LECTURE NOTES

On

GREEN'S, STOKE'S AND DIVERGENCE THEOREM

Semester: 4

Subject: Mathematics (Honours)

Paper: C9T , Unit-IV (Marks: 09)

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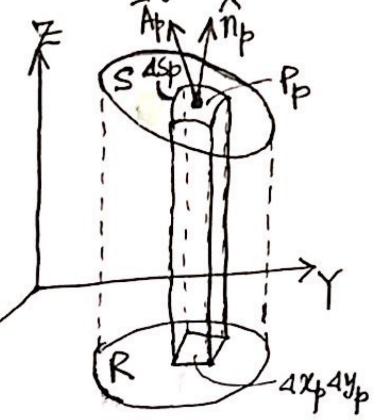
VECTOR SURFACE INTEGRALS AND VOLUME INTEGRALS

Mathematics Honours, Semester - 4
Paper - C9T, Unit - IV (Marks: 09).

Green's, Stoke's and Divergence Theorem

SURFACE INTEGRALS :

- Give a definition of $\iint_S \bar{A}\cdot\hat{n}\,ds$ over a surface S in terms of limit of a sum.



Let S be the surface in a vector field $\bar{A}$. If $\hat{n}$ be an outward drawn unit normal to an element of surface ds such that $\overline{ds} = \hat{n}ds$, where $\overline{ds}$ is the vector area of ds.
The total outward flux through the whole surface S is given by

$$\iint_S \bar{A}\cdot\overline{ds} = \iint_S \bar{A}\cdot\hat{n}\,ds.$$

Other type of surface integrals: $\iint_S \phi\,ds$, $\iint_S \phi\hat{n}\,ds$, $\iint_S \bar{A}\times\overline{ds}$, where ϕ is a scalar function of position.

Definition :

Let us subdivide the area S into M elements of area ΔS_p where $p = 1, 2, \cdots, M$.
Let us choose any point P_p within ΔS_p whose coordinates are (x_p, y_p, z_p).
Let us define $\bar{A}(x_p, y_p, z_p) = \bar{A}_p$. Let $\hat{n}_p$ be the outward (positive) unit normal to ΔS_p at P_p.
The normal flux of $\bar{A}_p$ at P_p is $\bar{A}_p\cdot\hat{n}_p\Delta S_p$.

The normal flux of $\bar{A}_p$ for $p = 1, 2, \cdots, M$ elements is $\sum_{p=1}^{M} \bar{A}_p\cdot\hat{n}_p\Delta S_p$. Taking the limit of this sum as $M\to\infty$ in such a way that the largest $\Delta S_p \to 0$.

This limit, if exists, is called the surface integral of the normal component of $\bar{A}$ over S and is denoted by $\iint_S \bar{A}\cdot\hat{n}\,ds$.

- To show that $\iint_S \bar{A}\cdot\hat{n}\, ds = \iint_R \bar{A}\cdot\hat{n}\, \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$; where R is the projection of S on the xy-plane.

Let the surface S have projection R on the xy plane. The surface integral is the limit of the sum

$$\sum_{p=1}^{M} \bar{A}_p\cdot\hat{n}_p\, \Delta S_p \longrightarrow (1)$$

The projection of ΔS_p on the xy plane is given by

$$|(\hat{n}_p \Delta S_p)\cdot\hat{k}| \quad \text{or,} \quad |\hat{n}_p\cdot\hat{k}|\Delta S_p = \Delta x_p \Delta y_p$$

$$\therefore \quad \Delta S_p = \frac{\Delta x_p \Delta y_p}{|\hat{n}_p\cdot\hat{k}|}$$

Then the sum (1) becomes

$$\sum_{p=1}^{M} \bar{A}_p\cdot\hat{n}_p \frac{\Delta x_p \Delta y_p}{|\hat{n}_p\cdot\hat{k}|}.$$

Taking limit as $M\to\infty$ in such a way that the largest Δx_p and Δy_p approaches zero, we get

$$\iint_R \bar{A}\cdot\hat{n}\, \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$; by the fundamental theorem of integral calculus

$$\therefore \quad \iint_S \bar{A}\cdot\hat{n}\, ds = \iint_R \bar{A}\cdot\hat{n}\, \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}.$$

- **VOLUME INTEGRALS**

Express $\iiint_V \phi\, dV$ as the limit of a sum; [where ϕ is a scalar function, and V be the volume of the closed region.]

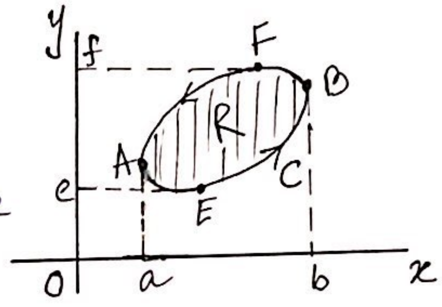
Let us subdivide region V into M cubes having volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_p$, $p = 1, 2, \cdots, M$, where (x_p, y_p, z_p) be a point within this cube. Let us define $\phi(x_p, y_p, z_p) = \phi_p$.

Let us consider the sum $\sum_{p=1}^{M} \phi_p \Delta V_p$ taken over all possible cubes in the region.

Taking limit as $M\to\infty$ in such a manner that the largest of $\Delta V_p \to 0$, if it exists, is called the volume integral, and is denoted by $\iiint_V \phi\, dV$.

GREEN'S THEOREM in the Plane :-

If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$, where C is traversed in the positive (counterclockwise) direction.



Proof:- Let us consider a region R in the xy plane bounded by a simple closed curve C which has the property that any straight line parallel to the coordinate axes cuts C in at most two points.

Let the equations of the curve AEB and AFB be $y = f_1(x)$ and $y = f_2(x)$ respectively.

We have $$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b} \left[\int_{y=f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\right] dx$$

$$= \int_{x=a}^{b} M(x,y)\Big|_{y=f_1(x)}^{f_2(x)} dx = \int_a^b \left[M(x,f_2) - M(x,f_1)\right] dx.$$

$$= -\int_a^b M(x,f_1)\,dx - \int_b^a M(x,f_2)\,dx = -\oint_C M\,dx.$$

$$\therefore \oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy \longrightarrow (1)$$

Similarly let the equations of curves EAF and EBF be $x = g_1(y)$ and $x = g_2(y)$ respectively. Then we have

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=e}^{f} \left[\int_{x=g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x}\,dx\right] dy$$

$$= \int_{y=e}^{f} \left[N(g_2,y) - N(g_1,y)\right] dy = \int_{y=f}^{e} N(g_1,y)\,dy + \int_{y=e}^{f} N(g_2,y)\,dy.$$

$$= \oint_C N\,dy.$$

$$\therefore \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy \longrightarrow (2)$$

Adding (1) & (2), $$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

(proved)

Express Green's theorem in the plane in Vector notation:-

We have $Mdx + Ndy = (M\hat{i} + N\hat{j})\cdot(dx\hat{i} + dy\hat{j}) = \bar{A}\cdot d\bar{r}$,
where $\bar{A} = M\hat{i} + N\hat{j}$ and $\bar{r} = x\hat{i} + y\hat{j}$ so that $d\bar{r} = dx\hat{i} + dy\hat{j}$.

Now
$$\bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z}\hat{i} + \frac{\partial M}{\partial z}\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

So that $(\bar{\nabla}\times\bar{A})\cdot\hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

Then Green's theorem in the plane can be written
$$\oint_C \bar{A}\cdot d\bar{r} = \iint_R (\bar{\nabla}\times\bar{A})\cdot\hat{k}\, dR \text{ ; where } dR = dxdy.$$
(Vector form)

Remark: If $\bar{A}$ denotes the force field acting on a particle, then $\oint_C \bar{A}\cdot d\bar{r}$ is the work done in moving the particle around a closed path C, which is determined by the value of $\bar{\nabla}\times\bar{A}$. [i.e., $|(\bar{\nabla}\times\bar{A})\cdot\hat{k}| = |\bar{\nabla}\times\bar{A}|$, since $\bar{\nabla}\times\bar{A}$ & $\hat{k}$ have the parallel direction].
If $\bar{\nabla}\times\bar{A} = \bar{0}$, or equivalently $\bar{A} = \bar{\nabla}\phi$, then the integral around a closed path is zero. So, work done is independent of the path in the plane and that the force field is conservative.
Conversely, if the integral around any closed path is zero, then $\bar{\nabla}\times\bar{A} = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the condition for exact differential $Mdx + Ndy$.

STOKES' Theorem :-

If S is an open, twosided surface bounded by a simple closed curve* C then if A has continuous derivatives

$$\oint_C \bar{A}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{A})\cdot\hat{n}\,ds = \iint_S (\bar{\nabla}\times\bar{A})\cdot d\bar{s}$$

Where C is traversed in the positive direction.*

* The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of the positive normal to S, has the surface on his left.

* simple closed curve means closed, non-intersecting curve.

THE DIVERGENCE THEOREM OF GAUSS :-

If V is the volume bounded by a closed surface S and $\bar{A}$ is a vector function of position with continuous derivatives, then

$$\iiint_V \bar{\nabla}\cdot\bar{A}\,dv = \iint_S \bar{A}\cdot\hat{n}\,ds = \oiint_S \bar{A}\cdot d\bar{s},$$ where

$\hat{n}$ is the unit positive (outward drawn) normal to S.

Remarks:

1. Green's theorem in the plane is a special case of Stokes' theorem.

2. The Divergence theorem is often called Green's theorem in space.
   Gauss' divergence theorem is a generalisation of Green's theorem in the plane, where the plane region R and its closed boundary curve C are replaced by a space region V and its closed boundary surface S.

3. Green's theorem in the plane also holds for regions bounded by a finite number of simple closed curves which do not intersect.

Related Integral Theorems :-

① $$\iiint_V [\phi \nabla^2\psi + (\bar{\nabla}\phi \cdot \bar{\nabla}\psi)]\, dv = \iint_S (\phi \bar{\nabla}\psi)\cdot d\bar{s}.$$ [Green's first identity]

② $$\iiint_V (\phi \nabla^2\psi - \psi \nabla^2\phi)\, dv = \iint_S (\phi\bar{\nabla}\psi - \psi\bar{\nabla}\phi)\cdot d\bar{s}.$$ [Green's 2nd identity]

③ $$\iiint_V \bar{\nabla}\times\bar{A}\, dv = \iint_S (\hat{n}\times\bar{A})\, ds = \iint_S d\bar{s}\times\bar{A}.$$

④ $$\oint_C \phi\, d\bar{r} = \iint_S (\hat{n}\times\bar{\nabla}\phi)\, ds = \iint_S d\bar{s}\times\bar{\nabla}\phi.$$

Proof of ① & ②.

