① As α, β, γ are the roots of $x^3+px^2+qx+r=0$, then we have

$\alpha+\beta+\gamma=-p$

$\alpha\beta+\beta\gamma+\gamma\alpha=q$

$\alpha\beta\gamma=-r$

$$(\alpha+\beta+\gamma)^2=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$\Rightarrow (-p)^2=\alpha^2+\beta^2+\gamma^2+2\times q$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2=p^2-2q$$

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② As each roots of the eqn $x^3-6x^2+11x-6=0$ is increased by 1, then the new eqn is

Theory of Equations

$$(x-1)^3-6(x-1)^2+11(x-1)-6=0$$

$$\Rightarrow (x^3-3x^2+3x-1)-6(x^2-2x+1)+11(x-1)-6=0$$

$$\Rightarrow x^3-3x^2+3x-1-6x^2+12x-6+11x-11-6=0$$

$$\Rightarrow x^3-9x^2+26x-24=0$$

③/④ the given eqn is

Theory of Equations

$$\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}+\frac{4}{x-4}+\frac{5}{x-5}=6$$

If possible let the root is $\alpha+i\beta$, then other root must be $\alpha-i\beta$. Now we have

$$\frac{1}{\alpha+i\beta-1}+\frac{2}{\alpha+i\beta-2}+\frac{3}{\alpha+i\beta-3}+\frac{4}{\alpha+i\beta-4}+\frac{5}{\alpha+i\beta-5}=6 \rightarrow ①$$

$$\frac{1}{\alpha-i\beta-1}+\frac{2}{\alpha-i\beta-2}+\frac{3}{\alpha-i\beta-3}+\frac{4}{\alpha-i\beta-4}+\frac{5}{\alpha-i\beta-5}=6 \rightarrow ②$$

① - ② we have

$$\left[\frac{1}{(\alpha-1)+i\beta}-\frac{1}{(\alpha-1)-i\beta}\right]+2\left[\frac{1}{(\alpha-2)+i\beta}-\frac{1}{(\alpha-2)-i\beta}\right]+3\left[\frac{1}{(\alpha-3)+i\beta}-\frac{1}{\alpha-3-i\beta}\right]$$
$$+4\left[\frac{1}{(\alpha-4)+i\beta}-\frac{1}{(\alpha-4)-i\beta}\right]+5\left[\frac{1}{(\alpha-5)+i\beta}-\frac{1}{(\alpha-5)-i\beta}\right]=0$$

$$\Rightarrow \frac{-2i\beta}{(\alpha-1)^2+\beta^2}-\frac{4i\beta}{(\alpha-2)^2+\beta^2}-\frac{6i\beta}{(\alpha-3)^2+\beta^2}-\frac{8i\beta}{(\alpha-4)^2+\beta^2}-\frac{10i\beta}{(\alpha-5)^2+\beta^2}=0$$

$\Rightarrow -2i\beta\left[\frac{1}{(\alpha-1)^2+\beta^2}+\frac{2}{(\alpha-2)^2+\beta^2}+\frac{3}{(\alpha-3)^2+\beta^2}+\frac{4}{(\alpha-4)^2+\beta^2}+\frac{5}{(\alpha-5)^2+\beta^2}\right]=0$

$\Rightarrow \beta=0$ [as other expression never becomes zero)

therefore all roots of the given equⁿ is real.

(5) As $\alpha_1, \alpha_2, \alpha_3$ are the roots of $x^3+x+1=0$, then using reddation between roots and coefficient we have.

$\alpha_1+\alpha_2+\alpha_3=0$

$\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1=1$

$\alpha_1\alpha_2\alpha_3=-1$

$\alpha_1+\alpha_2=-\alpha_3$

$\alpha_1\alpha_2+\alpha_3(\alpha_1+\alpha_2)=1$

$\Rightarrow \alpha_1\alpha_2+\alpha_3(-\alpha_3)=1$

$\Rightarrow \alpha_1\alpha_2-\alpha_3^2=1$

$\Rightarrow 1+\alpha_3^2=\alpha_1\alpha_2$

Similarly we have.

