Maxwell's Stress Tensor (Based on D. J. Griffiths, Introduction to electrodynamics, Chapter-8)**:**

Total electromagnetic force on a distribution of charge in a volume τ:

$$\vec{\boldsymbol{F}} = \int_\tau \rho\, d\tau\big(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\big) = \int_\tau \rho\big(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\big) d\tau = \int_\tau \big(\rho\vec{\boldsymbol{E}} + \vec{\boldsymbol{J}} \times \vec{\boldsymbol{B}}\big) d\tau = \int_\tau \vec{\boldsymbol{f}}\, d\tau \dots\dots\dots\dots (1)$$

Where the integrand

$$\vec{\boldsymbol{f}} = \rho\big(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\big) = \rho\vec{\boldsymbol{E}} + \vec{\boldsymbol{J}} \times \vec{\boldsymbol{B}} \dots\dots\dots\dots (\mathbf{2})$$

can be identified as the force per unit volume. We can express $\vec{\boldsymbol{f}}$ in terms of fields $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ only, by replacing ρ and $\vec{\boldsymbol{J}}$ with the help of Maxwell's 1^{st} and 4^{th} equations:

$$\rho = \epsilon_0\big(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}\big) \qquad \text{and} \quad \rho\vec{\boldsymbol{v}} = \vec{\boldsymbol{J}} = \frac{1}{\mu_0}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} - \epsilon_0 \frac{\partial \vec{\boldsymbol{E}}}{\partial t}$$

Thus:

$$\vec{\boldsymbol{f}} = \rho\vec{\boldsymbol{E}} + \vec{\boldsymbol{J}} \times \vec{\boldsymbol{B}} = \epsilon_0\big(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}\big)\vec{\boldsymbol{E}} + \left(\frac{1}{\mu_0}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} - \epsilon_0 \frac{\partial \vec{\boldsymbol{E}}}{\partial t}\right) \times \vec{\boldsymbol{B}}$$

$$= \epsilon_0\big(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}\big)\vec{\boldsymbol{E}} + \frac{1}{\mu_0}\big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}\big) \times \vec{\boldsymbol{B}} - \epsilon_0 \frac{\partial \vec{\boldsymbol{E}}}{\partial t} \times \vec{\boldsymbol{B}}$$

Now, using Maxwell' s 3rd eqn.:

$$\frac{\partial \vec{\boldsymbol{B}}}{\partial t} = -\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}$$

We can write

$$\frac{\partial}{\partial t}\big(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}\big) = \vec{\boldsymbol{E}} \times \frac{\partial \vec{\boldsymbol{B}}}{\partial t} + \frac{\partial \vec{\boldsymbol{E}}}{\partial t} \times \vec{\boldsymbol{B}} = -\vec{\boldsymbol{E}} \times \big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}\big) + \frac{\partial \vec{\boldsymbol{E}}}{\partial t} \times \vec{\boldsymbol{B}}$$

$$\Rightarrow \frac{\partial \vec{\boldsymbol{E}}}{\partial t} \times \vec{\boldsymbol{B}} = \frac{\partial}{\partial t}\big(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}\big) + \vec{\boldsymbol{E}} \times \big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}\big)$$

Then:

$$\vec{\boldsymbol{f}} = \epsilon_0\big(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}\big)\vec{\boldsymbol{E}} + \frac{1}{\mu_0}\big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}\big) \times \vec{\boldsymbol{B}} - \epsilon_0 \left[\frac{\partial}{\partial t}\big(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}\big) + \vec{\boldsymbol{E}} \times \big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}\big)\right]$$

$$= \epsilon_0\big[\big(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}\big)\vec{\boldsymbol{E}} - \vec{\boldsymbol{E}} \times \big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}\big)\big] - \frac{1}{\mu_0}\vec{\boldsymbol{B}} \times \big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}\big) - \epsilon_0 \frac{\partial}{\partial t}\big(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}\big)$$

The expression of $\vec{\boldsymbol{f}}$ can look more symmetrical if we add a term $\frac{1}{\mu_0}\big(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}}\big)\vec{\boldsymbol{B}}$. This will not hamper anything since from Maxwell's 2^{nd} equation $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} = 0$. Thus:

$$\vec{\boldsymbol{f}} = \epsilon_0\big[\big(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}\big)\vec{\boldsymbol{E}} - \vec{\boldsymbol{E}} \times \big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}}\big)\big] + \frac{1}{\mu_0}\big[\big(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}}\big)\vec{\boldsymbol{B}} - \vec{\boldsymbol{B}} \times \big(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}\big)\big] - \epsilon_0 \frac{\partial}{\partial t}\big(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}\big) \dots\dots\dots (3)$$

Again, from vector identities:

$$\vec{\nabla}(\vec{E}\cdot\vec{E}) = 2(\vec{E}\cdot\vec{\nabla})\vec{E} + 2\vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\Rightarrow \vec{E}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{2}\vec{\nabla}(E^2) - (\vec{E}\cdot\vec{\nabla})\vec{E}$$