In Gauss divergence theorem, let $A = \phi\bar{\nabla}\psi$.

$$\iiint_V \bar{\nabla}\cdot(\phi\bar{\nabla}\psi)\, dv = \iint_S (\phi\bar{\nabla}\psi)\cdot\hat{n}\, ds = \iint_S (\phi\bar{\nabla}\psi)\cdot d\bar{s}.$$

Now $\bar{\nabla}\cdot(\phi\bar{\nabla}\psi) = \phi(\bar{\nabla}\cdot\bar{\nabla}\psi) + (\bar{\nabla}\phi\cdot\bar{\nabla}\psi) = \phi\nabla^2\psi + (\bar{\nabla}\phi\cdot\bar{\nabla}\psi).$

Thus $$\iiint_V [\phi\nabla^2\psi + (\bar{\nabla}\phi\cdot\bar{\nabla}\psi)]\, dv = \iint_S (\phi\bar{\nabla}\psi)\cdot d\bar{s}. \longrightarrow ①$$ [proved ①]

Now to prove ②, interchange ϕ & ψ in ①, we get

$$\iiint_V [\psi\nabla^2\phi + (\bar{\nabla}\psi\cdot\bar{\nabla}\phi)]\, dv = \iint_S (\psi\bar{\nabla}\phi)\cdot d\bar{s} \longrightarrow ②$$

Subtracting ② from ①, we have

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dv = \iint_S (\phi\bar{\nabla}\psi - \psi\bar{\nabla}\phi)\cdot d\bar{s}$$ [proved ②]

Note: ϕ and ψ are scalar functions of position with continuous derivatives of the second order at least.

Proof of ③: To prove: $\iiint_V \bar{\nabla}\times\mathbf{B}\, dv = \iint_S (\hat{n}\times\mathbf{B})\, ds.$

In the divergence theorem, let $\bar{A} = \bar{B}\times\bar{C}$ where $\bar{C}$ is a constant vector. Then

$$\iiint_V \bar{\nabla}\cdot(\bar{B}\times\bar{C})\, dv = \iint_S (\bar{B}\times\bar{C})\cdot\hat{n}\, ds$$

Now, $\bar{\nabla}\cdot(\bar{B}\times\bar{C}) = \bar{B}\cdot(\bar{\nabla}\times\bar{C}) + \bar{C}\cdot(\bar{\nabla}\times\bar{B}) = 0 + \bar{C}\cdot(\bar{\nabla}\times\bar{B}).$

Also $(\bar{B}\times\bar{c})\cdot\hat{n} = \bar{B}\cdot(\bar{c}\times\hat{n}) = (\bar{c}\times\hat{n})\cdot\bar{B} = \bar{c}\cdot(\hat{n}\times\bar{B})$.

$$\therefore \iiint_V \bar{c}\cdot(\bar{\nabla}\times\bar{B})\,dV = \iint_S \bar{c}\cdot(\hat{n}\times\bar{B})\,ds$$

Taking $\bar{c}$ outside the integrats,

$$\bar{c}\cdot\iiint_V(\bar{\nabla}\times\bar{B})\,dV = \bar{c}\cdot\iint_S(\hat{n}\times\bar{B})\,ds$$

Since $\bar{c}$ is an arbitrary constant vector,

$$\iiint_V(\bar{\nabla}\times\bar{B})\,dV = \iint_S(\hat{n}\times\bar{B})\,ds.$$

Proof of (4): That is to prove $\oint_C \phi\, d\bar{r} = \iint_S(\hat{n}\times\bar{\nabla}\phi)\,ds$.

Let $\bar{A} = \bar{c}\phi$, where $\bar{c}$ is an arbitrary constant vector and ϕ is a continuously differentiable scalar point function.

Then $\oint_C \bar{A}\cdot d\bar{r} = \iint_S(\bar{\nabla}\times\bar{A})\cdot\hat{n}\,ds$

$$\Rightarrow \oint_C \bar{c}\phi\cdot d\bar{r} = \iint_S(\bar{\nabla}\times\bar{c}\phi)\cdot\hat{n}\,ds.$$

$$\Rightarrow \bar{c}\cdot\oint_C \phi\, d\bar{r} = \iint_S[\phi(\bar{\nabla}\times\bar{c}) + \bar{\nabla}\phi\times\bar{c}]\cdot\hat{n}\,ds$$

$$= \iint_S(\bar{\nabla}\phi\times\bar{c})\cdot\hat{n}\,ds \quad [\because \bar{\nabla}\times\bar{c} = \bar{0}]$$

$$= -\iint_S(\bar{c}\times\bar{\nabla}\phi)\cdot\hat{n}\,ds = -\bar{c}\cdot\iint_S(\bar{\nabla}\phi\times\hat{n})\,ds.$$

Since $\bar{c}$ is an arbitrary constant vector,

$$\oint_C \phi\, d\bar{r} = -\iint_S(\bar{\nabla}\phi\times\hat{n})\,ds = \iint_S(\hat{n}\times\bar{\nabla}\phi)\,ds.$$

(proved)

Ex: Prove that a necessary and sufficient condition that $\oint_C \bar{A}\cdot d\bar{r} = 0$ for every closed curve C is that $\bar{\nabla}\times\bar{A} = \bar{0}$ identically.

Sufficiency:- Let $\bar{\nabla}\times\bar{A} = \bar{0}$, then by Stokes' theorem

$$\oint_C \bar{A}\cdot d\bar{r} = \iint_S(\bar{\nabla}\times\bar{A})\cdot\hat{n}\,ds = 0$$

around every closed path C,

Necessity:- Let $\oint_C \bar{A}\cdot d\bar{r} = 0$ also let $\bar{\nabla}\times\bar{A} \neq 0$ at some point P, which is an interior point of the surface S whose unit normal is $\hat{n} \parallel \bar{\nabla}\times\bar{A}$, i.e.,

$$\bar{\nabla}\times\bar{A} = \alpha\hat{n},\ \alpha \text{ is a +ve constant.}$$

Let C be the boundary of S. Then by Stokes' theorem

$$\oint_C \bar{A}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{A})\cdot\hat{n}\, ds = \alpha \iint_S \hat{n}\cdot\hat{n}\, ds > 0$$

which contradicts the hypothesis that $\oint_C \bar{A}\cdot d\bar{r} = 0$

$\therefore\ \bar{\nabla}\times\bar{A} = \bar{0}$ identically.

Spiegel Exercise:

(55) If S be any closed surface enclosing a volume V and $\bar{A} = ax\hat{i} + by\hat{j} + cz\hat{k}$, prove that $\iint_S \bar{A}\cdot\hat{n}\, ds = (a+b+c)V$.

The divergence theorem gives that

$$\iint_S \bar{A}\cdot\hat{n}\, ds = \iiint_V \bar{\nabla}\cdot\bar{A}\, dV = \iiint_V \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(ax\hat{i} + by\hat{j} + cz\hat{k})\, dV$$

$$= \iiint_V (a+b+c)\, dV = (a+b+c)V.$$

(58). Prove $\iiint_V \frac{dV}{r^2} = \iint_S \frac{\bar{r}\cdot\hat{n}}{r^2}\, ds$

By the divergence theorem, we have

$$\iint_S \frac{\bar{r}\cdot\hat{n}}{r^2}\, ds = \iiint_V \bar{\nabla}\cdot\left(\frac{\bar{r}}{r^2}\right) dV = \iiint_V \sum \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right)(\hat{i}\cdot\hat{i})\, dV$$

$$= \iiint_V \sum\left(\frac{1}{r^2} - \frac{2\cdot x}{r^3}\frac{\partial r}{\partial x}\right) dV = \iiint_V \sum\left(\frac{1}{r^2} - \frac{2}{r^3}x\cdot\frac{x}{r}\right) dV$$

$$= \iiint_V \left[\frac{3}{r^2} - \frac{2}{r^4}\cdot(x^2+y^2+z^2)\right] dV = \iiint_V \left[\frac{3}{r^2} - \frac{2}{r^4}\cdot r^2\right] dV$$

$$= \iiint_V \frac{1}{r^2}\, dV. \text{ (proved)}$$

(59) Prove $\iint_S r^5\hat{n}\, ds = \iiint_V 5r^3\bar{r}\, dV$.

First to prove: $\iiint_V \bar{\nabla}\phi\, dV = \iint_S \phi\hat{n}\, ds$.

In the divergence theorem, let $\bar{A} = \phi\bar{c}$, where $\bar{c}$ is a constant vector. Then

$$\iiint_V \bar{\nabla}\cdot(\phi\bar{c})\, dV = \iint_S \phi\bar{c}\cdot\hat{n}\, ds$$

$$\Rightarrow \iiint_V (\bar{\nabla}\phi\cdot c + \phi\bar{\nabla}\cdot\bar{c})\, dV = \iint \bar{c}\cdot(\phi\hat{n})\, ds$$

or, $\iiint_V (\bar{\nabla}\phi \cdot \bar{c})\, dv = \iint_S \bar{c}\cdot(\phi\hat{n})\, ds$, since $\bar{\nabla}\cdot\bar{c} = 0$.

or, $\iiint_V (\bar{c}\cdot\bar{\nabla}\phi)\, dv = \iint_S \bar{c}\cdot(\phi\hat{n})\, ds$

or, $\bar{c}\cdot\iiint_V \bar{\nabla}\phi\, dv = \bar{c}\cdot\iint_S \phi\hat{n}\, ds$ [Taking $\bar{c}$ outside the integral]

or, $\iiint_V \nabla\phi\, dv = \iint_S \phi\hat{n}\, ds$. [Since $\bar{c}$ is an arbitrary constant vector]

Now in the given problem, let us take $\phi = r^5$.

Then we have, $\iint_S r^5\hat{n}\, ds = \iiint_V \bar{\nabla} r^5\, dv$

or, $\iint_S r^5\hat{n}\, ds = \iiint_V \left(\sum \hat{i}\frac{\partial}{\partial x} r^5\right) dv = \iiint_V \left(\sum \hat{i}\, 5r^4 \frac{\partial r}{\partial x}\right) dv$

$= \iiint_V \left(\sum 5r^4 \frac{x}{r}\hat{i}\right) dv = \iiint_V \left(\sum 5r^3 x\hat{i}\right) dv$

$= \iiint_V 5r^3\left(\sum x\hat{i}\right) dv = \iiint_V 5r^3\bar{r}\, dv$. (proved)

(62). Prove $\iint_S \bar{r}\times d\bar{s} = \bar{0}$ for any closed surface.

