Student's t-distribution.

Let X and Y be independent random variables where $X \sim N(0,1)$ and $Y \sim \chi^2_n$ then $T = \frac{X}{\sqrt{Y/n}}$ is said to have t-distn on n d.f.

$T \sim t_n$.

$$f_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)\sqrt{\pi n}} \left(1+t^2/n\right)^{-\left(\frac{n+1}{2}\right)}, \quad -\infty < t < \infty$$

$$= \frac{1}{\sqrt{n}\, B\left(\frac{n}{2}, \frac{1}{2}\right)} \left(1+t^2/n\right)^{-\left(\frac{1+n}{2}\right)}, \quad -\infty < t < \infty.$$

The pdf of T is symmetric about 0.

Hence moments of odd order $= 0$, provided they exist.

For even ordered moment

$$E(T^k) = \frac{n^{k/2}\, \Gamma\frac{k+1}{2}\, \Gamma\frac{n-k}{2}}{\Gamma 1/2\; \Gamma n/2}, \quad k < n.$$

$E(T) = 0$, $E(T^2) = V(T) = \frac{n}{n-2}$, $n > 2$.

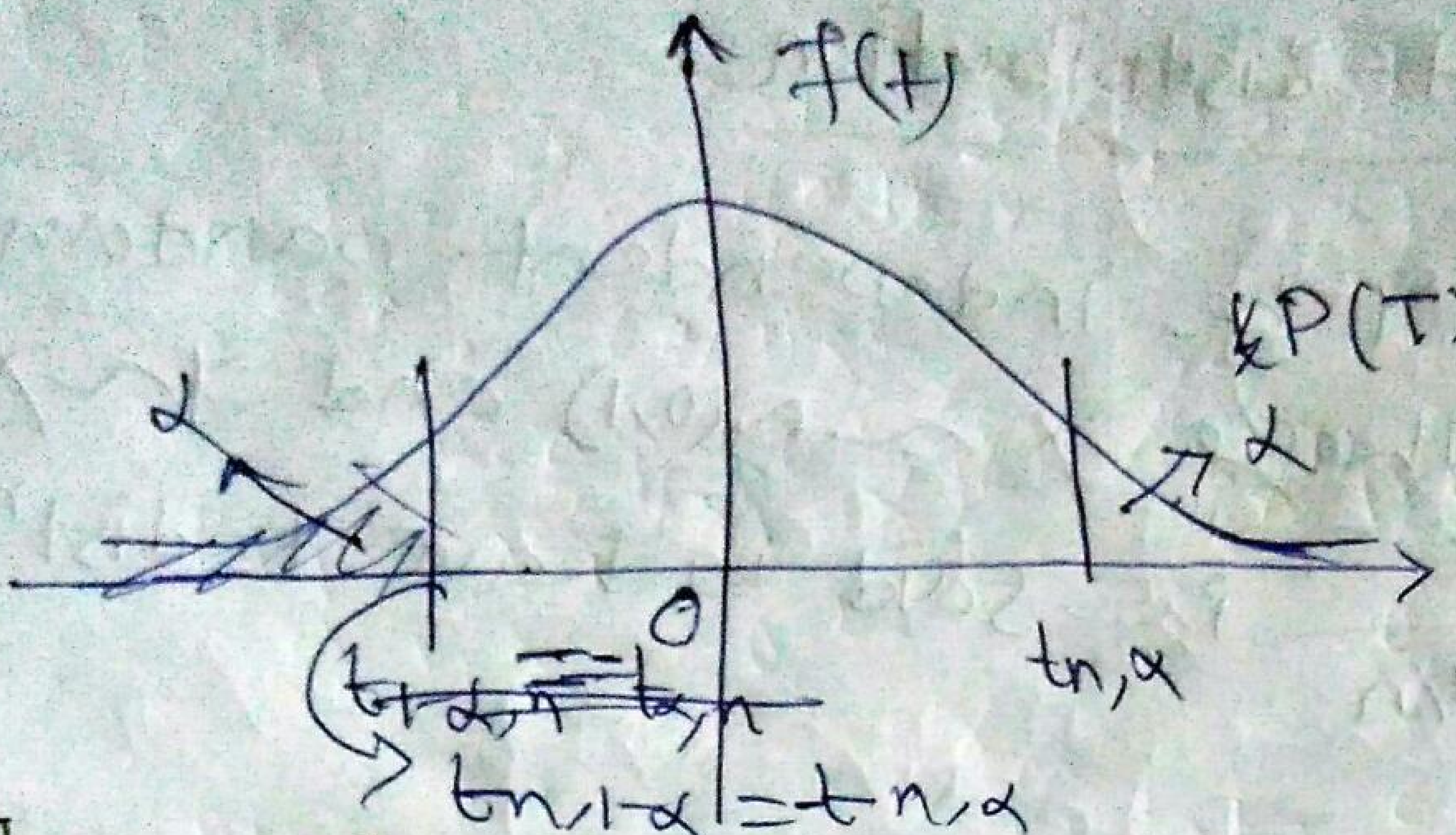
$\mu_4 = E(T^4) = \frac{3n^2}{(n-2)(n-4)}$, $n > 4$.

