# STUDY MATERIALS

# VECTOR FIELDS

Mathematics Honours
Semester – 4
Paper – C9T Unit - 3

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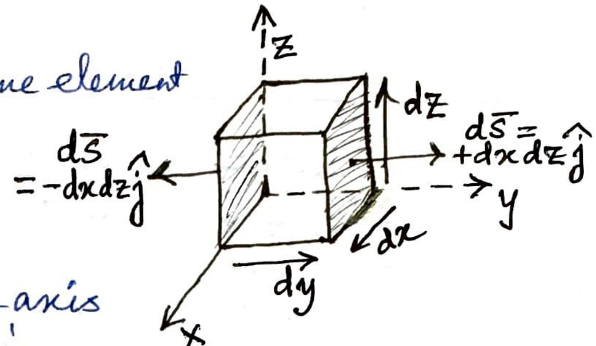
- **The significance of divergence (or, div):-**

Let us consider a typical vector field, water flow, and let it be denoted by $\bar{a}(\bar{r})$. It has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of $\bar{a}$ per unit time. Let us take an infinitesimal volume element $dV$ and calculate the balance of the flow of $\bar{a}$ in and out of $dV$.




Let us consider the volume element $dV = dxdydz$.
First, let us consider: the face of area $dxdz$ perpendicular to the y-axis and facing outwards in the (-ve) y-direction, i.e., $d\bar{s} = -dxdz\hat{j}$.

The component of the vector $\bar{a}$ normal to $d\bar{s}(= -dxdz\hat{j})$ is $\bar{a}\cdot\hat{j} = a_y$ (say), and is pointing inwards. So the outward flux from this surface is

$$\bar{a}\cdot d\bar{s} = -a_y(x,y,z)\,dz\,dx \longrightarrow ① \quad \text{[flux means mass/time]}$$

From the opposite face, i.e., $d\bar{s} = +dxdz\hat{j}$, the outward flux amount will be

$$a_y(x, y+dy, z)\,dz\,dx = \left(a_y + \frac{\partial a_y}{\partial y}dy\right)dxdz \quad \text{[by Taylor's theorem]} \longrightarrow ②$$

$\therefore$ The total outward amount from these two faces is

$$-a_y\,dzdx + \left(a_y + \frac{\partial a_y}{\partial y}dy\right)dzdx$$
$$= \frac{\partial a_y}{\partial y}dxdydz = \frac{\partial a_y}{\partial y}dV \longrightarrow ③$$

Summing the other faces gives a total outward flux of

$$\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)dV = (\bar{\nabla}\cdot\bar{a})\,dV \longrightarrow ④$$

Therefore,
The divergence of a vector field represents the flux generation per unit volume at each point of the field.

Note: ① Divergence because it is an efflux not an influx.
② the total efflux from the infinitesimal volume is equal to the flux integrated over the surface of the volume.

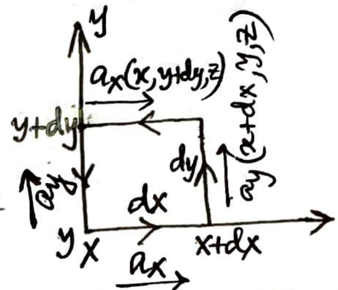
- The curl of a vector field

$$\bar{\nabla} \times \bar{a} \equiv \text{curl}(\bar{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix} = \sum \hat{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right).$$

The circulation of a vector $\bar{a}$ around any closed curve $C$ is defined to be $\oint_C \bar{a}\cdot d\bar{r}$.

The curl of the vector field $\bar{a}$ represents the vorticity, or, circulation per unit area, of the field.

Let us consider the small rectangular element $dx\,dy$, and the circulation around the perimeter of a rectangular element.




The fields in the x-direction at the bottom & top are:

$$a_x(x,y,z) \quad \text{and} \quad a_x(x,y+dy,z) = a_x(x,y,z) + \frac{\partial a_x}{\partial y}dy. \quad \text{--- (1)}$$

And the fields in the y-direction at the left and right are:

$$a_y(x,y,z) \quad \text{and} \quad a_y(x+dx,y,z) = a_y(x,y,z) + \frac{\partial a_y}{\partial x}dx. \quad \text{--- (2)}$$

