabla}\cdot\bar{c} = 0$. $\left[\bar{\nabla}\cdot\bar{c} = \frac{\partial}{\partial x}(c_1) + \frac{\partial}{\partial y}(c_2) + \frac{\partial}{\partial z}(c_3) = 0+0+0 = 0\right]$

⑦ $\bar{\nabla}\times\bar{r} = \bar{0}$. $\left[\bar{\nabla}\times\bar{r} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0}\right]$.

⑧ $\bar{\nabla}\times\bar{c} = \bar{0}$. $\left[\bar{\nabla}\times\bar{c} = \sum \hat{i}\times\frac{\partial}{\partial x}(\bar{c}) = \sum \hat{i}\times\frac{\partial\bar{c}}{\partial x} = \sum \hat{i}\times\bar{0} = \bar{0}\right]$

⑨ $\bar{\nabla}\cdot(\bar{r}\times\bar{c}) = 0$. $\left[\bar{\nabla}\cdot(\bar{r}\times\bar{c}) = \bar{c}\cdot(\bar{\nabla}\times\bar{r}) - \bar{r}\cdot(\bar{\nabla}\times\bar{c}) = \bar{c}\cdot\bar{0} - \bar{r}\cdot\bar{0} = 0\right]$

⑩ $\bar{\nabla}\times(\bar{r}\times\bar{c}) = -2\bar{c}$.

$$\bar{\nabla}\times(\bar{r}\times\bar{c}) = (\bar{c}\cdot\bar{\nabla})\bar{r} - \bar{c}(\bar{\nabla}\cdot\bar{r}) - (\bar{r}\cdot\bar{\nabla})\bar{c} + \bar{r}(\bar{\nabla}\cdot\bar{c})$$

$$= \sum c_1\frac{\partial}{\partial x}(x\hat{i}) - 3\bar{c} - \sum x\frac{\partial}{\partial x}(c_1\hat{i}) + \bar{r}0$$

$$= \sum \hat{i}c_1 - 3\bar{c} - 0 + 0 = \bar{c} - 3\bar{c}$$

$$= -2\bar{c}$$

(11) $\nabla^2\left(\frac{1}{r}\right) = 0.$

$$\left[\nabla^2\left(\frac{1}{r}\right) = \bar{\nabla}\cdot\bar{\nabla}\left(\frac{1}{r}\right) = \bar{\nabla}\cdot\left\{\frac{d}{dr}\left(\frac{1}{r}\right)\bar{\nabla}r\right\} = \bar{\nabla}\cdot\left\{-\frac{1}{r^2}\frac{\bar{r}}{r}\right\}\right.$$

$$= \bar{\nabla}\cdot\left(-\frac{1}{r^3}\bar{r}\right) = \bar{\nabla}\left(-\frac{1}{r^3}\right)\cdot\bar{r} + \left(-\frac{1}{r^3}\right)(\bar{\nabla}\cdot\bar{r})$$

$$= \frac{d}{dr}\left(-\frac{1}{r^3}\right)(\bar{\nabla}r\cdot\bar{r}) - \frac{3}{r^3}.$$

$$= \frac{3}{r^4}\sum\frac{\partial r}{\partial x}x - \frac{3}{r^3} = \frac{3}{r^4}\sum\frac{x}{r}\cdot x - \frac{3}{r^3}$$

$$= \frac{3}{r^4}\sum\frac{x^2}{r} - \frac{3}{r^3} = \frac{3}{r^5}(x^2+y^2+z^2) - \frac{3}{r^3}$$

$$\left.= \frac{3}{r^5}\times r^2 - \frac{3}{r^3} = 0.\right]$$

(12) $\nabla^2(r^n\bar{r}) = n(n+3)r^{n-2}\bar{r}.$

$$\nabla^2(r^n\bar{r}) = \nabla\left\{\bar{\nabla}\cdot(r^n\bar{r})\right\}$$

$$= \nabla\left\{(\bar{\nabla}r^n)\cdot\bar{r} + r^n(\bar{\nabla}\cdot\bar{r})\right\}$$

$$= \nabla\left\{nr^{n-2}\bar{r}\cdot\bar{r} + r^n 3\right\}$$

$$= \nabla\left\{nr^{n-2}r^2 + 3r^n\right\}$$

$$= (n+3)\nabla(r^n) = \underline{(n+3)\,nr^{n-2}\bar{r}}.$$

Ex (3)(c) Prove that $\text{curl}\left\{\frac{\bar{a}\times\bar{r}}{r^3}\right\} = -\frac{\bar{a}}{r^3} + \frac{3}{r^5}(\bar{a}\cdot\bar{r})\bar{r}$; where $\bar{a}$ is a constant vector.

