CHAPTER

7 Matrices and Determinants

CONCEPT BOOSTER

INTRODUCTION

1. Matrix

A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called an $m \times n$ matrix. We read as m by n matrix.

An $m \times n$ matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It is also denoted as $[a_{ij}]_{m \times n}$.

> *Note* A matrix is not a number. It just an ordered collection of numbers arranged in the form of a rectangular array.

1.1 Order

If a matrix has m rows and n columns, the order of the matrix is m by n or $m \times n$.

(i) The order of the matrix $[1 \quad 2 \quad 3]$ is 1×3.

(ii) The order of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 5 & 6 & 4 \end{bmatrix}$ is 3×3.

(iii) The order of the matrix $\begin{bmatrix} 1 & 4 & 8 \\ 2 & 3 & 5 \end{bmatrix}$ is 2×3.

2. TYPES OF MATRICES

(i) Row matrix

A matrix having only one row is called a row matrix.

For example, Let $A = [1 \quad 2 \quad 3]$.

(ii) Column matrix

A matrix having only one column is called a column matrix.

For example, $B = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 9 \end{bmatrix}$.

(iii) Rectangular matrix

A matrix in which number of rows and number of columns are not equal is called a rectangular matrix.

For example, $C = \begin{bmatrix} 1 & 4 & 8 \\ 2 & 3 & 5 \end{bmatrix}$.

(iv) Square matrix

In a matrix, in which the number of rows is equal to the number of columns, it is called a square matrix.

For example, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 7 & 6 \end{pmatrix}$

are square matrices of order 2 and 3 respectively.

(v) Diagonal matrix

In a square matrix, if all the diagonal elements are non-zero and rest are zero is called a diagonal matrix.

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, etc.

(vi) Scalar matrix

In a square matrix, if all the diagonal elements are the same and rest of the elements are zero, it is called a scalar matrix.

For example, $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(vii) Identity matrix

In a scalar matrix, if all the diagonal elements are 1, it is called an identity matrix.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.

(viii) Non-zero matrix

In a matrix, if at-least one element is non-zero, it is called a non-zero matrix.

For example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

are non-zero matrices.

(ix) Zero matrix

In a matrix, if every elements are zero, it is known as zero matrix. It is denoted as **O**.

For example, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, etc.

(x) Upper triangular matrix

In a square matrix, if all the elements below the leading elements are zero, it is called a upper triangular matrix.

For example, $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix}$, etc.

(xi) Lower triangular matrix

In a square matrix, if all the elements above the leading elements are zero, it is called a lower triangular matrix.

For example, $A = \begin{bmatrix} 1 & 0 \\ 2 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 2 & 3 & 4 \end{bmatrix}$, etc.

(xii) Strictly triangular matrix

In a square matrix, if all the diagonal matrices are zero, it is called a strictly triangular matrix.

For example, $A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 3 \\ 5 & 0 & 7 \\ 3 & 6 & 0 \end{bmatrix}$, etc.

(xiii) Comparable matrices

Two matrices are said to be comparable matrices, if their orders are the same.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are two comparable matrices.

(xiv) Trace of a matrix

The sum of the diagonal elements of a matrix is known as the trace of a matrix.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\text{Tr}(A) = a_{11} + a_{22} + a_{33}$

(xv) Equality of two matrices

Two comparable matrices are said to be equal if their corresponding elements are the same.

If $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $a = 2, b = 3, c = 4$ and $d = 6$.

(xvi) Sub-matrix

Any matrix is obtained by eliminating some rows and some columns from a given matrix A, it is called a sub-matrix of A.

Let $A = \begin{bmatrix} 1 & 3 & 4 & 6 \\ 7 & 0 & 5 & 2 \\ 2 & 5 & 9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 0 & 5 \end{bmatrix}$,

B is a sub-matrix of A.

3. Addition of Matrices

We can find the addition of two or more matrices if they are comparable matrices otherwise addition is not defined.