$1+\alpha_2^2=\alpha_1\alpha_3$ / $1+\alpha_1^2=\alpha_3\alpha_2$

$(1+\alpha_1^2)(1+\alpha_2^2)(1+\alpha_3^2)=(\alpha_2\alpha_3)\cdot(\alpha_1\alpha_3)*(\alpha_1\alpha_2)$

$=(\alpha_1\alpha_2\alpha_3)^2=(-1)^2=1$

(6) As one root is $\sqrt{3}-2$ i.e $-2+\sqrt{3}$ then other root must be $-2-\sqrt{3}$

Now $(x-(-2+\sqrt{3}))(x-(-2-\sqrt{3}))$

$=(x+2-\sqrt{3})(x+2+\sqrt{3})$

$=(x+2)^2-(\sqrt{3})^2=x^2+4x+4-3$

$=x^2+4x+1$

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Let $f(x) = x^4+2x^3-5x^2+6x+2$, then x^2+4x+1 is a factor of $f(x)$, so

$x^4+2x^3-5x^2+6x+2 = (x^2+4x+1)\cdot q(x)$

$$
\begin{array}{r|l|l}
x^2+4x+1 & x^4+2x^3-5x^2+6x+2 & x^2-2x+2 \\
 & x^4+4x^3+x^2 & \\
\hline
 & -2x^3-6x^2+6x & \\
 & -2x^3-8x^2-2x & \\
\hline
 & 2x^2+8x+2 & \\
 & 2x^2+8x+2 & \\
\hline
 & \times &
\end{array}
$$

then $q(x) = x^2-2x+2$. Now $q(x)=0$ gives

$x^2-2x+2=0$

$x = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm 2i}{2}$

$= 1\pm i$

$\therefore$ the roots are $1\pm i,\ -2\pm\sqrt{3}$

(7) If the eqn

$x^3+px^2+qx+r=0$ whose roots are α,β,γ

$\Rightarrow \left(\frac{1}{x}\right)^3+p\left(\frac{1}{x}\right)^2+q\left(\frac{1}{x}\right)+r=0$ " " $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$

$\Rightarrow \frac{1}{x^3}+p/x^2+q/x+r=0$ " " "

$\Rightarrow 1+px+qx^2+rx^3=0$ " " "

$\Rightarrow r(-x)^3+q(-x)^2+p(-x)+1=0$ (a) " $-\frac{1}{\alpha},-\frac{1}{\beta},-\frac{1}{\gamma}$

$\Rightarrow rx^3-qx^2+px-1=0$ " " $-\frac{1}{\alpha},-\frac{1}{\beta},-\frac{1}{\gamma}$

(8) As two roots are equal, then we take roots are α,α,β, now we have.

$\alpha+\alpha+\beta = 3$

$\alpha\cdot\alpha + \alpha\cdot\beta + \beta\cdot\alpha = 0$

$\alpha\cdot\alpha\cdot\beta = -4$

$\Rightarrow 2\alpha+\beta = 3$ & $\alpha^2+2\alpha\beta = 0$

$\alpha(\alpha+2\beta) = 0$

$\alpha+2\beta = 0 \; (\text{as } \alpha \neq 0)$

$2\alpha+\beta = 3$

$\alpha+2\beta = 0$

$$\begin{array}{r} \cancel{2\alpha}+\beta = 3 \\ \underline{\cancel{2\alpha}+4\beta = 0} \\ -3\beta = 3 \\ \beta = -1 \end{array}$$

$\alpha = 2$

Hence roots are

$2, 2, -1$

⑨ Let $f(x) = x^4+16x^2+\underbrace{7x-10}_{+1} = 0$

as change in sign in $f(x)$ is one. then $f(x)=0$ has exactly one positive real root.

$f(-x) = (-x)^4+16(-x)^2+7(-x)-10$

$= x^4+\underbrace{16x^2-7x}_{+1}-10$

As change in sign in $f(-x)$ is one then $f(x)=0$ has exactly one negative real root.

As $f(x)=0$ is a polynomial of degree 4. they it has 4 roots. as one positive and one negative real roots. then other two roots must imaginary

⑪ Let α, β, γ are the roots. as one root is the sum of the other two roots

i.e $\alpha = \beta+\gamma$

As the eqn is $x^3+px^2+qx+r=0$, then we have.