$$\text{curl}\left\{\frac{\bar{a}\times\bar{r}}{r^3}\right\} = \sum\left\{\hat{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)\right\}$$

Now, $\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{3}{r^4}\frac{\partial r}{\partial x}(\bar{a}\times\bar{r}) + \frac{1}{r^3}\left(\bar{a}\times\frac{\partial\bar{r}}{\partial x}\right) + \frac{1}{r^3}\left(\frac{\partial\bar{a}}{\partial x}\times\bar{r}\right)$

$$= -\frac{3}{r^4}\left(\frac{x}{r}\right)(\bar{a}\times\bar{r}) + \frac{1}{r^3}(\bar{a}\times\hat{i}) + \frac{1}{r^3}(\bar{0}\times\bar{r})$$

$$= -\frac{3}{r^4}\frac{x}{r}(\bar{a}\times\bar{r}) + \frac{1}{r^3}(\bar{a}\times\hat{i}) + \bar{0}$$

$$\therefore\ \hat{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = -\frac{3x}{r^5}\left[\hat{i}\times(\bar{a}\times\bar{r})\right] + \frac{1}{r^3}\left[\hat{i}\times(\bar{a}\times\hat{i})\right].$$

$$= -\frac{3x}{r^5}\left[(\hat{i}\cdot\bar{r})\bar{a} - (\hat{i}\cdot\bar{a})\bar{r}\right] + \frac{1}{r^3}\left[(\hat{i}\cdot\hat{i})\bar{a} - (\hat{i}\cdot\bar{a})\hat{i}\right]$$

$$= -\frac{3x}{r^5}x\bar{a} + \frac{3x}{r^5}a_1\bar{r} + \frac{1}{r^3}\bar{a} - \frac{1}{r^3}a_1\hat{i}$$

$$\sum\left\{\hat{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)\right\} = \left(-\frac{3}{r^5}\sum x^2\right)\bar{a} + \left(\frac{3}{r^5}\sum a_1 x\right)\bar{r} + \frac{3}{r^3}\bar{a} - \frac{1}{r^3}\bar{a}.$$

$$= -\frac{3}{r^5}r^2\bar{a} + \frac{3}{r^5}(\bar{r}\cdot\bar{a})\bar{r} + \frac{2}{r^3}\bar{a}$$

$$\therefore\ \text{curl}\left\{\frac{\bar{a}\times\bar{r}}{r^3}\right\} = -\frac{\bar{a}}{r^3} + \frac{3}{r^5}(\bar{a}\cdot\bar{r})\bar{r}.$$

Prove: $\bar{\nabla}\times(\bar{\nabla}\times\bar{A}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A}$

Let $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, where A_1, A_2, A_3 are real-valued differentiable functions of real variables x, y, z.

$$\bar{\nabla}\times\bar{A} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{A}) = \left[\frac{\partial}{\partial y}\left\{\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right\} - \frac{\partial}{\partial z}\left\{\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right\}\right]\hat{i} + \left[\frac{\partial}{\partial z}\left\{\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right\} - \frac{\partial}{\partial x}\left\{\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right\}\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}\left\{\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right\} - \frac{\partial}{\partial y}\left\{\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right\}\right]\hat{k}$$

$$= \sum\left[\hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right\}\right]$$

$$= \sum\left[\hat{i}\left(\frac{\partial^2 A_2}{\partial y\partial x} + \frac{\partial^2 A_3}{\partial z\partial x}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}\Bigg]$$

$$= \sum\hat{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}$$

$$= \sum\hat{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}$$

$$= \sum\hat{i}\left\{\frac{\partial}{\partial x}(\bar{\nabla}\cdot\bar{A}) - \nabla^2 A_1\right\} = \sum\hat{i}\left\{\frac{\partial}{\partial x}(\bar{\nabla}\cdot\bar{A})\right\} - \nabla^2\sum A_1\hat{i}$$

$$= \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A} = \text{R.H.S. proved.}$$

Prob: Show that $\bar{\nabla}\phi$ is a vector normal to the surface $\phi(x, y, z) = \text{constant}$.