Let $A = \begin{bmatrix} 1 & 3 \\ 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$,

then $A + B = \begin{bmatrix} 3 & 6 \\ 10 & 6 \end{bmatrix}$

Properties of Addition of Matrices

(i) Matrix addition is commutative.
(ii) Matrix addition is associative.
(iii) Additive identity of a matrix exists.
(iv) Additive inverse of a matrix exists.

3.1 Scalar Multiplication

If $A = \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$, then $kA = \begin{bmatrix} 2k & 4k \\ 6k & 7k \end{bmatrix}$,

where k is the scalar multiple of A.

3.2 Negative of a matrix

If $A = \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & -4 \\ -5 & -7 \end{bmatrix}$

4. Multiplication of Matrices

If the number of columns of a first matrix is equal to the number of rows of a second matrix, we can find out the product, otherwise product is not defined.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ji}]_{n \times p}$,

then $AB = [c_{ij}]_{m \times p}$

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

then $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

Properties of matrix multiplication

(i) In general, matrix multiplication is not commutative.
(ii) Matrix multiplication is associative.
(iii) Matrix multiplication is distributive over addition
(iv) Multiplicative identity of a matrix exists.

5. Transpose of a Matrix

Transpose of a matrix is obtained by interchanging rows into columns and columns into rows.

If A be any matrix, its transpose is denoted by A^T or A'.

For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

5.1 Properties of Transpose of Matrices

(i) $(A + B)^T = A^T + B^T$
(ii) $(A^T)^T = A$
(iii) $(kA)^T = k(A)^T$
(iv) $(AB)^T = B^T A^T$

5.2 Symmetric Matrix

A square matrix A is said to be symmetric $A^T = A$.

For example, $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 \\ 5 & 7 \end{bmatrix},$

$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$ and $D = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

etc., are symmetric matrices.

Example: If $A = \begin{pmatrix} x & x+3 \\ 6x-2 & 10x \end{pmatrix}$ be a symmetric matrix, find x.

5.3 Skew-symmetric Matrix

A square matrix is said to be skew-symmetric, if $A^T = -A$.

For example, $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$ are skew-symmetric matrices.

Example: If $A = \begin{pmatrix} x & x+2 \\ 5x-2 & 9x \end{pmatrix}$ is a skew-symmetric matrix, find x.

5.4 Properties of Symmetric and Skew-symmetric Matrices

(i) In skew-symmetric matrix, all the diagonal elements are zero.

Let $A = (a_{ij})$
Given $A^T = -A$
$\Rightarrow (a_{ji}) = -(a_{ij})$
Put $i = j$
$\Rightarrow (a_{ii}) = -(a_{ii})$
$\Rightarrow 2(a_{ii}) = \mathbf{O}$
$\Rightarrow (a_{ii}) = \mathbf{O}$
$\Rightarrow a_{11} = 0 = a_{22} = a_{33} = \ldots = a_{nn}$

(ii) For any square matrix A, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

(iii) Any square matrix can be expressed uniquely as a sum of a symmetric and a skew-symmetric matrices.

Let A be a square matrix.

Consider $A = \frac{1}{2}(2A)$

$= \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$= P + Q$, (say)

To prove $P' = P$ and $Q' = -Q$

Now, $P^T = \left[\frac{1}{2}(A + A^T)\right]^T$

$= \frac{1}{2}(A + A^T)^T$

$= \frac{1}{2}[A^T + (A^T)^T]$

$= \frac{1}{2}(A^T + A)$

$= \frac{1}{2}(A + A^T) = P$

Thus P is a symmetric matrix.

Also, $Q^T = \left(\frac{1}{2}(A - A^T)\right)^T$

$= \frac{1}{2}(A - A^T)^T$

$= \frac{1}{2}[A^T - (A^T)^T]$

$= \frac{1}{2}(A^T - \text{A})$

$= -\frac{1}{2}(A - A^T)$

$= -Q$

Thus Q is skew-symmetric matrix
Hence, the result.