The total circulation $dc$ starting from the bottom and working round in the anticlockwise sense, we have:

$$dc = +[a_x dx] + [a_y(x+dx,y,z)dy] - [a_x(x,y+dy,z)dx] - [a_y dy]$$

$$= [a_x dx] + \left[\left(a_y + \frac{\partial a_y}{\partial x}dx\right)dy\right] - \left[\left(a_x + \frac{\partial a_x}{\partial y}dy\right)dx\right] - [a_y dy]$$

$$= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)dx\,dy = (\bar{\nabla}\times\bar{a})\cdot d\bar{s}, \text{ where}$$

$$d\bar{s} = dx\,dy\,\hat{k}.$$

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Some Important definitions:

1. A vector field with zero divergence is called Solenoidal.
2. " " " " " curl " " irrotational.
3. " Scalar " " " gradient " " Constant.

74. Prove that $\nabla^2 r^n = n(n+1)r^{n-2}$, $r$ is a constant.

$$\nabla^2 r^n = \bar{\nabla}\cdot\bar{\nabla} r^n = \bar{\nabla}\cdot\left\{\frac{d}{dr}(r^n)\,\bar{\nabla} r\right\}$$
$$= \bar{\nabla}\cdot\left(n r^{n-1}\frac{\bar{r}}{r}\right) = \bar{\nabla}\cdot\left(n r^{n-2}\bar{r}\right)$$
$$= \bar{\nabla}(n r^{n-2})\cdot\bar{r} + n r^{n-2}(\bar{\nabla}\cdot\bar{r})$$
$$= n\left\{\frac{d}{dr}(r^{n-2})\,\bar{\nabla} r\right\}\cdot\bar{r} + n r^{n-2}\,3.$$
$$= n(n-2)r^{n-3}\frac{\bar{r}}{r}\cdot\bar{r} + 3n r^{n-2}$$
$$= n(n-2)r^{n-4}(\bar{r}\cdot\bar{r}) + 3n r^{n-2}$$
$$= n(n-2)r^{n-4}r^2 + 3n r^{n-2}$$
$$= n(n-2)r^{n-2} + 3n r^{n-2}$$
$$= n r^{n-2}(n-2+3) = n(n+1)r^{n-2}.$$

$\therefore\ \nabla^2 r^n = n(n+1)r^{n-2}$ (proved).

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67. Find the constants $a, b$ so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$.

Let the surface $\phi_1(x,y,z) = ax^2 - byz - (a+2)x = 0$,

& " " " $\phi_2(x,y,z) = 4x^2y + z^3 = 4$.

Normal vector at $(1,-1,2)$ to $\phi_1 = \bar{\nabla}\phi_1\Big]_{(1,-1,2)}$

and " " " " to $\phi_2 = \bar{\nabla}\phi_2\Big]_{(1,-1,2)}$.

Now $\bar{\nabla}\phi_1\Big]_{(1,-1,2)} = (2ax - a - 2,\ -bz,\ -by)\Big]_{(1,-1,2)}$
$$= (a-2,\ -2b,\ b)$$

$\bar{\nabla}\phi_2\Big]_{(1,-1,2)} = (8xy,\ 4x^2,\ 3z^2)\Big]_{(1,-1,2)} = (-8, 4, 12).$

If $\phi_1 = 0$, $\phi_2 = 4$ be orthogonal, then

$\bar{\nabla}\phi_1\cdot\bar{\nabla}\phi_2 = 0 \Rightarrow (a-2)(-8) - 2b\cdot 4 + b\cdot 12 = 0$
$$\Rightarrow -8a - 8b + 12b + 16 = 0$$
$$\Rightarrow 2a - b = 4 \longrightarrow (1).$$

Again the surface $\phi_1(x,y,z)=0$ passes through the point $(1,-1,2)$, i.e.,

$$a+2b-(a+2)=0 \Rightarrow 2b=2 \Rightarrow \underline{b=1} \rightarrow ②.$$

Using ② in ①, we obtain

$$2a=b+4=1+4 \Rightarrow \underline{a=5/2}.$$

(61) Find equations for the tangent plane and normal line to the surface $z=x^2+y^2$ at the point $(2,-1,5)$.

Let the surface be $\phi(x,y,z)\equiv x^2+y^2-z=0$.