We have $\iint_S \bar{r}\times d\bar{s} = \iint_S (\bar{r}\times\hat{n})\, ds$ ~~$= -\iint_S (\hat{n}\times\bar{r})\, ds$~~.

To show first that $\iint_S (\hat{n}\times\bar{r})\, ds = \iiint_V (\bar{\nabla}\times\bar{r})\, dv$.

In the divergence theorem, let $\bar{A} = \bar{r}\times\bar{c}$, where $\bar{c}$ is an arbitrary constant vector.

$\iiint_V \bar{\nabla}\cdot(\bar{r}\times\bar{c})\, dv = \iint_S (\bar{r}\times\bar{c})\cdot\hat{n}\, ds$

or, $\iiint_V [\bar{c}\cdot(\bar{\nabla}\times\bar{r}) + \bar{r}\cdot(\bar{\nabla}\times\bar{c})]\, dv = -\iint_S (\bar{c}\times\bar{r})\cdot\hat{n}\, ds$

or, $\iiint_V \bar{c}\cdot(\bar{\nabla}\times\bar{r})\, dv = -\iint_S \bar{c}\cdot(\bar{r}\times\hat{n})\, ds$ ~~$= \iint_S \bar{c}\cdot(r$~~

or, $\bar{c}\cdot\iiint_V (\bar{\nabla}\times\bar{r})\, dv = -\bar{c}\cdot\iint_S (\bar{r}\times\hat{n})\, ds$ [Since $\bar{c}$ is an arbitrary constant vector]

$\Rightarrow \iint_S (\bar{r}\times\hat{n})\, ds = -\iiint_V (\bar{\nabla}\times\bar{r})\, dv = \bar{0}$

or, $\underline{\iint_S \bar{r}\times d\bar{s} = \bar{0}}$. for any closed surface. (proved)

(67) A vector $\bar{B}$ is always normal to a given closed surface S. Show that $\iiint \text{curl}\,\bar{B}\,dv = \bar{0}$, where V is the region bounded by S.

First to show that $\iiint_V (\bar{\nabla}\times\bar{B})\,dv = \iint_S (\hat{n}\times\bar{B})\,ds$.

In the divergence theorem, let $\bar{A} = \bar{B}\times\bar{C}$ where $\bar{C}$ is an arbitrary constant vector.

$$\therefore \iiint_V \bar{\nabla}\cdot(\bar{B}\times\bar{C})\,dv = \iint_S (\bar{B}\times\bar{C})\cdot\hat{n}\,ds$$

$$\text{or, } \iiint_V [\bar{c}\cdot(\bar{\nabla}\times\bar{B}) + \bar{B}\cdot(\bar{\nabla}\times\bar{c})]\,dv = -\iint_S (\bar{c}\times\bar{B})\cdot\hat{n}\,ds$$

$$\text{or, } \iiint_V \bar{c}\cdot(\bar{\nabla}\times\bar{B})\,dV = -\iint_S \bar{c}\cdot(\bar{B}\times\hat{n})\,ds \quad \left[\text{Since } \bar{C} \text{ is an arbitrary constant vector}\right]$$

$$\text{or, } \iiint_V (\bar{\nabla}\times\bar{B})\,dV = +\iint_S (\hat{n}\times\bar{B})\,ds.$$

$$= \bar{0} \quad \left[\text{Since } \hat{n}\times\bar{B} = \bar{0}, \text{ as } \bar{B} \text{ is always normal to S, } \hat{n} \parallel \bar{B}\right]$$

$$\therefore \iiint_V \text{curl}\,\bar{B} = \bar{0}. \text{ (proved)}$$

(68) If $\oint_C \bar{E}\cdot d\bar{r} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \bar{H}\cdot d\bar{s}$, where S is any surface bounded by the curve C, show that $\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t}$.

By Stoke's theorem, we have

$$\oint_C \bar{E}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{E})\cdot d\bar{s}$$

Given that $\oint_C \bar{E}\cdot d\bar{r} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \bar{H}\cdot d\bar{s}$

$$\therefore \iint_S (\bar{\nabla}\times\bar{E})\cdot d\bar{s} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \bar{H}\cdot d\bar{s} = \iint_S \left(-\frac{1}{c}\frac{\partial \bar{H}}{\partial t}\right)\cdot d\bar{s}$$

$$\Rightarrow \bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t} \quad [\text{Since S is any arbitrary surface}]$$

(proved)

Ex. Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C x\,dy - y\,dx$.

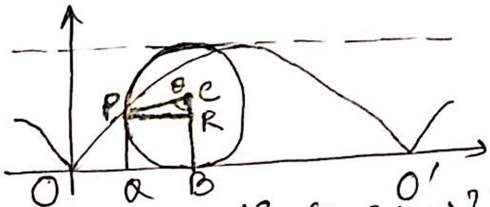
In Green's theorem, let us put $M = -y$, $N = x$. Then

$$\oint_C x\,dy - y\,dx = \iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dx\,dy = 2\iint_R dx\,dy$$

$= 2A$, where A is the required area

$\therefore$ $A = \frac{1}{2}\oint_C x\,dy - y\,dx$. (proved)

(43) Find the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$, $a>0$, and the x axis.



Area $(A) = \frac{1}{2}\oint_C x\,dy - y\,dx$

$$= -\frac{1}{2}\int_{\theta=0}^{2\pi} \left[a(\theta-\sin\theta)\,d\{a(1-\cos\theta)\} - a(1-\cos\theta)\,d\{a(\theta-\sin\theta)\}\right]$$

[negative sign is taken as θ moves from 0 to 2π, C describes clockwise direction]

$$= -\frac{1}{2}\int_{\theta=0}^{2\pi} \left[a(\theta-\sin\theta)\,a\sin\theta\,d\theta - a(1-\cos\theta)\cdot(a - a\cos\theta)\,d\theta\right]$$

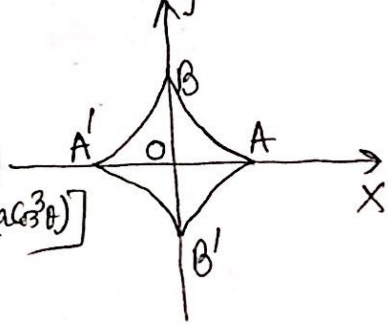
$$= -\frac{1}{2}\int_{\theta=0}^{2\pi} \left[a^2\{\theta\sin\theta - \sin^2\theta - (1+\cos^2\theta - 2\cos\theta)\}\,d\theta\right]$$

$$= -\frac{1}{2}\int_{\theta=0}^{2\pi} a^2\left[\theta\sin\theta - 2(1-\cos\theta)\right]d\theta = -\frac{a^2}{2}\left[-\theta\cos\theta + \int\cos\theta\,d\theta - 2\theta + 2\sin\theta\right]_{\theta=0}^{2\pi}$$

$$= -\frac{a^2}{2}\left[-\theta\cos\theta - 2\theta + 3\sin\theta\right]_{\theta=0}^{2\pi}$$

$$= -\frac{a^2}{2}\left[-2\pi - 4\pi\right] = \underline{3\pi a^2}. \text{ (Ans).}$$

Let parametric equations be $x = a\cos^3\theta$, $y = a\sin^3\theta$. $(0 \le \theta \le 2\pi)$.



$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_{\theta=0}^{2\pi}\left[a\cos^3\theta\,d(a\sin^3\theta) - a\sin^3\theta\,d(a\cos^3\theta)\right]$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi}\left[3a^2\{\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta\}\,d\theta\right]$$

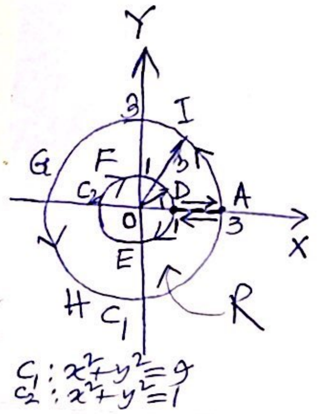
$$= \frac{1}{2}\times 3a^2 \int_{\theta=0}^{2\pi} \cos^2\theta \sin^2\theta \, d\theta = \frac{3a^2}{8}\int_0^{2\pi} \sin^2 2\theta \, d\theta = \frac{3a^2}{8}\cdot\frac{1}{2}\int_0^{2\pi}(1-\cos 4\theta)\,d\theta$$

$$= \frac{3a^2}{16}\left[\theta - \frac{\sin 4\theta}{4}\right]_{\theta=0}^{2\pi} = \frac{3a^2}{16}\times 2\pi = \underline{\frac{3\pi a^2}{8}} \quad \text{(Ans.)}$$

(49) Verify Green's theorem in the plane for $\oint_C (2x-y^3)dx - xy\,dy$ where C is the boundary of the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.

Let us construct a line AD, called a cross-cut, connecting the exterior and interior boundaries.

The region bounded by ADEFAIGHA is simply-connected, so Green's theorem is valid.



$$\oint_{ADEFAIGHA} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

L.H.S. $= \left[\int_{AD} + \int_{DEFD} + \int_{DA} + \int_{AIGHA}\right]$, leaving out the integrand,

$$= \int_{DEFD} + \int_{AIGHA} \qquad \left[\text{Since} \int_{AD} = -\int_{DA}\right]$$

Let C_1 be the curve AIGHA, C_2 be the curve DEFD and C is the boundary of R consisting of C_1 & C_2, then $\int_{C_1} + \int_{C_2} = \int_C$ and so $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$.