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the p.v. to any point $P(x, y, z)$ on the surface.

Then $d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ lies on the tangent plane to the surface at P.

But $d\phi = 0 = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$\Rightarrow \bar{\nabla}\phi\cdot d\bar{r} = 0$.

$\therefore$ $\bar{\nabla}\phi$ is $\perp^r$ to $d\bar{r}$ and therefore to the surface $\phi(x,y,z) = c$

① Find the unit vector $\perp^r$ to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point $(1, 2, 5)$.

Ans: $\dfrac{2\hat{i} + 4\hat{j} - \hat{k}}{\pm\sqrt{21}}$. find: $\dfrac{\nabla\phi}{|\nabla\phi|}$ at $(1, 2, 5)$

② Find equations for the tangent plane and normal line to the surface $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$

$\phi = xz^2 + x^2y - z = -1$ (constant)

Tangent plane: $(\bar{r} - \bar{r}_0)\cdot\bar{N} = 0$; $\bar{N} = \bar{\nabla}\phi$ at $(1, -3, 2)$ $= (z^2 + 2xy,\ x^2,\ 2xz - 1)\Big|_{(1,-3,2)}$

Normal line: $(\bar{r} - \bar{r}_0)\times\bar{N} = \bar{0}$; $= (-2, 1, 3)$.

$r_0 = (1, -3, 2)$

Ans: Tangent plane: $2x - y - 3z + 1 = 0$. $\Leftarrow \{(x, y, z) - (1, -3, 2)\}\cdot(-2, 1, 3)$

Normal line: $\dfrac{x - 1}{-2} = \dfrac{y + 3}{1} = \dfrac{z - 2}{3}$

(8) Let $\phi(x,y,z)$ and $\phi(x+\Delta x, y+\Delta y, z+\Delta z)$ be the temperatures at two nbd. points $P(x,y,z)$ and $Q(x+\Delta x, y+\Delta y, z+\Delta z)$ of a certain region.

(a) Interpret physically the quantity:

$$\frac{\Delta\phi}{\Delta s} = \frac{\phi(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(x,y,z)}{\Delta s}, \text{ where } \Delta s = PQ.$$

(b) Evaluate: $\lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s} = \frac{d\phi}{ds}$ and interpret physically

(c) Show that $\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\bar{r}}{ds}$.

(a) Since $\Delta\phi$ is the change in temperature between P & Q and Δs is the distance between these points, $\frac{\Delta\phi}{\Delta s}$ represents the average rate of change in temp. per unit distance in the direction from P to Q.

(b) From the calculus,

$$\Delta\phi = \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z + \text{infinitesimal of order higher than } \Delta x, \Delta y, \Delta z.$$

Then $\lim_{\Delta s\to 0} \frac{\Delta\phi}{\Delta s} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} = \frac{d\phi}{ds}$.

(c) $\boxed{\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\bar{r}}{ds}}$ $\longrightarrow$ Directional derivative

found $\longrightarrow$ The rate of change of ϕ at P.

$\frac{d\bar{r}}{ds}$ is a unit vector. $\frac{d\phi}{ds}$ is called the directional derivative of ϕ.

Directional derivative: $\nabla\phi \cdot \frac{d\bar{r}}{ds}$ is the component of $\bar{\nabla}\phi$ in the direction of the unit vector $\frac{d\bar{r}}{ds}$.

Show that the maximum directional derivative, takes place in the direction of and has the magnitude of the vector $\bar{\nabla}\phi$.

Now, $\frac{d\phi}{ds} = \bar{\nabla}\phi \cdot \frac{d\bar{r}}{ds}$ is the projection of $\bar{\nabla}\phi$ in the direction $\frac{d\bar{r}}{ds}$. This will be maximum when $\bar{\nabla}\phi$ & $\frac{d\bar{r}}{ds}$ have the same direction. Then the max. value of $\frac{d\phi}{ds}$ takes place in the direction of $\bar{\nabla}\phi$ and its magnitude is $|\bar{\nabla}\phi|$.

Eg. (62) Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$.