Normal vector $\bar{N}$ to $\phi=0$ at $(2,-1,5)$ is

$$\bar{N}=\bar{\nabla}\phi\Big|_{(2,-1,5)}=(2x,2y,-1)\Big|_{(2,-1,5)}=(4,-2,-1).$$

Eqn. of the tangent plane at $(2,-1,5)$ is

$(\bar{r}-\bar{r}_0)\cdot\bar{N}=0$, where $\bar{r}=(x,y,z)$ is arbitrary point on the tangent plane, and $\bar{r}_0=(2,-1,5)$.

$$\Rightarrow \{(x,y,z)-(2,-1,5)\}\cdot(4,-2,-1)=0$$

$$\Rightarrow 4(x-2)-2(y+1)-1(z-5)=0$$

$$\Rightarrow \underline{4x-2y-z=8+2-5=5}.$$

Eqn. of the normal line at $(2,-1,5)$ is

$(\bar{r}-\bar{r}_0)\times\bar{N}=\bar{0}$, where $\bar{r}=(x,y,z)$ is any point on the normal line and $\bar{r}_0=(2,-1,5)$.

$$\Rightarrow (x-2,\ y+1,\ z-5)\times(4,-2,-1)=\bar{0}$$

$$\Rightarrow \underline{\frac{x-2}{4}=\frac{y+1}{-2}=\frac{z-5}{-1}}.\ \text{OR},\ \underline{x=2+4t,\ y=-1-2t,\ z=5-t}$$

($t$ is a parameter)

76. If $\bar{\omega}$ is a constant vector and $\bar{v}=\bar{\omega}\times\bar{r}$, prove that $\operatorname{div}\bar{v}=0$.

$$\operatorname{div}\bar{v}=\bar{\nabla}\cdot(\bar{\omega}\times\bar{r})=\bar{r}\cdot(\bar{\nabla}\times\bar{\omega})-\bar{\omega}\cdot(\bar{\nabla}\times\bar{r})$$

$$=\bar{r}\cdot\bar{0}-\bar{\omega}\cdot\bar{0}=0-0=0.$$

Ex (105) If $f(r)$ is differentiable, prove that $f(r)\bar{r}$ is irrotational

$$\bar{\nabla}\times\{f(r)\bar{r}\} = \bar{\nabla}f(r)\times\bar{r} + f(r)\bar{\nabla}\times\bar{r} \quad [\because \bar{\nabla}\times(\phi\bar{A}) = \bar{\nabla}\phi\times\bar{A}+\phi(\bar{\nabla}\times\bar{A})]$$

$$=\{f'(r)\bar{\nabla}r\}\times\bar{r} + f(r)\bar{0} \quad [\because \bar{\nabla}f(u) = f'(u)\bar{\nabla}u]$$

$$=\left\{f'(r)\frac{\bar{r}}{r}\right\}\times\bar{r} + \bar{0} \quad \left[\because \bar{\nabla}r = \sum\hat{i}\frac{\partial r}{\partial x} = \sum\hat{i}\frac{x}{r} = \frac{\bar{r}}{r}\right]$$

$$= \frac{f'(r)}{r}(\bar{r}\times\bar{r}) + \bar{0}$$

$$= \frac{f'(r)}{r}\bar{0} + \bar{0}$$

$$= \bar{0} + \bar{0}$$

$$= \bar{0}$$

$\therefore$ $f(r)\bar{r}$ is irrotational. proved.

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Maity & Ghosh

(16) Find the angle of intersection at the point $(2,-1,2)$ of the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$.

The angle of intersection of the given surfaces is equal to the angle of intersection of their normals.

Let $\phi_1(x,y,z) = x^2+y^2+z^2 = 9 \longrightarrow (1)$

& $\phi_2(x,y,z) = x^2+y^2-z = 3 \longrightarrow (2)$.

$$\bar{\nabla}\phi_1\Big]_{(2,-1,2)} = (2x, 2y, 2z)\Big]_{(2,-1,2)} = (4,-2,4) \longrightarrow \text{Normal to } \phi_1$$

$$\bar{\nabla}\phi_2\Big]_{(2,-1,2)} = (2x, 2y, -1)\Big]_{(2,-1,2)} = (4,-2,-1) \longrightarrow \text{Normal to } \phi_2$$

$\therefore$ Angle between their normals at $(2,-1,2)$ is given by

$$\cos\theta = \frac{\bar{\nabla}\phi_1\cdot\bar{\nabla}\phi_2}{|\bar{\nabla}\phi_1||\bar{\nabla}\phi_2|} = \frac{16+4-4}{\sqrt{36}\cdot\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}},$$

a, $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$, which is the required angle.