Now $$\oint_C M\,dx + N\,dy = \int_{C_1:(x^2+y^2=9)} M\,dx + N\,dy + \int_{C_2:(x^2+y^2=1)} M\,dx + N\,dy$$

Now, $$\int_{C_1: x^2+y^2=9\,(x=3\cos t,\, y=3\sin t)} (2x-y^3)dx - xy\,dy = \int_{t=0}^{2\pi}\left\{(2\cdot 3\cos t - 3^3\sin^3 t)\,d(3\cos t) - 3\cos t\cdot 3\sin t\, d(3\sin t)\right\}$$

$$= \int_{t=0}^{2\pi}\left[(6\cos t - 27\sin^3 t)(-3\sin t) - 9\sin t\cos t\cdot 3\cos t\right] dt$$

$$= \int_{t=0}^{2\pi}\left[-9\sin 2t + 81\sin^4 t - 9\sin t\cdot 3(1-\sin^2 t)\right] dt$$

$$= \int_{t=0}^{2\pi}\left[-9\sin 2t - 27\sin t + 27\sin^3 t + 81\sin^4 t\right] dt$$

$$= \int_{t=0}^{2\pi}\left[-9\sin 2t + 27\sin t + 27\cdot\frac{1}{4}(3\sin t - \sin 3t) + \frac{81}{4}(1-\cos 2t)^2\right] dt$$

$$= \int_{t=0}^{2\pi} \left[-9\sin 2t + \frac{27\cdot 7}{4}\sin t - \frac{27}{4}\sin 3t + \frac{81}{4} - \frac{81}{2}\cos 2t + \frac{81}{8}(1+\cos 4t)\right] dt$$

$$= \left[\frac{9}{2}\cos 2t - \frac{27\cdot 7}{4}\cos t + \frac{27}{4\cdot 3}\cos 3t + \frac{81}{4}t + \frac{81}{4}\sin 2t + \frac{81}{8}t - \frac{81}{32}\sin 4t\right]_{t=0}^{2\pi}$$

$$= \frac{9}{2}\times 0 - \frac{27\cdot 7}{4}\times 0 + \frac{27}{4\cdot 3}\cdot 0 + \frac{81}{4}\times 2\pi + \frac{81}{8}\times 2\pi$$

$$= \frac{81}{4}\pi\left(2 + \frac{2}{2}\right) = \boxed{\frac{243\pi}{4}}$$

Now, $\int_{C_2} \left[(2\cos t - \sin^3 t)\, d(\cos t) - \cos t \sin t \, d(\sin t)\right] = \int_{t=2\pi}^{0} \left[-\sin 2t + \sin^4 t - \cos^2 t \sin t\right] dt$ [Since c_2 traverses (-ve) direction]

$C_2 : x^2+y^2=1 \ (x=\cos t, y=\sin t)$

$$= \int_{t=2\pi}^{0} \left[-\sin 2t + \frac{1}{4}(1-\cos 2t)^2 - \sin t + \sin^3 t\right] dt$$

$$= \int_{t=2\pi}^{0} \left[-\sin 2t + \frac{1}{4} - \frac{1}{2}\cos 2t + \frac{1}{8}(1+\cos 4t) - \sin t + \frac{1}{4}(3\sin t - \sin 3t)\right] dt$$

$$= \int_{t=2\pi}^{0} \left[-\sin 2t + \left(\frac{1}{4}+\frac{1}{8}\right) - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t - \frac{1}{4}\sin t - \frac{1}{4}\sin 3t\right] dt$$

$$= \left[\frac{\cos 2t}{2} + \left(\frac{1}{4}+\frac{1}{8}\right)t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + \frac{1}{4}\cos t + \frac{1}{12}\cos 3t\right]_{t=2\pi}^{0}$$

$$= -\frac{3}{8}\times 2\pi = \underline{-\frac{3}{4}\pi}.$$

$$\therefore \oint_C = \int_{C_1} + \int_{C_2} = \frac{243\pi}{4} - \frac{3}{4}\pi = \frac{240\pi}{4} = \underline{60\pi}.$$

Now, $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$, by Green's theorem

Here $M = 2x - y^3$, $N = -xy$. Let $x = r\cos\theta$, $y = r\sin\theta$.

$\therefore \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = -y - 3y^2$; $\quad \therefore dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$.

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-y - 3y^2)\, dx\,dy$$

$$= \int_{r=1}^{3}\int_{\theta=0}^{2\pi} (-r\sin\theta - 3r^2\sin^2\theta)\, r\,dr\,d\theta = -\int_{r=1}^{3}\int_{\theta=0}^{2\pi} \left(r^2 \sin\theta\, d\theta + \frac{3r^3}{2} 2\sin^2\theta\right) dr\,d\theta$$

$$= +\left[\frac{r^3}{3}\right]_{r=1}^{3} \cdot \left[\cos\theta\right]_{\theta=0}^{2\pi} - \frac{3}{2}\left[\frac{r^4}{4}\right]_{r=1}^{3} \cdot \int_{\theta=0}^{2\pi} (1-\cos 2\theta)\, d\theta$$

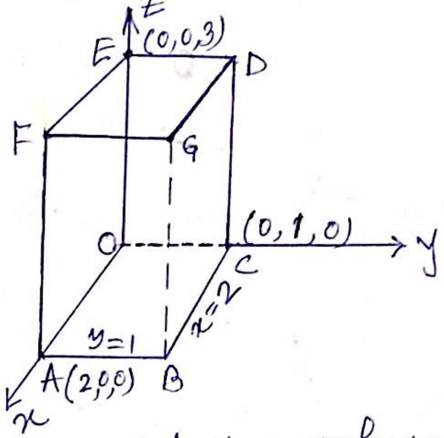
$$= \frac{1}{3}\times 26 \times (1-1) - \frac{3\cdot(81-1)}{2\cdot 4} \cdot \left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta=0}^{2\pi} = \frac{3\times \cancel{80}^{10}}{\cancel{8}} \times 2\pi = \underline{60\pi}$$

$$\therefore \oint_C M\,dx + N\,dy = 60\pi = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$\therefore$ Green's theorem is verified. (Proved)

(52) Evaluate $\iint_S \bar{F}\cdot\hat{n}\,ds$, where $\bar{F} = 2xy\,\hat{i} + yz^2\,\hat{j} + xz\,\hat{k}$ and S is the surface of the parallelopiped bounded by $x=0, y=0, z=0;\ x=2, y=1, z=3$.

Plane OCDE : $x = 0$;
Plane AOEF : $y = 0$;
Plane OABC : $z = 0$;
" ABGF : $x = 2$;
" BCDG : $y = 1$;
" DEFG : $z = 3$;



By the Divergence theorem,

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \bar{\nabla}\cdot\bar{F}\,dV,$$ where V is the volume of the given parallelopiped.

$$= \iiint_V \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(2xy\,\hat{i} + yz^2\,\hat{j} + xz\,\hat{k}\right)dV$$

$$= \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3}(2y + z^2 + x)\,dx\,dy\,dz$$

$$= \left[y^2\right]_0^1\left[x\right]_0^2\left[z\right]_0^3 + \left[x\right]_0^2\left[y\right]_0^1\left[\frac{z^3}{3}\right]_0^3 + \left[\frac{x^2}{2}\right]_0^2\left[y\right]_0^1\left[z\right]_0^3$$

$$= 1.2.3 + 2.1.3^2 + 2.1.3 = 6 + 18 + 6 = \underline{30}.\ \text{(Ans)}$$

(54) Evaluate $\iint_S \bar{r}\cdot\hat{n}\,ds$, where

(a) S is the surface of the sphere of radius 2 and centre $(0,0,0)$,
(b) S " " " " cube bounded by $x=-1, y=-1, z=-1;\ x=1, y=1, z=1$.

(a) By the divergence theorem, $\iint_S \bar{r}\cdot\hat{n}\,ds = \iiint_V \bar{\nabla}\cdot\bar{r}\,dV$

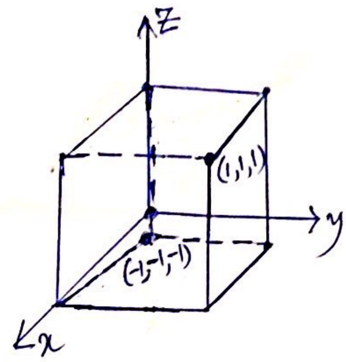
$$= \iiint_V (1+1+1)\,dV = 3\iiint_V dV.$$

Now the eqn of the sphere is given by $x^2+y^2+z^2=4$.
Into polar coordinates, $x = \rho\sin\theta\cos\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\theta$. $\therefore |J| = \rho^2\sin\theta$, $dV = dx\,dy\,dz$

$$\therefore 3\iiint_V dV = 3\int_{r=0}^{2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\rho^2\sin\theta\,d\rho\,d\theta\,d\phi = \left[\frac{\rho^3}{3}(-\cos\theta)\,\phi\right]_{r=0,\theta=0,\phi=0}^{2,\pi,2\pi}$$

$$= 3\times\frac{8}{3}\times 2\times 2\pi = \underline{32\pi}.\ \text{(Ans)}$$

(b) By the divergence theorem,



$$\iint_S \bar{r}\cdot\hat{n}\,ds = \iiint_V \bar{\nabla}\cdot\bar{r}\,dV$$

$$= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} 3\,dz\,dy\,dx$$

$$= 3\times 2\times 2\times 2 = \underline{\underline{24}}. \text{ (Ans)}$$

Ex: Use Stokes' theorem to prove that $\int_C \bar{r}\cdot d\bar{r} = 0$.

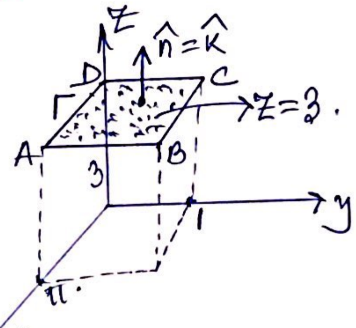
$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{F})\cdot\hat{n}\,ds$$

Let us put $\bar{F} = \bar{r}$, $\oint_C \bar{r}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{r})\cdot\hat{n}\,ds = \iint_S \bar{0}\cdot\hat{n}\,ds$

$$= 0 \text{ (proved)}.$$

Ex: Evaluate by Stokes' theorem

$\oint_\Gamma (\sin z\,dx - \cos x\,dy + \sin y\,dz)$, where Γ is the boundary of the rectangle $0\le x\le \pi,\ 0\le y\le 1,\ z=3$.



By Stokes' theorem,

$$\oint_{\Gamma:(ABCD)} \bar{F}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{F})\cdot\hat{n}\,ds, \text{ where}$$

$$\bar{F} = \sin z\,\hat{i} - \cos x\,\hat{j} + \sin y\,\hat{k}.$$

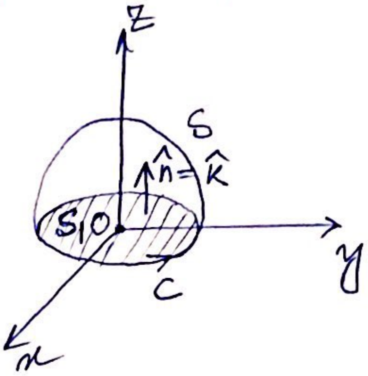
$$\therefore \bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin z & -\cos x & \sin y \end{vmatrix} = \cos y\,\hat{i} + \cos z\,\hat{j} + \sin x\,\hat{k}.$$

$$\hat{n} = \bar{\nabla}(z-3) = \hat{k}. \qquad \therefore (\bar{\nabla}\times\bar{F})\cdot\hat{n} = \sin x$$

$$\therefore \oint_\Gamma \bar{F}\cdot d\bar{r} = \iint_S \sin x\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \int_{y=0}^{1}\int_{x=0}^{\pi} \sin x\,dx\,dy = \int_{y=0}^{1} \left(-\cos x\right)_0^{\pi} dy$$

$$= 2\int_0^1 dy = \underline{2}. \text{ (Ans)}$$

Ex: Verify Stokes' theorem for $\bar{A} = (x-yz)\hat{i} + yx^2\hat{j} + y^2z\hat{k}$ over S where S is the upper half of the sphere $x^2+y^2+z^2=1$, C is its boundary.