(iv) The sum of skew-symmetric matrices is again a skew-symmetric matrix.
(v) The square of a skew-symmetric matrix is not a skew symmetric.
(vi) The cube of a skew symmetric matrices is again skew-symmetric matrix.
(vii) If A is symmetric matrix, then kA is also symmetric.
(viii) If A is skew-symmetric matrix, then kA is also skew-symmetric matrix.
(ix) If A is symmetric, then AA^T and A^TA are symmetric matrices.

(xii) If A is symmetric (skew-symmetric) matrix, then B^TAB is symmetric (skew-symmetric) matrix.

6. DETERMINANT

For every square matrix of order n, there is associated number (real or complex) is called a determinant of the same order.

A determinant is a polynomial of the elements of a square matrix. It is scalar.

It has some finite values. Determinants are defined only for square matrices. Determinants of a non-square matrix is not defined.

Determinant of a square matrix A is denoted by det A or $|A|$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

6.1 Minor of an Element of a Matrix

The minor of an element of a matrix is obtained after deleting the corresponding rows and corresponding columns.

Let $A = \begin{bmatrix} 1 & 3 \\ 5 & 8 \end{bmatrix}$.

Then
minor of 1 = 8
minor of 3 = 5
minor of 5 = 3, etc.

Also, let $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Then

minor of 1 = $\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 45 - 48 = -3$

minor of 2 = $\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36 - 42. = -6$ and so on.

6.2 Co-factor of an Element of a Matrix

The co-factor of an element of a matrix is obtained after deleting the corresponding rows and corresponding columns with a proper sign. The sign scheme can be used for 2nd order matrix $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$ and for 3rd order matrix $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

Then

co-factor of 2 = 5
co-factor of 3 = –4
co-factor of 4 = –3
co-factor of 5 = 2

Also, if $B = \begin{pmatrix} 1 & 0 & -2 \\ 4 & -5 & 6 \\ -7 & -3 & 6 \end{pmatrix}$,

the co-factor of

$1 = \begin{vmatrix} -5 & 6 \\ -3 & 6 \end{vmatrix} = -30 + 18 = -12.$

co-factor of 0 = $\begin{vmatrix} 4 & 6 \\ -7 & 6 \end{vmatrix} = 24 + 42 = -66$ and so on.

6.3 Expansion of a Determinant

Expansion of a determinant is the sum of the product of the elements of any one row or column with their corresponding co-factors.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

then $|A| = a.d + b.(-c) = ad - bc$

Also, if $B = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$

then $|B| = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$$a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$$= a(bc - f^2) - h(ch - fg) + g(hf - bg)$$
$$= abc - af^2 - ch^2 + fgh + fgh - bg^2$$
$$= abc + 2fgh - af^2 - bg^2 - ch^2$$

6.5 Properties of Determinants

1. The value of a determinant remains unchanged, if the rows and columns are interchanged.

   If A is any square matrix, then $|A| = |A^T|$.

2. If any two rows or columns of a determinant are interchanged, the value of determinant remains same but in opposite sign.

   Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

   and $D_2 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

   Then $D_1 = -D_2$

3. If every element of a row or column of a determinant is zero, the value of a determinant is zero.

   For example, $D_1 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

4. If any two rows (or columns) of a determinant are identical, the value of determinant is zero.

   Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$,

   then $D_1 = 0$
5. If any two rows or columns of a determinant are proportional, the value of a determinant is 0.

   Let $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 5 & 7 \end{vmatrix}$,

   then $D_1 = 0$
6. If every elements of any one row or column is multiplied by a non-zero constant, the value of a determinant is multiplied by that number.

   Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

   and $D_2 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $D_1 = kD_2$
7. If every element of a row or a column as a sum of two or more terms, the given determinant is equal to the sum of two or more determinants.

   For example,

   $$\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
8. The value of a determinant is unchanged by adding to the elements of any row or column with the same multiples of the corresponding elements of any other row or column.