Ex: (83)

(a) Prove $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$. (b) Find $f(r)$ s.t. $\nabla^2 f(r) = 0$.

$$\nabla^2 f(r) = \bar{\nabla}\cdot\bar{\nabla} f(r) = \bar{\nabla}\cdot f'(r)\,\bar{\nabla} r \quad [\because \nabla f(u) = f'(u)\,\bar{\nabla} u].$$

$$= \bar{\nabla}\cdot \frac{f'(r)}{r}\bar{r} \qquad \left[\because \bar{\nabla} r = \frac{\bar{r}}{r}\right].$$

$$= \frac{f'(r)}{r}\bar{\nabla}\cdot\bar{r} + \bar{r}\cdot\bar{\nabla}\frac{f'(r)}{r} \quad [\because \bar{\nabla}(f\bar{F}) = f(\bar{\nabla}\cdot\bar{F}) + \bar{F}\cdot(\bar{\nabla} f)]$$

$$= \frac{3f'(r)}{r} + \bar{r}\cdot\left\{-\frac{f'(r)}{r^2} + \frac{f''(r)}{r}\right\}\bar{\nabla} r$$

$$= \frac{3f'(r)}{r} + \bar{r}\cdot\left\{\frac{f''(r)}{r} - \frac{f'(r)}{r^2}\right\}\frac{\bar{r}}{r}$$

$$= \frac{3f'(r)}{r} + \left\{\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right\}(\bar{r}\cdot\bar{r})$$

$$= \frac{3f'(r)}{r} + \left\{\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right\}r^2 = \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$\therefore \nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}.$$

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For $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0 \longrightarrow (1)$

we substitute $z = \log r$. Then $\frac{dz}{dr} = \frac{1}{r}$.

Now $\frac{df}{dr} = \frac{df}{dz}\cdot\frac{dz}{dr} = \frac{1}{r}\frac{df}{dz} \Rightarrow$

or, $\frac{d^2 f}{dr^2} = \frac{d}{dr}\left(\frac{1}{r}\frac{df}{dz}\right) = -\frac{1}{r^2}\frac{df}{dz} + \frac{1}{r}\frac{d^2 f}{dz^2}\cdot\frac{dz}{dr}$

$= -\frac{1}{r^2}\frac{df}{dz} + \frac{1}{r^2}\frac{d^2 f}{dz^2} = \frac{1}{r^2}\left(\frac{d^2 f}{dz^2} - \frac{df}{dz}\right).$

From (1), we get,

$$\frac{1}{r^2}\left(\frac{d^2 f}{dz^2} - \frac{df}{dz}\right) + \frac{2}{r}\cdot\frac{1}{r}\frac{df}{dz} = 0$$

or, $\frac{d^2 f}{dz^2} + \frac{df}{dz} = 0 \Rightarrow \frac{\frac{d^2 f}{dz^2}}{\frac{df}{dz}} = -1$

Integrating both sides w.r.t. $z$, we get

$\log\left(\frac{df}{dz}\right) = -z + \log C$, or, $\frac{df}{dz} = C\cdot e^{-z}$

Again integrating w.r.t. $z$, we get

$f = -Ce^{-z} + D$; or, $f(r) = D - \frac{C}{r}$; where $C$ & $D$ are arbitrary constants.

102. Show that $\bar{A} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$ is irrotational. Find $\phi$ s.t. $\bar{A} = \nabla\phi$.

$$\text{curl } \bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \hat{i}(-1+1) + \hat{j}(3z^2-3z^2) + \hat{k}(6x-6x)$$

$$= \hat{i}\,0 + \hat{j}\,0 + \hat{k}\,0$$

$$= \bar{0}$$

$\therefore$ $\bar{A}$ is irrotational.

Now $\bar{A} = \bar{\nabla}\phi$

$\Rightarrow \bar{\nabla}\phi \cdot d\bar{r} = \bar{A} \cdot d\bar{r}$

$\Rightarrow d\phi = (6xy+z^3)dx + (3x^2-z)dy + (3xz^2-y)dz$

$= (6xy\,dx + 3x^2dy) + (z^3dx + 3xz^2dz) - (z\,dy + y\,dz)$

$= d(3x^2y) + d(xz^3) - d(yz)$.

$\therefore \quad d\phi = d(3x^2y + xz^3 - yz)$

Integrating $\Rightarrow \underline{\phi = 3x^2y + xz^3 - yz + C}$,

(C is a constant).