C: $x^2+y^2=1$, $z=0$, in the xy plane.

S_1: the projection of S on the xy plane bounded by the circle C.

By Stokes' theorem,

$$\oint_C \bar{A}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{A})\cdot\hat{n}\, ds$$

Now to evaluate $\oint_C \bar{A}\cdot d\bar{r}$, let $x=\cos t$, $y=\sin t$, $z=0$ where $0\le t\le 2\pi$.

$$\therefore \oint_C \bar{A}\cdot d\bar{r} = \oint_C (x-yz)\,dx + yx^2\,dy + y^2z\,dz$$

$$= \oint_C x\,dx + yx^2\,dy \quad [\text{Since on } C,\ z=0;\ dz=0]$$

$$= \int_{t=0}^{2\pi} \cos t\, d(\cos t) + \sin t\cdot\cos^2 t\, d(\sin t)$$

$$= \int_{t=0}^{2\pi} [(-\sin t\cos t) + \sin t\cos^3 t]\, dt = \underline{0.}$$

Again, to evaluate the surface integral,

$$\bar{\nabla}\times\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \frac{\partial}{\partial z} \\ x-yz & yx^2 & y^2z \end{vmatrix} = 2yz\,\hat{i} - y\,\hat{j} + (2xy+z)\hat{k}$$

$$\iint_S (\bar{\nabla}\times\bar{A})\cdot\hat{n}\, ds = \iint_{S_1} (2xy+z)\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \left[\because \hat{n}=\hat{k},\ (\bar{\nabla}\times\bar{A})\cdot\hat{n} = 2xy+z,\ ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}\right]$$

$$= \iint_{S_1} 2xy\, dx\,dy \quad [\because z=0].$$

Let us put $x = r\cos\theta$, $y = r\sin\theta$, $|J| = r$, $dx\,dy = r\,dr\,d\theta$.

$$\therefore \iint_S (\bar{\nabla}\times\bar{A})\cdot\hat{n}\, ds = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta\cdot r\sin\theta\cdot r\, dr\, d\theta = \cancel{2}\left[\frac{r^4}{4}\right]_0^1 \int_{\theta=0}^{2\pi} \sin 2\theta\, d\theta$$

$$= \frac{1}{4}\times\frac{1}{2}(-\cos 2\theta)_0^{2\pi} = \frac{1}{8}(1-1) = \underline{0.}$$

$\therefore$ Stokes' theorem is verified, (proved).

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Ex: If $\bar{F}(x,y,z) = (x^2-y^2,\ 5-2xy,\ z)$, then show that $\iint_S F.\hat{n}\,ds = \frac{4}{3}\pi a^3$; S is the surface of a sphere of radius a.

By divergence theorem,

$$\iint_S \bar{F}.\hat{n}\,ds = \iiint_V \bar{\nabla}.\bar{F}\,dv = \iiint_V (\cancel{2x}-\cancel{2x}+1)\,dv$$

$$= \iiint_V dv = \underline{\frac{4}{3}\pi a^3}\ \left[\text{volume of a sphere of radius } a\right]$$

Ex. If $\bar{F} = \frac{\bar{r}}{r^3}$, evaluate $\iint_S \bar{F}.\hat{n}\,ds$, where S is the surface of the sphere $x^2+y^2+z^2=4$.

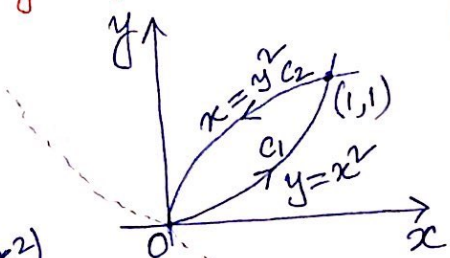
By divergence theorem, $\iint_S \bar{F}.\hat{n}\,ds = \iiint_V \bar{\nabla}.\bar{F}\,dv$

$$= \iiint_V \left[\sum \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right]dV = \iiint_V \sum\left[\frac{1}{r^3} - \frac{3x}{r^4}\cdot\frac{\partial r}{\partial x}\right]dv$$

$$= \iiint_V \sum\left[\frac{1}{r^3} - \frac{3x}{r^4}\cdot\frac{x}{r}\right]dv = \iiint_V \left[\frac{3}{r^3} - \frac{3}{r^5}(x^2+y^2+z^2)\right]dv$$

$$= \iiint_V \left[\frac{3}{r^3} - \frac{3}{r^5}\cdot r^2\right]dv = \underline{0}.$$

Ex. 37 Verify Green's theorem in the plane for $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$, where C is the boundary of the region defined by
(a) $y=\sqrt{x}$, $y=x^2$ (b) $x=0, y=0, x+y=1$.



(a) $y=\sqrt{x}$ & $y=x^2$ intersects at $(0,0)$ & $(1,1)$.

$$\oint_C = \oint_{C_1} + \oint_{C_2}.$$

Now $\oint_{C_1:(y=x^2)} (3x^2-8x^4)\,dx + (4x^2-6x^3)\,d(x^2)$

$$= \int_{x=0}^{1} (3x^2-8x^4+8x^3-12x^4)\,dx$$

$$= \int_{x=0}^{1} (3x^2-20x^4+8x^3)\,dx$$

$$= \left(x^3-4x^5+2x^4\right)_{x=0}^{1}$$

$$= 1-4+2 = -1.$$

$\oint_{C_2:(y=\sqrt{x})} (3y^4-8y^2)\,d(y^2)+(4y-6y^3)\,dy = \int_{y=1}^{0}(6y^5-16y^3+4y-6y^3)\,dy$

$$= \left[y^6-\frac{11}{2}y^4+2y^2\right]_{y=1}^{0} = -1+\frac{11}{2}-2 = \frac{5}{2}$$

$$\therefore \oint_C = \oint_{C_1} + \oint_{C_2} = -1 + \frac{5}{2} = \frac{3}{2}.$$

Now
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(4y-6xy) - \frac{\partial}{\partial y}(3x^2-8y^2)\right] dx\,dy.$$

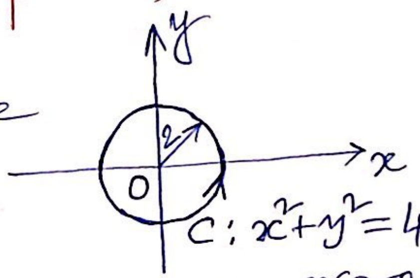
$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (-6y+16y)\,dy\,dx = \int_{x=0}^{1}\left[5y^2\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= 5\int_{x=0}^{1}(x-x^4)\,dx = 5\left[\frac{x^2}{2}-\frac{x^5}{5}\right]_{x=0}^{1} = 5\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$= 5\times\frac{3}{10} = \frac{3}{2}$$

$\therefore$ Green's theorem is verified.

Ex. 38. Evaluate $\oint_C (3x+4y)\,dx + (2x-3y)\,dy$, where C, a circle of radius 2 with centre at the origin of the xy plane, is traversed in the +ve sense.



$C: x^2+y^2=4$

By Green's theorem, we have

$$\oint_C (3x+4y)\,dx + (2x-3y)\,dy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} |J| = r.$

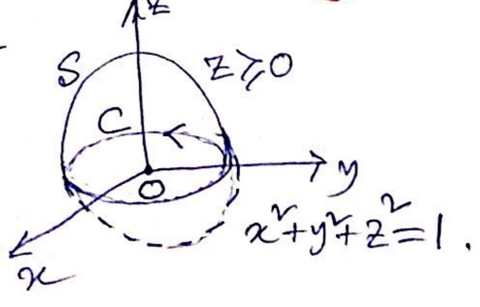
$$= \int_{r=0}^{2}\int_{\theta=0}^{2\pi} (2-4)\,r\,dr\,d\theta = -2\left[\frac{r^2}{2}\right]_0^2\left[\theta\right]_0^{2\pi} = -2\cdot 2\cdot 2\pi$$

$$= -8\pi. \quad \text{Ans}$$

Ex. Verify Stokes' theorem for the vector function $\bar{F} = (2x-y, -yz^2, -y^2z)$ where S is the surface of $x^2+y^2+z^2=1$, $z \geq 0$ and C is its boundary.

Stokes' theorem states that

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds$$



C is the perimeter of $x^2+y^2=1$, $z=0$.
Let $x=\cos\theta$, $y=\sin\theta$ be the parametric eqns of $x^2+y^2=1$.

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C (2x-y)\,dx \quad [\text{Since } z=0]$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)\, d(\cos\theta) = \int_{\theta=0}^{2\pi} (-\sin 2\theta + \sin^2\theta)\, d\theta$$

$$= \int_{\theta=0}^{2\pi} (-\sin 2\theta)\, d\theta + \frac{1}{2}\int_{\theta=0}^{2\pi} (1-\cos 2\theta)\, d\theta$$

$$= \left[\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{2\cdot 2}\right]_{\theta=0}^{2\pi} = \frac{2\pi}{2} = \pi.$$

Now $\hat{n} = \dfrac{\bar{\nabla}\phi}{|\bar{\nabla}\phi|} = \dfrac{2(x\hat{i}+y\hat{j}+z\hat{k})}{2\sqrt{x^2+y^2+z^2}} = x\hat{i}+y\hat{j}+z\hat{k}$,

Since $\phi = x^2+y^2+z^2-1=0$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = (-2yz+2yz)\hat{i} + 0\hat{j} + 1\hat{k} = \hat{k}.$$

$\therefore \iint_S (\bar{\nabla}\times\bar{F})\cdot\hat{n}\, ds$,

$$= \iint_S \hat{k}\cdot(x\hat{i}+y\hat{j}+z\hat{k})\,ds = \iint_S z\, ds$$

$$= \iint_R z\cdot\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}, \text{ where } R \text{ is the projection of } S \text{ on the } xy \text{ plane: } x^2+y^2=1, z=0.$$

$$= \iint_R \frac{z}{z}\,dx\,dy = \iint_R dx\,dy = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r\,dr\,d\theta = \left[\frac{r^2}{2}\right]_0^1 \left[\theta\right]_0^{2\pi} = \frac{1}{2}\times 2\pi = \pi.$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{F})\cdot\hat{n}\, ds.$$

Stokes' theorem is hereby verified.

