Buddhadeb Mondal 16/04/2020

Theory of Equation

Polynomials: The algebric expression $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, is called a polynomial inn with degree n, provided the power of in each term is a non negative integer and $a_n \neq 0$

Remainder theorem: If a polynomial $f(x)$ is divided by $(x-\alpha)$: then the remainder is $f(\alpha)$.

If the polynomial $f(x)$ is totally divided by $(x-\alpha)$ then we have $f(\alpha) = 0$

Quadratic equation: An equation is of the form $ax^2 + bx + c = 0$ where $(a \neq 0)$, b, c both are real numbers.

For any polynomial $f(x)$, if $f(\alpha) = 0$, then α is a root of the equation $f(x) = 0$.

Now the roots of the eqn $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(By Sridharacharya)}$$

i.e $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

Nature of roots:

Let $D = b^2 - 4ac$ (Discriminant of quadratic)

Case-I : If $D > 0$, then roots are real and distinct.

Case-II : If $D = 0$, then roots are real and equal.

Case-III : If $D < 0$, then roots are Imaginary and distinct.

** If $\alpha+i\beta$ be a roots of a quadratic equ^n, then other root must be $\alpha-i\beta$

** If $p+\sqrt{q}$ be a root of quadratic equ^n, then other root must be $p-\sqrt{q}$, where q is not a perfect square.

Relation between roots and coefficients

For quadratic equ^n:

$ax^2+bx+c=0$, & if α, β be two roots of the given equation, then we have

$$\boxed{\alpha+\beta=-b/a} \text{ and } \boxed{\alpha\beta=c/a}$$

For Cubic polynomial equation:

If $ax^3+bx^2+cx+d=0\ (a\neq 0)$ has roots α, β and γ then we have

$$\Sigma\alpha=\alpha+\beta+\gamma=-b/a$$

$$\Sigma\alpha\beta=\alpha\beta+\beta\gamma+\gamma\alpha=c/a$$

$$\alpha\beta\gamma=-d/a$$

Similary, for biquadratic equ^n, $ax^4+bx^3+cx^2+dx+e=0$ has roots α, β, γ and δ, then we have

$$\alpha+\beta+\gamma+\delta=-b/a,\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=c/a$$

$$\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\gamma\delta\alpha=-d/a$$

$$\alpha\beta\gamma\delta=e/a$$

** If α, β be two roots of an equation $f(x)=0$ then the equ^n is $x^2-(\alpha+\beta)x+\alpha\beta=0$

"Descartes' Rule of Signs":

It asserts that the number of positive real roots is atmost the number of sign changes in the sequence of polynomial's coefficients (omitting the zero coefficients), or it is less than by an even number.

Similarly, the number of negative real roots is atmost the number of sign changes in the sequence of polynomial's coefficients ($f(-x)$) or it is less than by an even number.

In particular, that if the number of sign changes is zero or one, there are exactly zero or one positive (negative) roots, respectively

Ex Let the polynomial

$$f(x) = x^3 + \underbrace{x^2 - x}_{+1} - 1$$

We see that $f(x)$ has one change in sign between 2nd and 3rd term. Therefore it has exactly one positive roots.

Now we calculate $f(-x)$

$$f(-x) = (-x)^3 + (-x)^2 + (-x) - 1$$

$$= \underbrace{-x^3 + x^2}_{+1} + \underbrace{x - 1}_{+1}$$

We see that in $f(-x)$ there are two change in sign, thus the original polynomial $f(x)=0$ has two or zero negative roots.

Now

$$f(x) = x^3 + x^2 x - 1$$
$$= (x+1)^2 (x-1)$$

$\therefore f(x) = 0$ has the roots $1, -1$ (two times)
So one positive and two negative roots.

finally we can conclude that Descartes' Rule of sign not give exact no. of roots of the equn $f(x) = 0$, so it gives maximum no. of roots.