$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{6}{n-4} > 0$

↳ measure of kurtosis

So, the density is leptokurtic.

(As $n \to \infty$ pdf $\to$ pdf of $N(0,1)$).



$P(T \geq t_{n,\alpha}) = \alpha$

upper $100\alpha\%$ point of t_n distn

Theorem: Let $T \sim t_n$, as $n \to \infty$ the pdf of T converges to $\phi(t)$.

Remarks: For $n > 30$ the approx is very good.

Let $X_1, X_2, \ldots, X_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$.

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$ and $\bar{X}$ and S^2 are independent

So, $$\frac{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}{\sqrt{\frac{(n-1)S^2}{\sigma^2(n-1)}}} \sim t_{n-1}$$

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

$$\therefore \sqrt{n}\frac{(\bar{X}-\mu)}{S} \sim t_{n-1}.$$

Snedcor's F-distribution

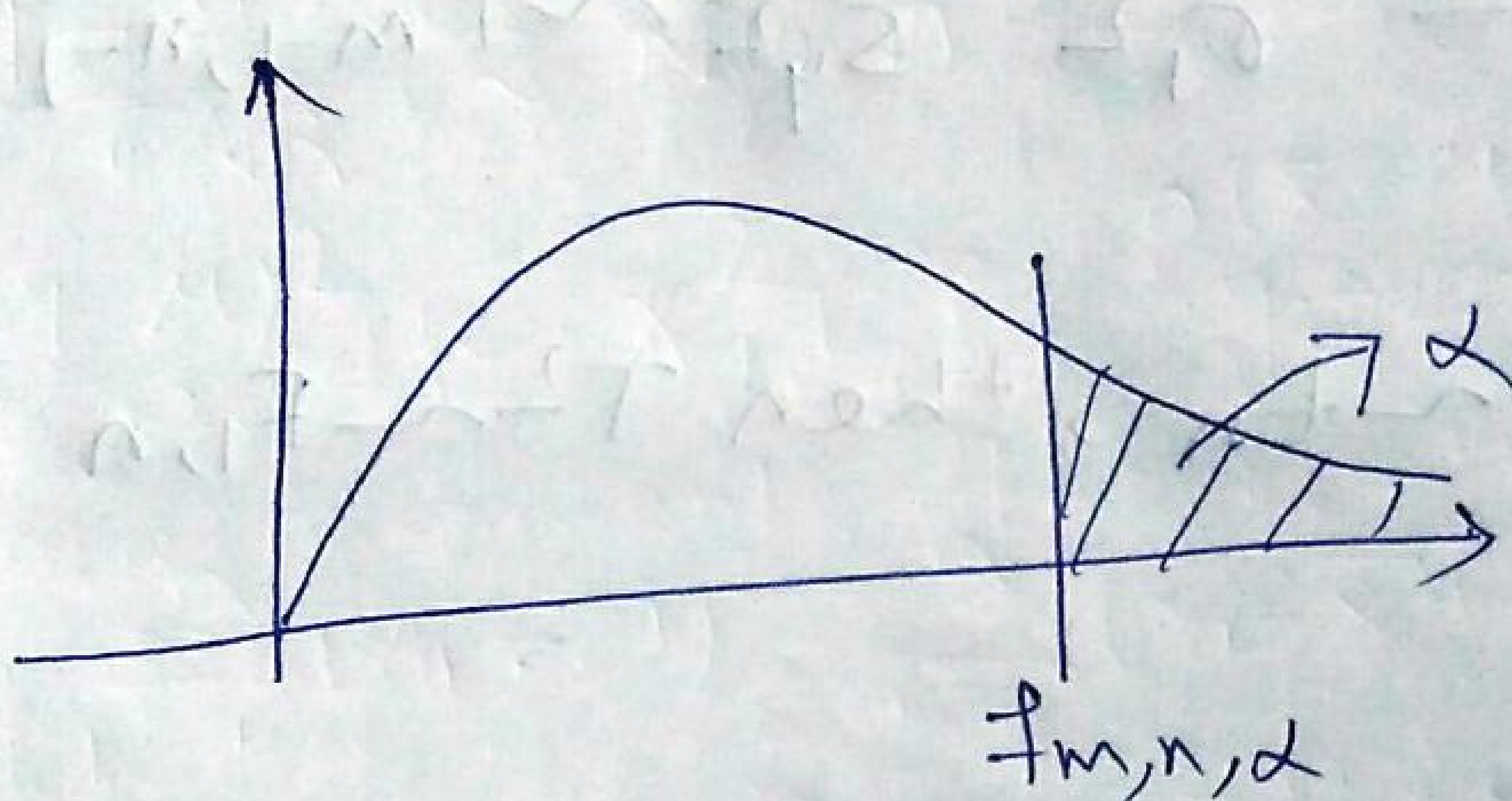
Let Y_1 and Y_2 be independent χ^2_m and χ^2_n r.v. Then