---

53. Verify the divergence theorem for $\bar{A} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$, taken over the region in the first octant bounded by $y^2+z^2=9$ and $x=2$.

By the divergence theorem, $\iiint_V \bar{\nabla}\cdot\bar{A}\,dV = \iint_S \bar{A}\cdot\hat{n}\,ds$.

Now
$$\iiint_V \bar{\nabla}\cdot\bar{A}\,dV = \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}}\int_{x=0}^{2} (4xy - 2y + 8xz)\,dx\,dy\,dz.$$

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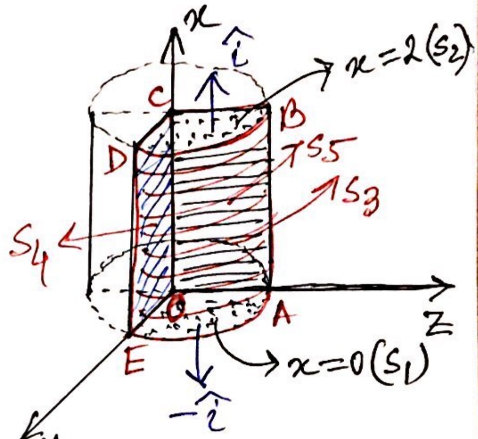
$$= \int_{z=0}^{3} \int_{y=0}^{\sqrt{9-z^2}} \left[2x^2y - 2xy + 4x^2z\right]_{x=0}^{2} dy\,dz = \int_{z=0}^{3} \int_{y=0} (4y+16z)\,dy\,dz.$$

$$= \int_{z=0}^{3} \left[2y^2 + 16yz\right]_{y=0}^{\sqrt{9-z^2}} dz = \int_{z=0}^{3} \left[2(9-z^2) + 16z\sqrt{9-z^2}\right] dz$$

$$= \left[18z - \frac{2}{3}z^3 - 8\int \sqrt{9-z^2}\,.\,d(9-z^2)\right]_{z=0}^{3} = 54 - 18 - 8\times\frac{2}{3}\left[(9-z^2)^{3/2}\right]_{z=0}^{3}$$

$$= 36 + \frac{16}{3}\times 9^{3/2} = 36 + \frac{16}{3}\times 3^3 = 36 + 16\times 9 = \underline{180}.$$

Evaluation of $\iint_S \bar{A}\cdot\hat{n}\,ds$.



Let OAEO be the S_1 surface
BCDB " " S_2 "
ABDEA " " S_3 "
OCDEO " " S_4 "
OABCO " " S_5 "

$\therefore S = S_1 + S_2 + S_3 + S_4 + S_5$.

On S_1: $x=0$, $\hat{n} = -\hat{i}$, $\bar{A}\cdot\hat{n} = -2x^2y = 0$.

$$\therefore \iint_{S_1} \bar{A}\cdot\hat{n}\,ds_1 = 0.$$

On S_2: $x=2$, $\hat{n} = \hat{i}$, $A\cdot\hat{n} = 2x^2y = 8y$.

$$\therefore \iint_{S_2} \bar{A}\cdot\hat{n}\,ds_2 = \iint_{S_2} 8y\,dy\,dz = \int_{r=0}^{3}\int_{\theta=0}^{\pi/2} 8r\cos\theta\; r\,dr\,d\theta$$

$$= \frac{8}{3}\left[r^3\right]_0^3 \cdot \left[\sin\theta\right]_0^{\pi/2} = \frac{8}{3}\times 3^3 = \underline{72}.$$

On S_3: $(y^2+z^2=9)$, $\hat{n} = \dfrac{2y\hat{j} + 2z\hat{k}}{\sqrt{4(y^2+z^2)}} = \dfrac{y\hat{j}+z\hat{k}}{3}$.

$\bar{A}\cdot\hat{n} = \dfrac{-y^3 + 4xz^3}{3}$; Let R_3 be the projection of S_3 taken over xy-plane.

then $$\iint_{S_3} \bar{A}\cdot\hat{n}\,ds_3 = \iint_{R_3} \frac{-y^3+4xz^3}{3}\cdot\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{1}{3}\iint_{R_3}(4xz^3 - y^3)\cdot\frac{dx\,dy}{z/3}$$

$$= \iint_{R_3}\left[4xz^2 - \frac{y^3}{\sqrt{9-y^2}}\right]dx\,dy = \iint_{R_3}\left[4x(9-y^2) - \frac{y^2\cdot y}{\sqrt{9-y^2}}\right]dx\,dy$$

$$= \int_{y=0}^{3}\int_{x=0}^{2} 4x(9-y^2)\,dx\,dy - \int_{y=0}^{3}\int_{x=0}^{2}\frac{y^2\cdot y}{\sqrt{9-y^2}}\,dx\,dy.$$

$$= \left[2x^2\left(9y - \frac{y^3}{3}\right)\right]_{x=0,\, y=0}^{2,\,3} - [x]_0^2 \int_{t=0}^{3} \frac{(9-t^2).\, t\,dt}{t} \quad \left[\begin{array}{l} \text{Let } 9-y^2=t^2 \\ -y\,dy = t\,dt \end{array}\right]$$

$$= 8(27-9) - 2.\left[9t - \frac{t^3}{3}\right]_{t=0}^{3} = 8\times 18 - 2[27-9] = 8\times 18 - 2\times 18$$

$$= 6\times 18 = \underline{108}.$$

On S_4: $\hat{n} = -\hat{k}$, $z=0$, $\bar{A}.\hat{n} = -4xz^2 = 0$.

$$\therefore \iint_{S_4} \bar{A}.\hat{n}\, ds_4 = \underline{0}.$$

On S_5: $\hat{n} = -\hat{j}$, $y=0$, $\bar{A}.\hat{n} = +y^2 = 0$

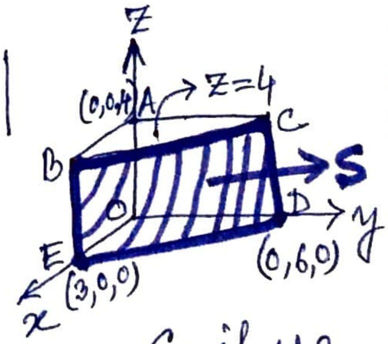
$$\therefore \iint_{S_5} \bar{A}.\hat{n}\, ds_5 = \underline{0}.$$

$$\therefore \iint_S \bar{A}.\hat{n}\, ds = 0 + 72 + 108 + 0 + 0 = \underline{180} \quad \left[\begin{array}{l}\text{Divergence} \\ \text{Th. is verified}\end{array}\right]$$
$$= \iiint_V \bar{\nabla}.\bar{A}\, dv.$$

58. Evaluate $\iint_S \bar{A}.\hat{n}\, ds$; (a) $\bar{A} = y\hat{i} + 2x\hat{j} - z\hat{k}$ and S is the surface of the plane $2x+y=6$ in the first octant cut-off by the plane $z=4$.
(b) $\bar{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the 1st octant.

(a) The given surface is S (BCDE).
$\hat{n}$ for S is given by $\dfrac{\bar{\nabla}(2x+y)}{|\bar{\nabla}(2x+y)|}$

i.e, $\hat{n} = \dfrac{2\hat{i}+\hat{j}}{\sqrt{2^2+1^2}} = \dfrac{2\hat{i}+\hat{j}}{\sqrt{5}}$.



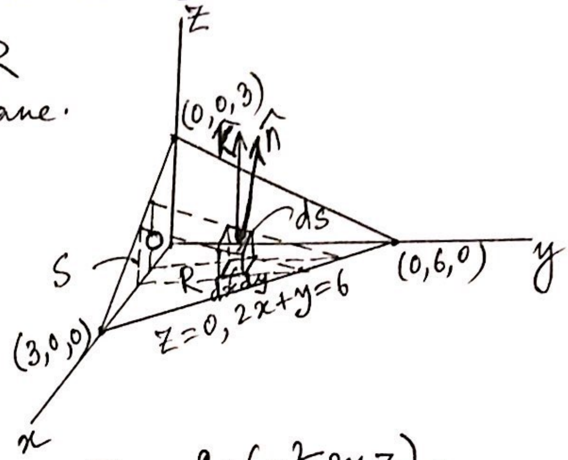
Here $\hat{n}$ is parallel to the xy-plane. So if we take the projection of S on the xy-plane, it will be zero. So projection of S on the xy-plane is not possible, as S is perpendicular to the xy-plane.
Let us take projection R of S on the xz-plane

$$\therefore \iint_S \bar{A}.\hat{n}\, ds = \iint_R \bar{A}.\hat{n}\, \frac{dx\,dz}{|\hat{n}.\hat{j}|} = \iint_R \frac{2y+2x}{\sqrt{5}} \cdot \frac{dx\,dz}{1/\sqrt{5}}$$

$$= 2\int_{x=0}^{3}\int_{z=0}^{4} (x + 6 - 2x)\, dx\,dz = 2\left[6x - \frac{x^2}{2}\right]_0^3 [z]_0^4 = \underline{108}.$$

(b) $\bar{A} = (x+y^2, -2x, 2yz)$; $S: 2x+y+2z=6$ in 1st octant.