   Let $D_1 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $D_2 = \begin{vmatrix} a + cm & b + dm \\ c & d \end{vmatrix}$,

   then $D_1 = D_2$
9. **Special determinants**

   (i) **Symmetric determinant**

   $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

   (ii) **Skew-symmetric determinant of odd order is zero, i.e.**

   $$\begin{vmatrix} 0 & b & c \\ -b & 0 & a \\ -c & -a & 0 \end{vmatrix} = 0$$

   (iii) **Circulant determinant**

   $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

   (iv) **Some important determinants to remember**

   1. $\begin{vmatrix} 1 & a & b + c \\ 1 & b & c + a \\ 1 & c & a + b \end{vmatrix} = 0$

   2. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a) = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

   3. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

   4. $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

      $= (a - b)(b - c)(c - a)(ab + bc + ca)$

   5. $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix} = 0$,

      where ω is the complex cube root of unity.

   6. $\begin{vmatrix} 1 & bc & ca + ab \\ 1 & ca & ab + bc \\ 1 & ab & bc + ca \end{vmatrix} = 0$

   7. $\begin{vmatrix} a - b & b - c & c - a \\ p - q & q - r & r - p \\ x - y & y - z & z - x \end{vmatrix} = 0$

   8. $\begin{vmatrix} \sin A & \cos A & \sin(A + \theta) \\ \sin B & \cos B & \sin(B + \theta) \\ \sin C & \cos C & \sin(C + \theta) \end{vmatrix} = 0$

7. Cramers Rule Statement

The solutions of the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

are given by

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D},$$

where

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_1=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_2=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Proof: Given $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\Rightarrow \quad xD = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix} \quad (C_1 \to C_1 + C_2 + C_3)$$

$$\Rightarrow \quad xD = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = D_1$$

$$\Rightarrow \quad x = \frac{D_1}{D}$$

Similarly, we can proved that

$$y = \frac{D_2}{D}, z = \frac{D_3}{D}$$

Hence, the result.

Nature of solutions of the system of equaions by Camers Rule

(i) If $D \neq 0$, the system of equations has a unique solution and is said to be consistent.

(ii) If $D = 0$ as well as $D_1 = 0 = D_2 = D_3$, the system of equations has infinitely many solutions and is said to be consistent.

(iii) If $D = 0$ and at least one of D_1, D_2, D_3 is non-zero, the system of equations has no solution and is said to be inconsistent.

8. Homogeneous System of Equations

The given homogenous system of equations are

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0.$$

Let $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_1=\begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix},$

$$D_2=\begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} \text{ and } D_3=\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}$$

Nature of solutions by homogeneous system of equations

(i) If $D \neq 0$, the system of equations has only trivial solution, say $x = 0 = y = z$, and the system of equations is said to be consistent.

(ii) If $D = 0$, the system of equations has non-trivial solution, i.e. infinite solutions and the system of equations is also said to be consistent.

9. Multiplication of Two Determinants

Two determinants can be multiplied by a variety of ways. row-by-column, row-by-row, column-by-column and column-by-row multiplication rule.

Let $A=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $B=\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}$.

Then $AB=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}$

10. Differentiation of Determinant

Let $F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$

Then
$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ p'(x) & q'(x) & r'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

or
$$F'(x)=\begin{vmatrix} f'(x) & g(x) & h(x) \\ p'(x) & q(x) & r(x) \\ u'(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g'(x) & h(x) \\ p(x) & q'(x) & r(x) \\ u(x) & v'(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h'(x) \\ p(x) & q(x) & r'(x) \\ u(x) & v(x) & w'(x) \end{vmatrix}$$

11. Integration of Determinant

If $F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix}$,

then
$$\int_a^b F(x)\,dx=\begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx & \int_a^b h(x)\,dx \\ a & b & c \\ 1 & m & n \end{vmatrix}$$

12. Summation of Determinants

Let $\Delta_r=\begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ p & q & r \end{vmatrix}$

Then $\sum_{r=1}^{n}\Delta_r=\begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ p & q & r \end{vmatrix}$

13. Adjoint of a Matrix

The adjoint of a matrix is the transpose of the co-factors of the corresponding elements of a given matrix.

If A be any square matrix, then

$$\text{adj } A=(C_{ij})^T$$

Example 1: Let $A=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$,

then $\text{adj}(A)=\begin{pmatrix} 5 & -4 \\ -3 & 2 \end{pmatrix}^T=\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$.