$\alpha+\beta+\gamma=-p \rightarrow ①$ | $\alpha=\beta+\gamma \longrightarrow Ⓐ$

$\alpha\beta+\beta\gamma+\gamma\alpha=q \rightarrow ②$ | From ① we have

$\alpha\beta\gamma=-r \rightarrow ③$

$$2\alpha=-p$$

$$\alpha=-p/2$$

$$\beta+\gamma=\alpha=-p/2$$

From ② $\beta\gamma+\alpha(\beta+\gamma)=q$

$\Rightarrow \beta\gamma+\alpha^2=q$

$\Rightarrow \beta\gamma+(-p/2)^2=q$

$\Rightarrow \beta\gamma=q-p^2/4$

From ③

$\alpha.\beta\gamma=-r$

$-p/2\left(q-p^2/4\right)=-r$

$\Rightarrow p\left(q-p^2/4\right)=2r$

$\Rightarrow 4pq-p^3=8r$

$\Rightarrow \boxed{p^3+8r=4pq}$

(ii) As roots are in G.P

Let roots are $\frac{a}{r}, a, ar$ | $27x^3+42x^2-28x-8=0$

$\frac{a}{r}+a+ar=-42/27$

$\frac{a}{r}.a+a.ar+ar.\frac{a}{r}=-28/27$

$\frac{a}{r}.a.ar=+8/27$

$a\left(\frac{1}{r}+1+r\right)=-42/27$

$a^2\left(\frac{1}{r}+1+r\right)=-28/27$

$a^3=+8/27=(+2/3)^3$

$a=+2/3$

$\frac{2}{3}\left(1+r+\frac{1}{r}\right)=-42/27$

$\Rightarrow 1+r+\frac{1}{r}=-\frac{21}{9}$

$\Rightarrow r+\frac{1}{r}=-\frac{4}{9}-1$

$=-30/9$

$\Rightarrow 9r^2+30r+9=0$

$\Rightarrow 3r^2+10r+3=0$

$\Rightarrow (3r+1)(r+3)=0$

$\Rightarrow r=-1/3 \mid r=-3$

	If
$\frac{a}{r}, a, ar$	When $r=-1/3$
$= \frac{4/3}{-3}, 4/3, 4/3\cdot(-3)$	$-4/9, 4/3, -2$
$= -4/9, 4/3, -2$	Hence roots are $-4/9, 4/3, -2$

(12) As

$x^3-3x^2+8x-5=0$ whose roots are α, β, γ

$\Rightarrow \left(\frac{x}{2}\right)^3-3\left(\frac{x}{2}\right)^2+8\left(\frac{x}{2}\right)-5=0$ whose roots are $2\alpha, 2\beta, 2\gamma$

$\Rightarrow \frac{x^3}{8}-\frac{3x^2}{4}+\frac{8x}{2}-5=0$ " " " "

$\Rightarrow x^3-6x^2+32x-40=0$ " " " "

$\Rightarrow (x-3)^3+6(x-3)^2+32(x-3)-40=0$ " " $2\alpha+3, 2\beta+3, 2\gamma+3$

$\Rightarrow x^3-9x^2+27x-27+6x^2-36x+54+32x-96-40=0$ "

$\Rightarrow x^3-3x^2+23x-109=0$ whose roots are $2\alpha+3, 2\beta+3, 2\gamma+3$

(13) $(x-1)(x-2)(x-3)(x-4)=3$

$\Rightarrow [(x-1)(x-4)][(x-2)(x-3)]=3$

$\Rightarrow (x^2-5x+4)(x^2-5x+6)=3$

$\Rightarrow (p+4)(p+6)=3$ (where $p=x^2-5x$)

$\Rightarrow p^2+10p+24-3=0$

$\Rightarrow p^2+10p+21=0$

$\Rightarrow (p+7)(p+3)=0$

$\Rightarrow p=-7 \quad | \quad p=-3$

$\Rightarrow x^2-5x=-7 \quad | \quad x^2-5x=-3$

$\Rightarrow x^2-5x+7=0 \quad | \quad x^2-5x+3=0$

$x=\dfrac{5\pm\sqrt{25-28}}{2} \quad | \quad x=\dfrac{5\pm\sqrt{25-12}}{2}$

$=\dfrac{5\pm i\sqrt{3}}{2} \quad | \quad =\dfrac{5\pm\sqrt{13}}{2}$

Hence roots are $\dfrac{5\pm i\sqrt{3}}{2}, \dfrac{5\pm\sqrt{13}}{2}$

(14)/(13) As $2x^4-7x^3+2x^2-x+1$ is divided by x^2-1 then reminder is $px+q$.