<u>Sign of the roots</u>:

i) <u>positive roots</u>: Both roots will be positive if $\alpha+\beta$ and $\alpha\beta$ are both positive, i.e $-b/a$ and c/a are positive.

ii) <u>Negative roots</u>: Both roots will be negative if $\alpha+\beta$ negative and $\alpha\beta$ positive, ie $-b/a$ is negative and c/a is positive.

iii) <u>Roots of opposit signs</u>: It will be occure when $\alpha\beta$ is negative.

iv) <u>Roots equal in magnatude but opposit in signs</u>:
It will be occure if $\alpha+\beta = 0$

<u>transformed Equations</u>:

If $f(x) = 0$ is a polynomial equation, then the equation whose roots are:

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a) Reciprocals of the roots of $f(x)=0$ is $f(1/x)=0$

b) Roots of $f(x)=0$ each increased by a constant k is $f(x-k)=0$

c) Roots of $f(x)=0$ each decreased by a constant k is $f(x+k)=0$

d) Roots of $f(x)=0$ with signs changed is $f(-x)=0$

e) Roots of $f(x)=0$ with each multiplied by $k(\neq 0)$ is $f(\frac{x}{k})=0$

Fundamental theorem on classical algebra:

Every algebric equation has a root, real or complex.

Algebric equation:

Let $f(x)$ be a polynomial in x of degree ≥ 1 whose coefficients are real or complex. then $f(x)=0$ is said to be an algebric equation or a polynomial equation.

** An algebric equation of degree n has atmost n roots.

* Every odd degree polynomial with rational coefficients has atleast one real root.

Some problems over theory of equation

① If α, β, γ are roots of the equation $x^3+px^2+qx+r=0$, find the value of $\alpha^2+\beta^2+\gamma^2$

② Find the equation whose roots are the roots of the equation $x^3-6x^2+11x-6=0$ each increased by 1.

③ If the roots of $x^4+ax^3+bx^2+cx+d=0$ are in G.P, then prove that $c^2=ad$.

④ Show that the roots of the equation.
$$\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}+\frac{4}{x-4}+\frac{5}{x-5}=6$$ are all real.

⑤ If $\alpha_1, \alpha_2, \alpha_3$ be the roots of the equⁿ $x^3+x+1=0$. then prove that
$$(\alpha_1^2+1)(\alpha_2^2+1)(\alpha_3^2+1)=1$$

⑥ Solve the equation $x^4+2x^3-5x^2+6x+2=0$ if one root is $\sqrt{3}-2$.

⑦ If α, β, γ are roots of the equation $x^3+px^2+qx+r=0$, find the equation whose roots are $-\frac{1}{\alpha}, -\frac{1}{\beta}, -\frac{1}{\gamma}$.

⑧ Solve the equation $x^3-3x^2+4=0$, where two roots are equal.

(9) State Descarte's Rule of signs regarding the existance of roots of polynomial equn with real coefficients.
Show that the equation $x^4+16x^2+7x-10=0$ has one positive, one negative and two imaginary roots.

(10) If one roots of the equation $x^3+px^2+qx+r=0$ equals the sum of the other two, then prove that $p^3+8r=4pq$

(11) Solve the equation : $27x^3+42x^2-28x-8=0$ whose roots are in G. P.

(12) If α, β, γ be the roots of $x^3-3x^2+8x-5=0$, find the equation whose roots are $\cancel{2\alpha+3}$, $2\alpha+3$, $2\beta+3$, $2\gamma+3$.

(13) Solve : $(x-1)(x-2)(x-3)(x-4)=3$.

(14) Find the values of m for which both roots of the equation $x^2-mx+1=0$ are less than unity.

(15) The expression $2x^4-7x^3+2x^2-x+1$ when divided by x^2-1 leaves a remainder $px+q$, then find the order pair (p,q)

(16) If the equation x^3+ax^2+bx-4 has two roots equal to 2, then find the order pair (a,b)

(17) Find the roots of the equation $4x^3+16x^2-9x-36=0$, given that sum of two roots of it is zero.

(18) If a and b are two roots of $x^2+px+1=0$ and c, d are roots of $x^2+qx+1=0$, then Show that $q^2-p^2=(a-c)(b-c)(a+d)(b+d)$

(19) Solve the equation $x^4+2x^3-16x^2-22x+7=0$ which has a root $2+\sqrt{3}$.

(20) Apply Descartes' rule of sign, to examine nature of the roots of the equation.

i) $x^4+2x^2+3x-1=0$

(21) Solve the equation $2x^4-5x^3-15x^2+10x+8=0$, the roots being in geometric progression.

(22) Find the equation whose roots are the roots of the equation $x^3+3x^2-8x+1=0$

i) each diminished by 4

ii) each increased by 1.

iii) each multiplied by 2.

(23) If α, β, γ be the roots of the equation $x^3+px^2+qx+r=0$, find the equation whose roots are $\alpha\beta+\beta\gamma$, $\beta\gamma+\gamma\alpha$, $\gamma\alpha+\alpha\beta$.

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