Example 2: Let $B=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix}$,
then

$$\text{adj}(B)=\begin{pmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}^T$$

$$=\begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}^T$$

$$=\begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}$$

Theorem: If A be any square matrix, then

$$A\cdot[\text{adj}(A)]=|A|\cdot I_n=[\text{adj}(A)]\cdot A$$

13.1 Properties of Adjoint of Matrix or Matrices

(i) If A be any square matrix of order n, then

$$|\text{adj}(A)|=|A|^{n-1}$$

Proof: We know that $A\cdot\text{adj}(A)=|A|\cdot I_n$

$\Rightarrow\quad |A\cdot\text{adj}(A)|=||A|\cdot I_n|=|A|^n$

$\Rightarrow\quad |A|\,|\text{adj}(A)|=||A|\cdot I_n|=|A|^n$ $\quad(\because |AB|=|A||B|)$

$\Rightarrow\quad |\text{adj}(A)|=\dfrac{|A|^n}{|A|}=|A|^{n-1}$

Hence, the result.

(ii) If $|A|=0$, then $|\text{adj}(A)|=0$
i.e. if A is singular, then adj(A) is also singular

(iii) $\text{adj}(kA)=k^{n-1}(\text{adj}(A))$

(iv) $\text{adj}(A^T)=(\text{adj } A)^T$

(v) $\text{adj}(AB)=(\text{adj } B)(\text{adj } A)$

Proof: We have

$(AB)\,\text{adj}(AB)=|AB|\,I_n$

$\Rightarrow\quad (\text{adj } B)(\text{adj } A)(AB)\,\text{adj}(AB)$
$=|AB|\,(\text{adj } B)(\text{adj } A)I_n$

$\Rightarrow\quad (\text{adj } B)\,|A|\,I_nB(\text{adj } AB)=|AB|(\text{adj } B)(\text{adj } A)$

$\Rightarrow\quad |A|\,(\text{adj } B)B(\text{adj } AB)=|AB|\,(\text{adj } B)(\text{adj } A)$

$\Rightarrow\quad |A||B|\,I_n(\text{adj } AB)=|AB|(\text{adj } B)(\text{adj } A)$

$\Rightarrow\quad |A|\,|B|\,(\text{adj } AB)=|AB|(\text{adj } B)(\text{adj } A)$

$\Rightarrow\quad (\text{adj } AB)=(\text{adj } B)(\text{adj } A)$

(vi) If A is a non-singular matrix, then

$$\text{adj}[\text{adj}(A)]=|A|^{n-2}\cdot A$$

Proof: We know that
$A.\text{adj}(A)=|A|.I_n$
Replace A by adj$A$, we get

$\text{adj}(A)\cdot\text{adj}[\text{adj}(A)]=|\text{adj}(A)|\cdot I_n$

$\Rightarrow\quad \text{adj}(A)\cdot\text{adj}[\text{adj}(A)]=|A|^{n-1}\,I_n$

$\Rightarrow\quad [A\,\text{adj}(A)]\cdot\text{adj}(\text{adj}(A))=|A|^{n-1}(AI_n)$

$\Rightarrow\quad |A|\cdot I_n\,\text{adj}[\text{adj}(A)]=|A|^{n-1}\cdot A$

$\Rightarrow\quad |A|\,\text{adj}[\text{adj}(A)]=|A|^{n-1}\cdot A$

$\Rightarrow\quad \text{adj}[\text{adj}(A)]=\dfrac{|A|^{n-1}\cdot A}{|A|}$

$\Rightarrow\quad \text{adj}[\text{adj}(A)]=|A|^{n-2}\cdot A$

(vii) $|\text{adj}[\text{adj}(A)]|=|A|^{(n-1)^2}$

Proof: We have $|[\text{adj}(A)]|=|A|^{n-1}$

Replace A by adj(A), we get,

$$|[\text{adj}\{\text{adj}(A)\}]| = |\text{adj}(A)|^{n-1}$$

$$\Rightarrow |[\text{adj}\{\text{adj}(A)\}]| = ||A|^{n-1}|^{n-1}$$

$$\Rightarrow |[\text{adj}\{\text{adj}(A)\}]| = |A|^{(n-1)^2}$$

(viii) If A be a square matrix of order n and B be its adjoint, then $\det(AB + kI_n) = (\det A + k)^n$.