Now $x^2-1=(x+1)(x-1)$

Now $f(1)=2(1)^4-7(1)^3+2(1)^2-1+1$

$=2-7+2-1+1$

$=-3$

$f(-1)=2(-1)^4-7(-1)^3+2(-1)^2-(-1)+1$

$=2+7+2+1+1$

$=13$

According to the given problem we have

$f(x)=(x^2-1)\cdot\phi(x)+(px+q)$

$\Rightarrow (2x^4-7x^3+2x^2-1+1)=(px+q)+(x^2-1)\cdot\phi(x)$

put $x=1$

$f(1) = p+q+0$

$p+q = -3$

put $x=-1$

$f(-1) = -p+q+0$

$13 = -p+q$

$$\begin{aligned} \cancel{p}+q &= -3 \\ -\cancel{p}+q &= 13 \\ \hline 2q &= 10 \\ q &= 5 \end{aligned}$$

$p = -8$

Hence $(p,q) = (-8, 5)$

(16) As the eqn $f(x) = x^3+ax^2+bx-4$ has two roots equal to 2 i.e 2 is a multiple root of order 2. then we have

$f(2) = 0$ and $f'(2) = 0$ [f' means derivative $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$]

$f(2) = 0$ gives

$(2)^3 + a(2)^2 + b(2) - 4 = 0$

$\Rightarrow 4a+2b = -4$

$\Rightarrow 2a+b = -2$

$f'(x) = 3x^2+2ax+b$

$f'(2) = 0$ gives

$3(2)^2+2a(2)+b = 0$

$\Rightarrow 4a+b = -12$

$$\begin{aligned} 4a+\cancel{b} &= -12 \\ 2a+\cancel{b} &= -2 \\ - \quad &\quad + \\ \hline 2a &= -10 \\ a &= -5 \end{aligned}$$

$b = -2-2a$
$= -2+10$
$= 8$

$\therefore (a,b) = (-5, 8)$

(17) As sum of two roots is zero, then the roots of the given equ^n is $\alpha, -\alpha, \beta$. Now we have.

$4x^3+16x^2-9x-36=0$

$\alpha+(-\alpha)+\beta = -16/4$

$\alpha\cdot(-\alpha) + -\alpha\cdot\beta + \beta\cdot\alpha = -9/4$

$\alpha\cdot -\alpha\cdot\beta = +36/4$

$\beta = -4$

$-\alpha^2 + \cancel{\alpha\beta} - \cancel{\alpha\beta} = -9/4$

$\alpha^2 = 9/4$

$\alpha = \pm 3/2$ Hence roots are

$3/2, -3/2, -4$

(18) As a and b are two roots of $x^2+px+1=0$

$a+b=-p$

$ab=1$

c and d are roots of $x^2+qx+1=0$ then

$c+d=-q$

$cd=1$

Now $q^2-p^2 = (c+d)^2-(a+b)^2$

$= (c+d+a+b)(c+d-a-b)$

Now $(a-c)(b-c)(a+d)(b+d)$

$= (a-c)(b+d)(b-c)(a+d)$

$= (ab-bc+ad-cd)(ab-ac+bd-cd)$

$= (1-bc+ad-1)(1-ac+bd-1)$

$= (ad-bc)(bd-ac)$

$= (ab-bc)(bd-ac)$

$= abd^2 - b^2cd - a^2cd + abc^2$

$= d^2 - b^2 - a^2 + c^2$ (using $abcd=1$)

$\Rightarrow c^2+d^2-(a^2+b^2)$

$= [(c+d)^2 - 2cd] - [(a+b)^2 - 2ab]$

$= [(-q)^2 - 2\cdot 1] - [(-p)^2 - 2\cdot 1]$

$= (q^2-2)-(p^2-2)$

$= q^2 - \cancel{2} - p^2 + \cancel{2}$

$= q^2 - p^2$

Hence $q^2-p^2 = (a-c)(b-c)(b+d)(a+d)$ (proved)

(19) the given equation is $x^4+2x^3-16x^2-22x+7=0$ whose one root is $2+\sqrt{3}$ then other root must be $2-\sqrt{3}$