Similarly:

$$\vec{B}\times(\vec{\nabla}\times\vec{B}) = \frac{1}{2}\vec{\nabla}(B^2) - (\vec{B}\cdot\vec{\nabla})\vec{B}$$

Then we can write $\vec{f}$ as:

$$\vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}(E^2)\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}(B^2)\right] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

… … … … (4)

Also we can replace $\vec{E}\times\vec{B}$ by $\mu_0\vec{\mathbf{S}}$, where $\vec{\mathbf{S}}$ is the Poynting vector. Then:

$$\vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}(E^2)\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}(B^2)\right] - \epsilon_0\mu_0\frac{\partial\vec{\mathbf{S}}}{\partial t}$$

… … … (5)

The expression within the first bracket can be expressed in terms of a tensor through few steps as follows. The x-component $\vec{f}$ of can be written as:

$$f_x = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})E_x + (\vec{E}\cdot\vec{\nabla})E_x - \frac{1}{2}\frac{\partial E^2}{\partial x}\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})B_x + (\vec{B}\cdot\vec{\nabla})B_x - \frac{1}{2}\frac{\partial B^2}{\partial x}\right] - \mu_0\epsilon_0\frac{\partial S_x}{\partial t}$$

$$= \epsilon_0\left(\vec{\nabla}\cdot(\vec{E}E_x) - \frac{1}{2}\vec{\nabla}\cdot(\hat{\boldsymbol{x}}E^2)\right) + \frac{1}{\mu_0}\left(\vec{\nabla}\cdot(\vec{B}B_x) - \frac{1}{2}\vec{\nabla}\cdot(\hat{\boldsymbol{x}}B^2)\right) - \mu_0\epsilon_0\frac{\partial S_x}{\partial t}$$

$$= \vec{\nabla}\cdot\left(\epsilon_0\left[\vec{E}E_x - \frac{1}{2}\hat{\boldsymbol{x}}E^2\right] + \frac{1}{\mu_0}\left[\vec{B}B_x - \frac{1}{2}\hat{\boldsymbol{x}}B^2\right]\right) - \mu_0\epsilon_0\frac{\partial S_x}{\partial t}$$

Also we can write:

$$\vec{E}E_x - \frac{1}{2}\hat{\boldsymbol{x}}E^2 = (E_x\hat{\boldsymbol{x}} + E_y\hat{\boldsymbol{y}} + E_z\hat{\boldsymbol{z}})E_x - \frac{1}{2}\hat{\boldsymbol{x}}E^2 = E_xE_x\hat{\boldsymbol{x}} + E_yE_x\hat{\boldsymbol{y}} + E_zE_x\hat{\boldsymbol{z}} - \frac{1}{2}\hat{x}E^2$$

$$= \sum_{k=x,y,z}\left(E_kE_x\hat{\boldsymbol{k}} - \frac{1}{2}\delta_{kx}\hat{\boldsymbol{k}}E^2\right) = \sum_{k=x,y,z}\left(E_kE_x - \frac{1}{2}\delta_{kx}E^2\right)\hat{\boldsymbol{k}}$$

Similarly:

$$\vec{B}B_x - \frac{1}{2}\hat{\boldsymbol{x}}B^2 = \sum_{k=x,y,z}\left(B_kB_x - \frac{1}{2}\delta_{kx}B^2\right)\hat{\boldsymbol{k}}$$

Thus:

$$f_x = \vec{\boldsymbol{\nabla}}\cdot\left(\sum_{k=x,y,z}\left\{\epsilon_0\left(E_kE_x-\frac{1}{2}\delta_{kx}E^2\right)+\frac{1}{\mu_0}\left(B_kB_x-\frac{1}{2}\delta_{kx}B^2\right)\right\}\widehat{\boldsymbol{k}}\right)-\mu_0\epsilon_0\frac{\partial S_x}{\partial t}\ \ldots\ldots\ldots(6)$$

Or:

$$f_x = \left(\vec{\boldsymbol{\nabla}}\cdot\sum_{k=x,y,z}T_{kx}\widehat{\boldsymbol{k}}\right)-\mu_0\epsilon_0\frac{\partial S_x}{\partial t}\ \ldots\ldots\ldots(7A)$$

$$f_y = \vec{\boldsymbol{\nabla}}\cdot\left(\sum_{k=x,y,z}T_{ky}\widehat{\boldsymbol{k}}\right)-\mu_0\epsilon_0\frac{\partial S_y}{\partial t}\ \ldots\ldots\ldots(7B)$$

$$f_z = \vec{\boldsymbol{\nabla}}\cdot\left(\sum_{k=x,y,z}T_{kz}\widehat{\boldsymbol{k}}\right)-\mu_0\epsilon_0\frac{\partial S_z}{\partial t}\ \ldots\ldots\ldots(7C)$$

$$\Rightarrow f_l = \vec{\boldsymbol{\nabla}}\cdot\left(\sum_{k=x,y,z}T_{kl}\widehat{\boldsymbol{k}}\right)-\mu_0\epsilon_0\frac{\partial S_l}{\partial t},\qquad l=x,y,z\ \ldots\ldots\ldots(7D)$$