$\nabla\phi \cdot \hat{a}$, where $\hat{a} = \frac{2\hat{i}-3\hat{j}+6\hat{k}}{|2\hat{i}-3\hat{j}+6\hat{k}|} = \frac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\hat{i}-3\hat{j}+6\hat{k})$

$\nabla\phi\big|_{(2,-1,2)} = (4z^3 - 6xy^2z, -6x^2yz, 12xz^2 - 3x^2y^2)\big|_{(2,-1,2)} = 4(2,12,21)$

Directional derivative $= \nabla\phi\cdot\hat{a} = \frac{4}{7}\times(4-36+126) = \frac{4\times 94}{7} = \frac{376}{7}$.

(63) Find the directional derivative of $\phi = 4e^{2x-y+z}$ at the point $(1,1,-1)$ in a direction toward the point $(-3,5,6)$.

$\bar{\nabla}\phi = \bar{\nabla}(4e^{2x-y+z}) = 4e^{2x-y+z}(2\hat{i} - \hat{j} + \hat{k})$

$\bar{\nabla}\phi$ at $(1,1,-1) = 4(2\hat{i} - \hat{j} + \hat{k})$.

Direction is $(-3-1)\hat{i} + (5-1)\hat{j} + (6+1)\hat{k} = -4\hat{i} + 4\hat{j} + 7\hat{k}$

Unit vector in this direction $= \dfrac{-4\hat{i} + 4\hat{j} + 7\hat{k}}{\sqrt{16+16+49}}$

$= \dfrac{-4\hat{i} + 4\hat{j} + 7\hat{k}}{9}$

$\therefore$ Directional derivative $= 4(2\hat{i} - \hat{j} + \hat{k}) \cdot \frac{1}{9}(-4\hat{i} + 4\hat{j} + 7\hat{k})$

$= \frac{4}{9}(-8 - 4 + 7) = \frac{-20}{9}$. (Ans).

(64) In what direction from the point $(1,3,2)$ is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of this maximum?

The directional derivative is maximum when it is in $\bar{\nabla}\phi$ direction and has the max. value $|\bar{\nabla}\phi|$

Now, $\bar{\nabla}\phi = 2z\hat{i} - 2y\hat{j} + 2x\hat{k}$

$\bar{\nabla}\phi$ at $(1,3,2) = 4\hat{i} - 6\hat{j} + 2\hat{k}$ in this direction it is max.

And the max. value is $|\bar{\nabla}\phi| = \sqrt{4^2 + (-6)^2 + 2^2} = 2\sqrt{14}$.

(65) Find the values of the constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis.

$\bar{\nabla}\phi = (ay^2 + 3z^2x^2c)\hat{i} + (2axy + bz)\hat{j} + (by + 2cz)\hat{k}$.

$\bar{\nabla}\phi\Big|_{(1,2,-1)} = (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}$

Maximum directional derivative at $(1,2,-1)$ in a direction parallel to z-axis is given by

$\bar{\nabla}\phi \cdot \hat{k} = 2b - 2c = 64$ (given) $\longrightarrow$ (1)

Now, $|\bar{\nabla}\phi| = 64$, or, $|\bar{\nabla}\phi|^2 = (64)^2$. [Maximum magnitude of D.D.]

or, $(4a + 3c)^2 + (4a - b)^2 + (2b - 2c)^2 = (64)^2 \longrightarrow$ (2).

or, $(4a + 3c)^2 + (4a - b)^2 + (64)^2 = (64)^2$ [From (1)]

or, $(4a + 3c)^2 + (4a - b)^2 = 0$

$\Rightarrow$ $4a + 3c = 0$, $4a - b = 0$; Also $b - c = 32$ [From (1)]

Solving these 3 equations, we get

$a = 6$, $b = 24$, $c = -8$. (Ans.)

(67) Find the constants a, b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1, -1, 2)$. Ans. $a = 5/2, b = 1$.

$$\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1||\nabla\phi_2| \cos 90^\circ = 0.$$

!

(74) Prove $\nabla^2 r^n = n(n+1) r^{n-2}$, n is a constant.