$$W = \frac{Y_1/m}{Y_2/n} = \frac{nY_1}{mY_2} \text{ follows an } F_{m,n} \text{ dist}$$

$$f_W(w) = \left(\frac{m}{n}\right)^{m/2} \frac{1}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \frac{w^{m/2 - 1}}{\left(1 + \frac{m}{n}w\right)^{\frac{m+n}{2}}} ; \; w > 0$$

$$E(W) = \frac{n}{n-2}, \; n > 2$$

$$V(W) = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}, \; n > 4$$



$$P(W \geq f_{m,n,\alpha}) = \alpha.$$

- If $U \sim F_{m,n}$ then $\frac{1}{U} \sim F_{n,m}$.

$$\alpha = P(U \geq f_{\alpha,m,n}) = P\left(\frac{1}{U} \leq \frac{1}{f_{\alpha,m,n}}\right)$$

$$= P\left(V \leq \frac{1}{f_{\alpha,m,n}}\right) \quad V \sim F_{n,m}$$

$$= 1 - P\left(V \geq \frac{1}{f_{\alpha,m,n}}\right)$$

So, $P\left(V \geq \frac{1}{f_{\alpha,m,n}}\right) = 1 - \alpha$

$$\Rightarrow \boxed{f_{1-\alpha,n,m} = \frac{1}{f_{\alpha,m,n}}}$$

Let $X_1, \dots, X_m \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$

independent $Y_1, \dots, Y_n \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$

$$S_x^2 = \frac{1}{m-1}\sum_{i=1}^{n}(X_i-\bar{X})^2 \qquad S_y^2 = \frac{1}{n-1}\Sigma(Y_j-\bar{Y})^2$$

$$U = \frac{(m-1)S_x^2}{\sigma_1^2} \sim \chi^2_{m-1}, \quad V = \frac{(n-1)S_y^2}{\sigma_2^2} \sim \chi^2_{n-1}$$

$$\cancel{\frac{U/(m-1)}{V/(n-1)}} =$$

$$\frac{U/(m-1)}{V/(n-1)} = \frac{\sigma_2^2}{\sigma_1^2}\,\frac{S_x^2}{S_y^2} \sim F_{m-1,n-1}.$$

<u>Thm:-</u> If $T \sim t_n$, then $T^2 \sim F_{1,n}$.