Let us take the projection R of the surface S on the xy-plane.



$$\hat{n} = \frac{\bar{\nabla}(2x+y+2z)}{|\bar{\nabla}(2x+y+2z)|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}.$$

$$\bar{A}.\hat{n} = \frac{1}{3}\left[2(x+y^2) - 2x + 2\cdot 2yz\right] = \frac{2}{3}(y^2+2yz).$$

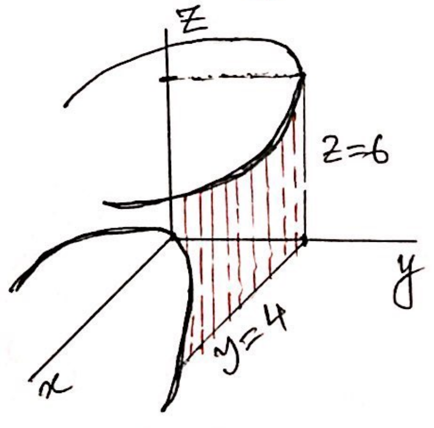
$$\iint_S \bar{A}.\hat{n}\,ds = \iint_R \frac{2}{3}(y^2+2yz)\cdot\frac{dx\,dy}{|\hat{n}.\hat{k}|} = \iint_R \frac{2}{3}(y^2+2yz)\frac{dx\,dy}{2/3}$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} y(y+2z)\,dx\,dy = \int_{x=0}^{3}\int_{y=0}^{6-2x} y\cdot(6-2x)\,dy\,dx \quad \left[\begin{array}{c}\text{Since}\\ y+2z=6-2x\end{array}\right]$$

$$= \int_{x=0}^{3}\left[\frac{y^2}{2}\right]_0^{6-2x}\cdot(6-2x)\,dx = \frac{1}{2}\int_{x=0}^{3}(6-2x)^3\,dx = \frac{1}{2}\times -\frac{1}{2}\left[\frac{(6-2x)^4}{4}\right]_{x=0}^{3}$$

$$= \frac{1}{16}\times 6^4 = \frac{\cancel{36}^{9}\times\cancel{36}^{9}}{\cancel{16}_{4}} = \underline{81}. \text{ (Ans)}.$$

59. If $F = (2y, -z, x^2)$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y=4$ and $z=6$, evaluate $\iint_S F.\hat{n}\,ds$.



$$\hat{n} = \frac{\bar{\nabla}(8x-y^2)}{|\bar{\nabla}(8x-y^2)|} = \frac{4\hat{i}-y\hat{j}}{\sqrt{16+y^2}}$$

$$\bar{F}.\hat{n} = \frac{8y+yz}{\sqrt{16+y^2}} = \frac{y(8+z)}{\sqrt{16+y^2}}.$$

Let R be the projection of S taken on the xz-plane.

$$\therefore \iint_S F.\hat{n}\,ds = \iint_R \frac{y(8+z)}{\sqrt{16+y^2}}\cdot\frac{dx\,dz}{|\hat{n}.\hat{j}|}$$

$$= \int_{x=0}^{2}\int_{z=0}^{6}\frac{y(8+z)}{\sqrt{16+y^2}}\cdot\frac{dx\,dz}{y/\sqrt{16+y^2}} = \int_{x=0}^{2}\int_{z=0}^{6}(8+z)\,dx\,dz$$

$$= \left[8xz + x\frac{z^2}{2}\right]_{x=0,\,z=0}^{2,\,6} = 8\times2\times6 + 2\times\frac{6^2}{2} = 96+36$$

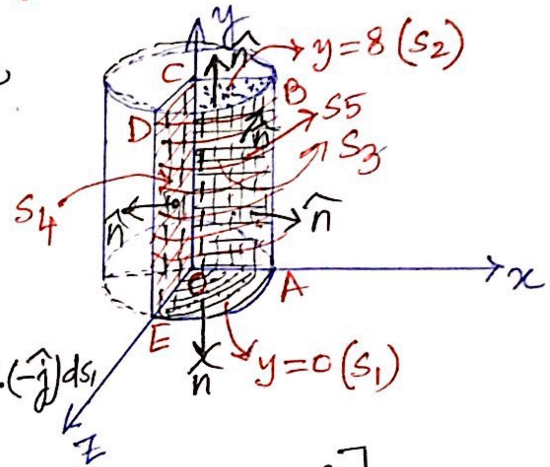
$$= \underline{\underline{132}}. \text{ (Ans)}$$

Spiegel. / Vector Surface Integrals. /

60. Evaluate $\iint_S \bar{A}\cdot\hat{n}\,ds$ over the entire surface S of the region bounded by the cylinder $x^2+z^2=9$, $x=0$, $y=0$, $z=0$ and $y=8$, if $\bar{A}=6z\hat{i}+(2x+y)\hat{j}-x\hat{k}$.

Let OAEO be the S_1 surface,
BCDB " " S_2 "
ABDEA " " S_3 "
OCDEO " " S_4 "
OABCO " " S_5 "

$\therefore$ $S=S_1+S_2+S_3+S_4+S_5$



$$\therefore \iint_{S_1} A\cdot\hat{n}\,dS_1 = \iint_{S_1:(y=0)} \left(6z\hat{i}+(2x+y)\hat{j}-x\hat{k}\right)\cdot(-\hat{j})\,dS_1$$

$$= \iint_{S_1}(-2x)\,dx\,dz \quad [\because y=0 \text{ on } S_1].$$

[put $x=r\cos\theta$, $z=r\sin\theta$]

$$= \int_{r=0}^{3}\int_{\theta=0}^{\pi/2}(-2r\cos\theta)\,r\,dr\,d\theta = \left[-\frac{2}{3}r^3\sin\theta\right]_{r=0,\,\theta=0}^{3,\,\pi/2}$$

$$= -\frac{2}{3}\cdot 3^3\cdot 1 = \underline{-18}.$$

$$\iint_{S_2}\bar{A}\cdot\hat{n}\,dS_2 = \iint_{S_2:(y=8)}(2x+y)\,dx\,dz = \iint_{S_2}(2x+8)\,dx\,dz, \left[\because \hat{n}=\hat{j} \ \& \ y=8\right]$$

$$= \int_{r=0}^{3}\int_{\theta=0}^{\pi/2}(2r\cos\theta+8)\,r\,dr\,d\theta = \left[2\cdot\frac{r^3}{3}\sin\theta+4r^2\theta\right]_{r=0,\,\theta=0}^{3,\,\pi/2}$$

$$= 2\cdot\frac{3^3}{3}\cdot 1 + 4\cdot 3^2\cdot\pi/2 = \underline{18+18\pi}.$$

On S_3: $(x^2+z^2=9)$, $\hat{n}=\dfrac{\bar{\nabla}(x^2+z^2)}{|\bar{\nabla}(x^2+z^2)|}=\dfrac{x\hat{i}+z\hat{k}}{3}$

$$\therefore A\cdot\hat{n} = \frac{6xz-xz}{3} = \frac{5}{3}xz.$$

Let us take projection R_3 of S_3 on the yz-plane

$$\therefore \iint_{S_3}\bar{A}\cdot\hat{n}\,dS_3 = \iint_{R_3}\frac{5}{3}xz\cdot\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \frac{5}{3}\iint_{R_3}xz\cdot\frac{dy\,dz}{x/3}$$

$$= 5\iint_{R_3}z\,dy\,dz = 5\int_{y=0}^{8}\int_{\theta=0}^{\pi/2}3\sin\theta\cdot dy\,d(3\sin\theta), \ \left[\text{where } x=3\cos\theta,\ z=3\sin\theta\right]$$

$$= 45\int_{y=0}^{8}\int_{\theta=0}^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta\, dy = \frac{45}{2}\left[y\right]_0^8 \left[-\frac{\cos 2\theta}{2}\right]_{\theta=0}^{\pi/2}$$

$$= \frac{45}{4}\times 8 \times (1+1) = 45\times 4 = \underline{180}.$$

<u>on S_4</u> : $\hat{n} = -\hat{i}$; $\bar{A}\cdot\hat{n} = -6z$

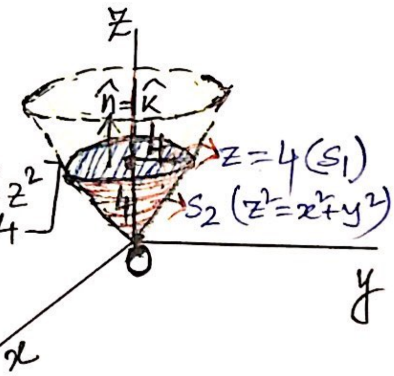
$$\iint_{S_4} \bar{A}\cdot\hat{n}\, ds_4 = \int_{y=0}^{8}\int_{z=0}^{3} (-6z)\, dy\, dz = -3\left[z^2\right]_0^3 \left[y\right]_0^8 = \underline{-216}.$$

<u>on S_5</u>; $\hat{n} = -\hat{k}$; $\bar{A}\cdot\hat{n} = x$.

$$\iint_{S_5} \bar{A}\cdot\hat{n}\, ds_5 = \int_{y=0}^{8}\int_{x=0}^{3} x\, dx\, dy = \left[y\right]_0^8 \left[\frac{x^2}{2}\right]_0^3 = 8\times\frac{3^2}{2} = \underline{36}.$$

$$\therefore \iint_S \bar{A}\cdot\hat{n}\, ds = -18 + (18 + 18\pi) + 180 - 216 + 36 = \underline{18\pi}. \text{ (Ans)}$$

<u>62</u>. Evaluate $\iint_S \bar{A}\cdot\hat{n}\, ds$ over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2+y^2$ and the plane $z=4$, if $\bar{A} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$.



$S = S_1 + S_2$.

on S_1 : $z=4$; $\hat{n} = \hat{k}$, $\bar{A}\cdot\hat{n} = 3z = 3\times 4 = 12$.

$$\iint_{S_1} \bar{A}\cdot\hat{n}\, ds_1 = \iint_{S_1} 12\, ds_1 = 12\,\pi\cdot 4^2 \quad \left[\because x^2+y^2=z^2 \ \&\ z=4\right]$$

$$= \underline{192\pi}.$$

on S_2 : $(z^2 = x^2+y^2)$;

$$\hat{n} = \frac{\bar{\nabla}(x^2+y^2-z^2)}{|\bar{\nabla}(x^2+y^2-z^2)|} = \frac{x\hat{i}+y\hat{j}-z\hat{k}}{\sqrt{2}\,z}\ ;\ \bar{A}\cdot\hat{n} = \frac{1}{\sqrt{2}}(4x^2+xy^2z-3z).$$

Let us take projection R on xy plane.

$$\iint_{S_2} \bar{A}\cdot\hat{n}\, ds_2 = \iint_R \frac{1}{\sqrt{2}}(4x^2+xy^2z-3z)\cdot\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \iint_R \frac{\frac{1}{\sqrt{2}}(4x^2+xy^2z-3z)\,dx\,dy}{(1/\sqrt{2})}$$

$$= \int_{z=0}^{4}\int_{\theta=0}^{2\pi} (4z^2\cos^2\theta + z^4\cos\theta\sin^2\theta - 3z)\, z\, d\theta\, dz \quad \left[\begin{array}{l}\text{put } x = z\cos\theta,\\ y = z\sin\theta\\ dx\,dy = z\,d\theta\,dz\end{array}\right]$$

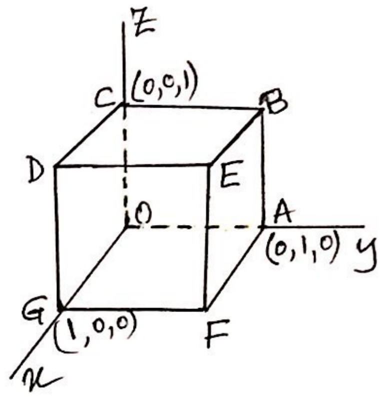
$$= \left[\frac{z^4}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + \left[\frac{z^6}{6}\right]\frac{\sin^3\theta}{3} - z^3\cdot\theta\right]_{z=0,\ \theta=0}^{4,\ 2\pi}$$

$$= \left[\frac{4^4}{2}\cdot 2\pi - 4^3\cdot 2\pi\right] = (256\pi - 128\pi) = 128\pi.$$

$$\therefore \iint_S \bar{A}\cdot\hat{n}\, ds = 192\pi + 128\pi = \underline{320\pi}. \text{ (Ans)}$$

(61). Evaluate $\iint_S \bar{r}\cdot\hat{n}\,ds$ over: (a) the surfaces of the unit cube bounded by the coordinate planes and the planes $x=1$, $y=1$, $z=1$; (b) the surface of a sphere of radius a with centre at $(0,0,0)$.