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$.

$$\nabla^2 r^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x^2+y^2+z^2)^{n/2}$$

$$= \sum \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{n/2} = \sum \frac{\partial}{\partial x}\left\{\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \times 2x\right\}$$

$$= \sum \frac{\partial}{\partial x}\left\{n x \times (x^2+y^2+z^2)^{\frac{n}{2}-1}\right\} = \sum\left\{n(x^2+y^2+z^2)^{\frac{n}{2}-1} + n\left(\frac{n}{2}-1\right)x \cdot 2x (x^2+y^2+z^2)^{\frac{n}{2}-2}\right\}$$

$$= \sum\left[n(x^2+y^2+z^2)^{n/2-1} + n(n-2)x^2(x^2+y^2+z^2)^{n/2-2}\right]$$

$$= \sum n(x^2+y^2+z^2)^{n/2-1} + \sum n(n-2)x^2(x^2+y^2+z^2)^{n/2-2}$$

$$= 3n(x^2+y^2+z^2)^{\frac{n-2}{2}} + n(n-2)(x^2+y^2+z^2)^{\frac{n-4}{2}}(x^2+y^2+z^2)$$

$$= 3n(x^2+y^2+z^2)^{\frac{n-2}{2}} + n(n-2)(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}[3 + (n-2)] = n(n+1) r^{n-2}.$$

Proved

(83) (a) Prove $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$. (b) find $f(r)$ s.t. $\nabla^2 f(r) = 0$.

(a)
$$\nabla^2 f(r) = \sum\left[\frac{\partial^2}{\partial x^2} f(r)\right] = \sum \frac{\partial}{\partial x}\left(\frac{df}{dr}\cdot\frac{\partial r}{\partial x}\right) = \sum \frac{\partial}{\partial x}\left\{\frac{df}{dr}\cdot x(x^2+y^2+z^2)^{-1/2}\right\}$$

$$= \sum \frac{\partial}{\partial x}\left\{\frac{df}{dr}\cdot\left(\frac{x}{r}\right)\right\} = \sum\left\{\frac{d^2 f}{dr^2}\cdot\frac{\partial r}{\partial x}\cdot\left(\frac{x}{r}\right) + \frac{df}{dr}\cdot\frac{1}{r} + x\frac{df}{dr}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right\}$$

$$= \sum\left\{\frac{d^2 f}{dr^2}\cdot\left(\frac{x}{r}\right)^2 + \frac{df}{dr}\cdot\frac{1}{r} - \frac{x^2}{r^3}\frac{df}{dr}\right\}$$

$$= \frac{d^2 f}{dr^2}\cdot\frac{1}{r^2}(x^2+y^2+z^2) + \frac{3}{r}\frac{df}{dr} - \frac{df}{dr}\cdot\frac{1}{r^3}(x^2+y^2+z^2)$$

$$= \frac{d^2 f}{dr^2} + \frac{3}{r}\frac{df}{dr} - \frac{1}{r}\frac{df}{dr} = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}.$$ Proved.

$r = (x^2+y^2+z^2)^{1/2}$

$\frac{\partial r}{\partial x} = x(x^2+y^2+z^2)^{-1/2} = \frac{x}{r}$.

(b)
$$\nabla^2 f(r) = 0$$

$$\Rightarrow \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0 \Rightarrow r^2\frac{d^2 f}{dr^2} + 2r\frac{df}{dr} = 0.$$

$$\Rightarrow \frac{d^2 f}{dt^2} - \frac{df}{dt} + 2\frac{df}{dt} = 0 \Rightarrow \frac{d^2 f}{dt^2} + \frac{df}{dt} = 0.$$

Let $f = ce^{mt}$ be the trial soln.

$m^2 + m = 0 \Rightarrow m(m+1) = 0$
$\Rightarrow m = 0, m = -1.$

$f(r) = A + Be^{-t} = A + \frac{B}{r}$, A, B are arbitrary constants.

Let $r = e^t$

$t = \log r$

$dt = \frac{1}{r}dr$.

$\frac{df}{dr} = \frac{df}{dt}\cdot\frac{dt}{dr} = \frac{1}{r}\frac{df}{dt}$.

102: Show that $\bar{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find ϕ s.t. $\bar{A} = \nabla\phi$. Ans: $\phi = 3x^2y + xz^3 - yz +$ Constant

104. If $\bar{A}$ and $\bar{B}$ are irrotational, prove that $\bar{A}\times\bar{B}$ is solenoidal.

Problem ① Show that if $\phi(x, y, z)$ is any solution of Laplace's eqn, then $\bar{\nabla}\phi$ is a vector which is both solenoidal and irrotational.