Now $(x-(2+\sqrt{3}))(x-(2-\sqrt{3})) = ((x-2)-\sqrt{3})(x-2+\sqrt{3})$

$= (x-2)^2 - (\sqrt{3})^2$

$= x^2-4x+4-3$

$= x^2-4x+1$

then x^2-4x+1 is a factor of $f(x)$. Now let

$f(x) = (x^2-4x+1)\,q(x)$

$$
\begin{array}{r|l}
x^2-4x+1 & x^4+2x^3-16x^2-22x+7 \;\big|\; x^2+6x+7 \\
 & \underline{x^4-4x^3+x^2} \\
 & 6x^3-17x^2-22x \\
 & \underline{6x^3-24x^2+6x} \\
 & 7x^2-28x+7 \\
 & \underline{7x^2-28x+7} \\
 & \times
\end{array}
$$

$\therefore q(x) = x^2+6x+7$

Now $\phi(x) = 0$ gives

$x^2+6x+7=0$

$$x = \frac{-6 \pm \sqrt{36-4\cdot1\cdot7}}{2\cdot1}$$

$$= \frac{-6 \pm \sqrt{36-28}}{2}$$

$$= \frac{-6 \pm 2\sqrt{2}}{2}$$

$$= -3 \pm \sqrt{2}$$

Hence roots are $2 \pm \sqrt{3}$; $-3 \pm \sqrt{2}$

(20) The given eqn is $x^4+2x^2+3x-1=0 = f(x)$ (say)

Now $f(x) = x^4+2x^2+3x-1$

Now change in singn in $f(x)$ is one. Hence $f(x)=0$ has exactly one positive real root.

Now $f(-x) = (-x)^4+2(-x)^2+3(-x)-1$

$$= x^4+\underbrace{2x^2-3x}_{+1}-1$$

As change in sign in $f(-x)$ is one, then $f(x)=0$ has exactly one negative real root.

As $f(x)$ is a polynomial of degree 4. then $f(x)$ has one positive and one negative, others two roots are imaginary.

(21) the given equation is $2x^4-5x^3-15x^2+10x+8=0$

As roots are in G.P. then we can take roots are a, ar, ar^2, ar^3

(21)/(22) the given equation is $x^3+3x^2-8x+1=0 \rightarrow (1)$

i) Now the eqn, whose roots are, the roots of (1) each diminished by 4 is

$$(x+4)^3+3(x+4)^2-8(x+4)+1=0$$

$$\Rightarrow x^3+12x^2+48x+64+3x^2+24x+48-8x-32+1=0$$

$$\Rightarrow x^3+15x^2-64x+81=0$$

ii) If each roots are increased by 1, then new eqn is

$$(x-1)^3+3(x-1)^2-8(x-1)+1=0$$

$$\Rightarrow x^3-\cancel{3x^2}+3x-1+\cancel{3x^2}-6x+3-8x+8+1=0$$

$$\Rightarrow x^3-11x-11=0$$

iii) If each roots are multiplied by 2, then new eqn is

$$(x/2)^3+3(x/2)^2-8(x/2)+1=0$$

$$\Rightarrow x^3/8+3x^2/4-8x/2+1=0$$

$$\Rightarrow x^3+6x^2-32x+8=0$$

(23)/(14) the given eqn is $x^2-mx+1=0$ as the given eqn whose roots are less than unity then $D \geq 0$ and $af(1) > 0$ and $-b/2a < 1$

$D = b^2-4ac \mid ax^2+bx+c=0$

$$(-m)^2-4 \geq 0$$

$$\Rightarrow m^2-4 \geq 0$$

$$\Rightarrow (m+2)(m-2) \geq 0 \Rightarrow m \leq -2 \text{ or } m \geq 2$$

$$\Rightarrow (-\infty,-2] \cup [2,\infty) \rightarrow (1)$$

Now $a f(1) \cancel{>} 0$

$\Rightarrow 1\,(1-m+1) \cancel{>} 0$

$\Rightarrow m-2 < 0$

$\Rightarrow m<2 \Rightarrow m \in (-\infty, 2) \rightarrow$ (2)

$-b/2a < 1 \Rightarrow m/2 < 1 \Rightarrow m < 2$ ~~[illegible]~~

$\Rightarrow (-\infty, 2) \longrightarrow$ (3)

Hence from (1), (2) and (3) the values of m are

$(-\infty, -2]$

The End

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10/05/2020