The cube has six (6) faces:



Face DEFG: $\hat{n}=\hat{i}$, $x=1$.

$\therefore \bar{r}=\hat{i}+y\hat{j}+z\hat{k}$

$\therefore \bar{r}\cdot\hat{n}=1$.

$\therefore \iint_{DEFG} \bar{r}\cdot\hat{n}\,ds = \int_0^1\int_0^1 1\,dy\,dz = \underline{1}.$

Face ABCO: $\hat{n}=-\hat{i}$, $x=0$.

$\therefore \bar{r}=y\hat{j}+z\hat{k}$; $\bar{r}\cdot\hat{n}=0$

$\therefore \iint_{ABCO} \bar{r}\cdot\hat{n}\,ds = \underline{0}$.

Face ABEF: $\hat{n}=\hat{j}$, $y=1$, $\bar{r}=x\hat{i}+\hat{j}+z\hat{k}$; $\bar{r}\cdot\hat{n}=1$.

$\therefore \iint_{ABEF} \bar{r}\cdot\hat{n}\,ds = \int_0^1\int_0^1 1\,dx\,dz = \underline{1}.$

Face OGDC: $\hat{n}=-\hat{j}$, $y=0$, $\bar{r}=x\hat{i}+z\hat{k}$; $\bar{r}\cdot\hat{n}=0$.

$\therefore \iint_{OGDC} \bar{r}\cdot\hat{n}\,ds = \underline{0}$.

Face AFGO: $\hat{n}=-\hat{k}$, $z=0$, $\bar{r}=x\hat{i}+y\hat{j}$; $\bar{r}\cdot\hat{n}=0$

$\therefore \iint_{AFGO} \bar{r}\cdot\hat{n}\,ds = \iint 0\,ds = \underline{0}$.

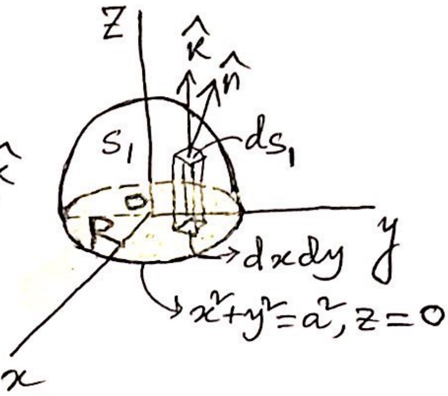
Face BCDE: $\hat{n}=\hat{k}$, $z=1$, $\bar{r}=x\hat{i}+y\hat{j}+\hat{k}$; $\bar{r}\cdot\hat{n}=1$.

$\iint_{BCDE} \bar{r}\cdot\hat{n}\,ds = \int_{x=0}^1\int_{y=0}^1 1\,dx\,dy = \underline{1}$,

Adding the integrals over all the faces, we get $\iint_S \bar{r}\cdot\hat{n}\,ds = 1+0+1+0+0+1 = \underline{\underline{3}}$. (Ans)

(b) Equation of the sphere is

$$x^2+y^2+z^2=a^2.$$



$$\hat{n} = \frac{\overline{\nabla}(x^2+y^2+z^2)}{|\overline{\nabla}(x^2+y^2+z^2)|} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

$$\overline{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

$$\therefore\ \overline{r}\cdot\hat{n} = \frac{x^2+y^2+z^2}{a}.$$

Let S_1 be the surface of the sphere above the xy-plane. Let us take the projection of S_1 on the xy-plane which is R bounded by the circle $x^2+y^2=a^2$, $z=0$.

Then $$\iint_{S_1} \overline{r}\cdot\hat{n}\, ds_1 = \iint_R \frac{x^2+y^2+z^2}{a}\cdot\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_R \frac{x^2+y^2+z^2}{a}\cdot\frac{dx\,dy}{z/a}.$$

$$= \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a^2}{\sqrt{a^2-x^2-y^2}}\,dx\,dy \quad ; \text{ Let us put } x = r\cos\theta,\ y = r\sin\theta.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{a^2}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta = -\int_{\theta=0}^{2\pi}\left[a^2\sqrt{a^2-r^2}\right]_{r=0}^{a} d\theta$$

$$= \int_{\theta=0}^{2\pi} a^3\, d\theta = a^3 \times 2\pi = 2\pi a^3.$$

Total surface of the sphere is $S = 2S_1$

$$\therefore\ \iint_S \overline{r}\cdot\hat{n}\, ds = 2\iint_{S_1} \overline{r}\cdot\hat{n}\, ds_1 = 2\times 2\pi a^3$$

$$= \underline{\underline{4\pi a^3}}.\ (\text{Ans}).$$

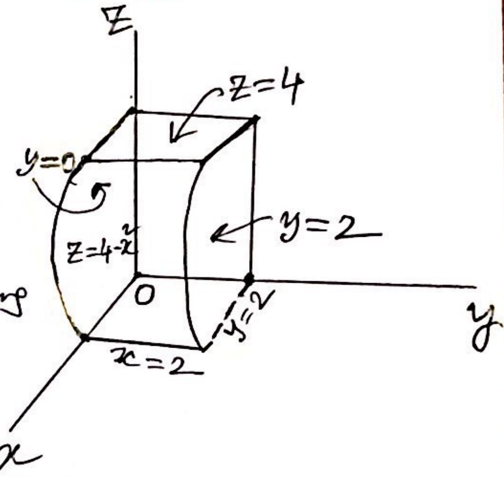
(69) Spiegel. Evaluate $\iiint_V (2x+y)\,dV$, where V is the closed region bounded by the cylinder $z=4-x^2$ and the planes $x=0$, $y=0$, $y=2$ and $z=0$.

The region V is covered by

(i) keeping x and y fixed and integrating from $z=0$ to $z=4-x^2$, then by

(ii) keeping x fixed and integrating from $y=0$ to $y=2$, and

(iii) finally integrating from $x=0$ to $x=2$ (where $z=4-x^2$ meets $z=0$).



Then the required integral is

$$\int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2}(2x+y)\,dz\,dy\,dx.$$

$$=\int_{x=0}^{2}\int_{y=0}^{2}(2x+y)\left[z\right]_0^{4-x^2}dy\,dx$$

$$=\int_{x=0}^{2}\int_{y=0}^{2}(2x+y)(4-x^2)\,dy\,dx$$

$$=\int_{x=0}^{2}\left[2x(4-x^2)y+\frac{y^2}{2}(4-x^2)\right]_{y=0}^{2}dx$$

$$=\int_{x=0}^{2}(4x+2)(4-x^2)\,dx=\int_{x=0}^{2}(16x+8-4x^3-2x^2)\,dx$$

$$=\left[8x^2+8x-x^4-\frac{2}{3}x^3\right]_{x=0}^{2}=32+\cancel{16}-\cancel{16}-\frac{2}{3}\times 2^3$$

$$=32-\frac{16}{3}=\frac{96-16}{3}$$

$$=\frac{80}{3}.\ \text{(Ans)}.$$

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Spiegel.

(63) Let R be the projection of a surface S on the xy-plane. Prove that the surface area of S is given by $\iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$, if the equation for S is $z = f(x,y)$.

Solution: given the equation for S is $z = f(x,y)$.

Let $\phi(x,y,z) = z - f(x,y)$.

Unit normal to the surface S is given by

$$\hat{n} = \frac{\bar{\nabla}\phi}{|\bar{\nabla}\phi|} = \frac{-\hat{i}\frac{\partial f}{\partial x} - \hat{j}\frac{\partial f}{\partial y} + \hat{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}}$$

The surface area S is given by

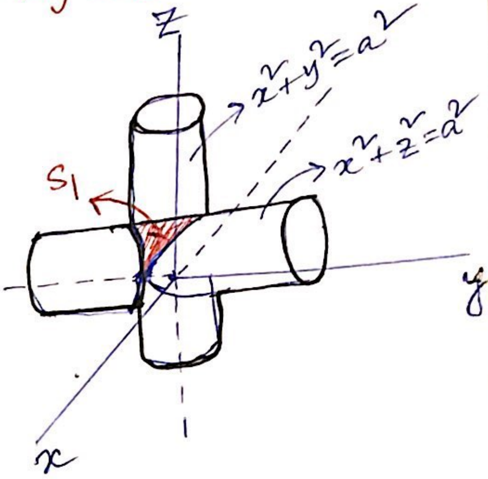
$$\iint_S dS = \iint_R \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \iint_R \frac{dx\,dy}{\frac{1}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}}}$$

Where R is the projection of S on the xy-plane

$$= \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy, \quad \text{where } \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x},\ \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}.$$

65. Find the surface area of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Its surface consists of 4 elements of equal area S_1. The projection of S_1 on the xy plane is a circle of radius a.



Let us consider the surface of the cylinder $x^2+z^2=a^2 \longrightarrow (1)$

$$a, \quad Z=\sqrt{a^2-x^2} \longrightarrow (2)$$

Let R be the projection of the surface S on the xy plane. The eqn of the surface is given by

$$Z=f(x,y) \longrightarrow (3)$$

Then we have the surface area S which is given by

$$\iint_R \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\, dx\,dy \longrightarrow (4)$$

$$= \iint_R \sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2+0}\; dx\,dy \quad [\text{using } (2) \,\&\, (3)]$$

$$= \iint_R \frac{a}{\sqrt{a^2-x^2}}\, dx\,dy \longrightarrow (5)$$

$$= \int_{x=-a}^{a} \frac{a}{\sqrt{a^2-x^2}} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \; dx$$ [Since the projection of S_1 on the xy plane is a circle of radius a.]

$$= \int_{x=-a}^{a} \frac{a}{\sqrt{a^2-x^2}} \cdot \Big[y\Big]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=-a}^{a} \frac{a}{\sqrt{a^2-x^2}} \cdot 2\sqrt{a^2-x^2}\, dx = 2a\Big[x\Big]_{-a}^{a} = 4a^2.$$

Since the total area consists of four areas $4a^2$, we have $S=4S_1=4\cdot 4a^2=\underline{16a^2}$. (Ans)