Problem ② Determine the constant a so that $\bar{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal. (Ans; $a=-2$)

Solution of ①: If $\phi(x,y,z)$ be any solution of Laplace's equation, then $\nabla^2\phi = 0$

$$\Rightarrow \bar{\nabla}\cdot\bar{\nabla}\phi = 0 \Rightarrow \bar{\nabla}\phi \text{ is solenoidal.}$$

For a vector $\bar{A}$, Condition of solenoidal $\bar{\nabla}\cdot\bar{A} = 0$. Condition of irrotational $\bar{\nabla}\times\bar{A} = 0$

Now
$$\bar{\nabla}\times\bar{\nabla}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)\hat{i} + \left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right)\hat{j} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)\hat{k}.$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} \quad \left[\text{Since } \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y}, \ldots \text{etc}\right]$$

$$= \bar{0}$$

[ϕ has continuous 2nd order partial derivatives.]

$\therefore \bar{\nabla}\times\bar{\nabla}\phi = \bar{0} \Rightarrow \bar{\nabla}\phi$ is irrotational.

$\therefore \bar{\nabla}\phi$ is both solenoidal & irrotational.

Solution of ②: If $\bar{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ be solenoidal, then $\bar{\nabla}\cdot\bar{V} = 0$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$\Rightarrow 1+1+a = 0 \Rightarrow \underline{a = -2}.$$

Solution of (10②) (from previous page);

Given $\bar{A} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$

To show $\bar{A}$ is irrotational, we calculate

$$\bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (6xy+z^3) & (3x^2-z) & (3xz^2-y) \end{vmatrix} = (-1+1)\hat{i} + (3z^2-3z^2)\hat{j} + (6x-6x)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \bar{0}$$

$\Rightarrow \bar{A}$ is irrotational.

Now to find ϕ s.t. $\bar{A} = \bar{\nabla}\phi$.

i.e., $\frac{\partial \phi}{\partial x} = 6xy + z^3 \to (1)$, $\frac{\partial \phi}{\partial y} = 3x^2 - z \to (2)$, $\frac{\partial \phi}{\partial z} = 3xz^2 - y \to (3)$

Integrating partially

(1) w.r.t. x, keeping y & z fixed; $\phi = 3x^2y + xz^3 + f_1(y,z)$.

(2) " y, " x & z " ; $\phi = 3x^2y - zy + f_2(x,z)$.

(3) " z, " x & y " ; $\phi = xz^3 - zy + f_3(x,y)$.

where f_1, f_2, f_3 are arbitrary functions.

If we choose $f_1(y,z) = -zy + c_1$

$f_2(x,z) = xz^3 + c_2$

$f_3(x,y) = 3x^2y + c_3$.

Then $\phi = 3x^2y + xz^3 - yz + c$, [where $c = c_1 + c_2 + c_3$], c being an arbitrary constant.

Solution of (104): (from previous page).

If $\bar{A}$ & $\bar{B}$ are irrotational then $\bar{A} \times \bar{B}$ is solenoidal — prove it.

$\bar{\nabla} \times \bar{A} = \bar{0} = \bar{\nabla} \times \bar{B}$. [Condition for irrotational]

Now $\bar{\nabla} \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\bar{\nabla} \times \bar{A}) - \bar{A} \cdot (\bar{\nabla} \times \bar{B})$

$= \bar{B} \cdot \bar{0} - \bar{A} \cdot \bar{0} = 0 - 0 = 0$.

$\Rightarrow \bar{A} \times \bar{B}$ is solenoidal.

Solution of (67): [from previous page].

Let the surface $\phi_1(x,y,z) = ax^2 - byz - (a+2)x = 0$.

& " " $\phi_2(x,y,z) = 4x^2y + z^3 = 4$

Normal to the surface ϕ_1 at $(1,-1,2) = \bar{\nabla}\phi_1\Big|_{(1,-1,2)}$

$= \left[2ax - (a+2), -bz, -by\right]_{(1,-1,2)}$

$= (a-2, -2b, b)$.

Normal to ϕ_2 at $(1,-1,2) = \nabla\phi_2\Big|_{(1,-1,2)} = (8xy, 4x^2, 3z^2)_{(1,-1,2)}$

$= (-8, 4, 12)$.

Now, $\bar{\nabla}\phi_1 \cdot \bar{\nabla}\phi_2 = 0$ [orthogonality condition]

$\Rightarrow -8(a-2) + 4(-2b) + 12b = 0$

$\Rightarrow 8a - 4b = 16 \Rightarrow 2a - b = 4 \to (1)$.