86. Find the most general differentiable function $f(r)$ so that $f(r)\bar{r}$ is solenoidal.

$f(r)\bar{r}$ is solenoidal $\Rightarrow \bar{\nabla}\cdot(f(r)\bar{r}) = 0$

$\Rightarrow \bar{\nabla}f(r)\cdot\bar{r} + f(r)\,\bar{\nabla}\cdot\bar{r} = 0$

$\Rightarrow f'(r)\,\bar{\nabla}r\cdot\bar{r} + 3f(r) = 0$

$\Rightarrow f'(r)\,\frac{\bar{r}}{r}\cdot\bar{r} + 3f(r) = 0$

$\Rightarrow \frac{f'(r)}{r}r^2 + 3f(r) = 0$

$\Rightarrow rf'(r) + 3f(r) = 0 \Rightarrow \frac{f'(r)}{f(r)} + \frac{3}{r} = 0$

Integrating w.r.t. $r$, $\int \frac{f'(r)}{f(r)}dr + \int \frac{3}{r}dr = 0$

$\Rightarrow \log f(r) + 3\log r = \log C \Rightarrow r^3 f(r) = C$.

$\therefore \quad f(r) = C/r^3$, (C is a constant)

103. Show that $\bar{E} = \bar{r}/r^2$ is irrotational. Find $\phi$ such that $\bar{E} = -\bar{\nabla}\phi$ and such that $\phi(a) = 0$ for $a > 0$.

Ans: $\bar{\nabla} \times \bar{E} = \bar{\nabla} \times \frac{\bar{r}}{r^2} = \bar{\nabla}\left(\frac{1}{r^2}\right) \times \bar{r} + \frac{1}{r^2}(\bar{\nabla} \times \bar{r})$

$= \frac{d}{dr}\left(\frac{1}{r^2}\right) \bar{\nabla} r \cdot \frac{1}{r^2}\bar{0} = -\frac{2}{r^3}\frac{\bar{r}}{r} \times \bar{r}$

$= -\frac{2}{r^4}(\bar{r} \times \bar{r}) = \bar{0}$

$\therefore$ E is irrotational.

To find $\phi$: $\bar{E} = -\bar{\nabla}\phi$

$\Rightarrow \frac{\bar{r}}{r^2} = -\bar{\nabla}\phi.$

$\Rightarrow \bar{\nabla}\phi \cdot d\bar{r} = -\frac{\bar{r}}{r^2} \cdot d\bar{r}$

$\Rightarrow \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = -\frac{1}{r^2}(x\,dx + y\,dy + z\,dz).$

$\Rightarrow d\phi = -\frac{1}{r^2} \cdot \frac{1}{2} d(x^2 + y^2 + z^2)$

$\Rightarrow d\phi = -\frac{1}{2}\frac{d(r^2)}{r^2}$

Integrating $\Rightarrow \phi = -\frac{1}{2}\log r^2 + \text{Constant}$

$\Rightarrow \phi(r) = -\log r + \text{Constant}.$

Now $\phi(a) = 0$ for $a > 0$

$\therefore \phi(a) = 0 = -\log a + \text{Constant}$

$\Rightarrow \text{Constant} = \log a.$

$\therefore \phi = -\log r + \log a = \log(a/r).$

$\therefore \phi = \log(a/r)$. (Ans).

Ex. 106 (a) Is there a differentiable vector function $\bar{v}$ such that curl $\bar{v} = \bar{r}$?

Let curl $\bar{v} = \bar{r}$, then $\bar{\nabla}\cdot(\bar{\nabla}\times\bar{V}) = \bar{\nabla}\cdot\bar{r}$

Now, L.H.S. $= \bar{\nabla}\cdot(\bar{\nabla}\times\bar{V}) = \bar{\nabla}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$ [Let $\bar{V}=(V_1,V_2,V_3)$]

$$= \bar{\nabla}\cdot\left\{\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)\hat{i}+\left(\frac{\partial V_1}{\partial z}-\frac{\partial V_3}{\partial x}\right)\hat{j}+\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right)\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z}-\frac{\partial V_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right)$$

$$= \frac{\partial^2 V_3}{\partial x\partial y}-\frac{\partial^2 V_2}{\partial x\partial z}+\frac{\partial^2 V_1}{\partial y\partial z}-\frac{\partial^2 V_3}{\partial y\partial x}+\frac{\partial^2 V_2}{\partial z\partial x}-\frac{\partial^2 V_1}{\partial z\partial y}$$

$$= 0$$

R.H.S. $= \bar{\nabla}\cdot\bar{r} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z) = 1+1+1 = 3.$

$\therefore$ L.H.S $= 0 \neq 3 =$ R.H.S.