Where,

$$T_{kl} = \epsilon_0\left(E_kE_l-\frac{1}{2}\delta_{kl}E^2\right)+\frac{1}{\mu_0}\left(B_kB_l-\frac{1}{2}\delta_{kl}B^2\right)\ldots\ldots\ldots(8)$$

T_{kl} are the elements of a 3×3 tensor $\overleftrightarrow{T}$, given by:

$$\overleftrightarrow{T} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz}\\ T_{yx} & T_{yy} & T_{yz}\\ T_{zx} & T_{zy} & T_{zz}\end{bmatrix}\ldots\ldots\ldots(9)$$

The diagonal and off diagonal elements of $\overleftrightarrow{T}$ look like:

$$T_{xx} = \frac{\epsilon_0}{2}\left({E_x}^2-{E_y}^2-{E_z}^2\right)+\frac{1}{2\mu_0}\left({B_x}^2-{B_y}^2-{B_z}^2\right)\ldots\ldots\ldots(10A)$$

$$T_{xy} = \epsilon_0E_xE_y+\frac{1}{\mu_0}B_xB_y\ \ldots\ldots\ldots(10B)$$

Using the property of divergence operation on a tensor, equation (1) can be written as:

$$\vec{\boldsymbol{f}} = \vec{\boldsymbol{\nabla}}\cdot\overleftrightarrow{\boldsymbol{T}}-\mu_0\epsilon_0\frac{\partial\vec{\mathbf{S}}}{\partial t}\ \ldots\ldots\ldots(11)$$

Therefore:

$$\vec{\boldsymbol{F}} = \int_\tau\vec{\boldsymbol{f}}d\tau = \int_\tau\left(\vec{\boldsymbol{\nabla}}\cdot\overleftrightarrow{\boldsymbol{T}}-\mu_0\epsilon_0\frac{\partial\vec{\mathbf{S}}}{\partial t}\right)d\tau = \int_\tau\vec{\boldsymbol{\nabla}}\cdot\overleftrightarrow{\boldsymbol{T}}d\tau-\int_\tau\mu_0\epsilon_0\frac{\partial\vec{\mathbf{S}}}{\partial t}d\tau\ \ \ldots\ldots\ldots(12)$$

To realize the interpretation of $\overleftrightarrow{\boldsymbol{T}}$, let us convert the first volume integralin the r.h.s of eqn. (12) to surface integral with the help of Gauss's divergence theorem. Then:

$$\vec{\boldsymbol{F}} = \oint\!\!\!\oint_S \overleftrightarrow{\boldsymbol{T}} \cdot \boldsymbol{d\vec{a}} - \mu_0\epsilon_0 \int_\tau \frac{\partial \vec{\mathbf{S}}}{\partial t} d\tau \ldots\ldots\ldots\ldots\ldots (13A)$$

$$\vec{\boldsymbol{F}} = \oint\!\!\!\oint_S \overleftrightarrow{\boldsymbol{T}} \cdot \boldsymbol{d\vec{a}} - \mu_0\epsilon_0 \frac{d}{dt}\int_\tau \vec{\mathbf{S}} d\tau \ldots\ldots\ldots\ldots\ldots (13B)$$

Where S is the closed surface bounding the volume τ. As seen from eqn. (13A), the second term vanishes in static case i.e. if $\vec{\mathbf{S}}$ does not depend explicitly on time. Eqn. (13B) shows that the second term will vanish if the volume integral $\int_\tau \vec{\mathbf{S}} d\tau$ is independent of time, even if $\vec{\mathbf{S}}$ has time dependence at different points within the volume τ. Thus in the cases, where the second term vanishes, we have:

$$\vec{\boldsymbol{F}} = \oint\!\!\!\oint_S \overleftrightarrow{\boldsymbol{T}} \cdot \boldsymbol{d\vec{a}} \ldots\ldots\ldots (14)$$

To understand $\overleftrightarrow{\boldsymbol{T}}$, we note that it has the dimension of stress, i.e. force per unit area. Now consider a fluid element, having an imaginary boundary surface S, at different points on which the (normally acting) pressure is P. The net force on the element will be given by:

$$\vec{\boldsymbol{F}} = \oint\!\!\!\oint_S P\boldsymbol{d\vec{a}}$$

Comparing eqn. (14) with the above one we can interpret the tensor $\overleftrightarrow{\boldsymbol{T}}$ as a stress tensor. The elements T_{ij} represent the force per unit area acting in the i-th direction on an element of surface oriented in the j-th direction. The diagonal elements of $\overleftrightarrow{\boldsymbol{T}}$ i.e. are $\boldsymbol{T_{xx}}, \boldsymbol{T_{yy}}, \boldsymbol{T_{zz}}$ **are pressures** and the off diagonal $\boldsymbol{T_{xy}}$ **etc. are shears**. $\overleftrightarrow{\boldsymbol{T}}$ is called **Maxwell's stress tensor**. Thus we see that the electromagnetic field has stress associated with it, which is given by Maxwell's stress tensor.