$\therefore$ There is no differentiable vector function $\bar{V}$ s.t. curl $\bar{V} = \bar{r}$.

Ex. 92 For what value of the constant $a$ will the vector $\bar{A} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ have its curl identically equal to zero?

$$\bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy-z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0\hat{i} + [-3z^2-(1-a)z^2]\hat{j} + [2x(a-2)-ax]\hat{k}$$

$$= (-4z^2+az^2)\hat{j} + (xa-4x)\hat{k}$$

$$= z^2(a-4)\hat{j} + x(a-4)\hat{k}$$

Now $\bar{\nabla}\times\bar{A} = \bar{0}$

$\Rightarrow z^2(a-4)\hat{j} + x(a-4)\hat{k} = 0\hat{i}+0\hat{j}+0\hat{k}.$

$\Rightarrow a-4=0 \Rightarrow \underline{a=4}$. (Ans.)

10(b) Is there a differentiable vector function $\bar{V}$ such that curl $\bar{V} = 2\hat{i} + \hat{j} + 3\hat{k}$ ? If so, find $\bar{V}$.

**Ans:** We have, $\bar{\nabla} \cdot \text{curl}\,\bar{V} = \bar{\nabla} \cdot (2\hat{i} + \hat{j} + 3\hat{k})$
$\Rightarrow \quad 0 = 0$ ; Equality holds.

$\therefore$ **Yes**, $\exists$ a vector function $\bar{V}$ s.t.
curl $\bar{V} = 2\hat{i} + \hat{j} + 3\hat{k}$.

To find $\bar{V}$ :

We have, $\bar{\nabla} \cdot \text{curl}\,\bar{V} = 0 \longrightarrow (1)$.
and $\text{curl}\,\bar{V} = 2\hat{i} + \hat{j} + 3\hat{k} = \bar{a}$ (say) $\longrightarrow (2)$.

With view of (1), (2) has all the solutions given by $\bar{V} = \bar{V}_0 + \bar{\nabla}\phi$, $\longrightarrow (3)$ where $\bar{V}_0$ is any vector function and $\phi(x, y, z)$ is an arbitrary scalar function.

Now curl $\bar{V} = \text{curl}(\bar{V}_0 + \bar{\nabla}\phi) = \text{curl}\,\bar{V}_0 + \text{Curl}\,\bar{\nabla}\phi$
$\Rightarrow \quad \bar{a} = \text{curl}\,\bar{V}_0 + \bar{0} = \text{curl}\,\bar{V}_0$.

$\therefore \quad \text{curl}\,\bar{V}_0 = \bar{a} \longrightarrow (4)$.

Since $\bar{V}_0$ is arbitrary, let us take
$\bar{V}_0 = (0, V_2, V_3) \longrightarrow (5)$.

Then $\text{curl}\,\bar{V}_0 = \bar{a} \Rightarrow \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}, -\frac{\partial V_3}{\partial x}; \frac{\partial V_2}{\partial x}\right) = (2, 1, 3)$

$\Rightarrow \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} = 2$ ; $-\frac{\partial V_3}{\partial x} = 1$ ; $\frac{\partial V_2}{\partial x} = 3. \longrightarrow (6)$

Now integrating $-\frac{\partial V_3}{\partial x} = 1$ partially w.r.t. $x$, we get
$V_3 = -x + f(y, z) \longrightarrow (7)$

Also integrating $\frac{\partial V_2}{\partial x} = 3$ " " " "
$V_2 = 3x + g(y, z) \longrightarrow (8)$ ; $f(y, z), g(y, z)$ are arbitrary functions.

Let us choose $g(y, z) = 0$ in (8), then we get $V_2 = 3x$. Then from (6), $\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} = 2 \Rightarrow \frac{\partial f}{\partial y} - 0 = 2$

Integrating partially w.r.t. $y$; $f = 2y + h(z)$; choosing $h(z) = 0$.

$\therefore \quad \bar{V}_0 = (0, 3x, -x + 2y)$

$\therefore \quad \bar{V} = \bar{V}_0 + \bar{\nabla}\phi = 3x\hat{j} + (2y - x)\hat{k} + \bar{\nabla}\phi$. (Ans).