Proof: We have,

$$AB = A[\text{adj}(A)]$$

$$\Rightarrow AB = |A|I_n$$

$$\Rightarrow AB + kI_n = |A|I_n + kI_n$$

$$\Rightarrow (AB + kI_n) = (|A| + k)I_n$$

$$\Rightarrow \det(AB + kI_n) = \det[(|A| + k)I_n]$$

$$\Rightarrow \det(AB + kI_n) = \det[(|A| + k)^n]$$

$(\because \det|kI_n| = k^n)$

14. Singular and Non-singular Matrices

Singular Matrix

If the determinant of a matrix is zero, it is called a singular matrix.

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, then $|A| = 4 - 4 = 0$.

Thus A is a singular matrix.

Non-singular matrix

If the determinant of a matrix is non-zero, it is called non-singular matrix.

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$,

then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

Thus A is non-singular.

15. Inverse of a Matrix

For every non-singular square matrix of order n, there exists another square matrix of the same order such that $AB = I_N$, then B is called the inverse of A.

As we know that, $A.[\text{adj}(A)] = |A|.I_n$

$$\Rightarrow A \cdot \left(\frac{[\text{adj}(A)]}{|A|}\right) = I_n$$

$$\Rightarrow A^{-1} = \left(\frac{[\text{adj}(A)]}{|A|}\right)$$

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$$

Thus, its inverse exists.

Now, $\text{adj}(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

Hence, $A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

15.1 Properties of Inverse of a Matrix or Matrices

(i) $(A^{-1})^{-1} = A$

(ii) $|A^{-1}| = \frac{1}{|A|}$

(iii) $(A^T)^{-1} = (A^{-1})^T$

(iv) If k is non-zero and A is non-singular, then $(kA)^{-1} = \frac{1}{k}(A^{-1})$

(v) $(I^{-1}) = I$

(vi) $(\text{adj } A^{-1}) = (\text{adj } A)^{-1} = \frac{A}{|A|}$

(vii) If A and B be two non singular matrices, then $(AB)^{-1} = (B^{-1}A^{-1})$.

(viii) Inverse of a non-singular diagonal matrix is again a diagonal matrix, i.e.

if $A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ where $a_{ii} \neq 0$

then $A^{-1} = \text{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \ldots, \frac{1}{a_{nn}}\right)$.

For example,

If $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$,

then $A^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$

(ix) If $A = \begin{pmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix} = F(x)$

then $A^{-1} = F(-x)$.

16. Solutions of the System of Equations by Matrix (Inverse) Method

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The given system of equations can be written in matrix form as

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$\Rightarrow \quad AX = B$

where

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Nature of solutions by the matrix method

(i) If $|A| \neq 0$, the system of equations has a unique solution and the system of equation is said to be consistent.

(ii) If $|A| = 0$ and (adj $A)B = O$, the system of equations has infinity many solutions and the system of equations is said to be consistent.

(iii) If $|A| = 0$ and (adj $A)B \neq O$, the system of equations has no solution and the system of equations is said to be inconsistent.

17. Elementary Transformations of a Matrix

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns, which are known as elementary transformations.

(i) The interchange of any two rows (or columns). Symbolically, $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

(ii) The multiplications of the elements of any one row (or column) by a non-zero constant, say k, i.e. symbolically,
$R_i \to kR_j$ or $C_i \to kC_j$

(iii) The addition to the elements of any one row (or column) with the corresponding elements of any other row (or column) multiplied by a non-zero number, i.e. symbolically,
$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Equivalent matrices

Two matrices are said to be equivalent, if one is obtained from other by elementary transformations.

We generally write it as

$A \sim B$.

18. Advance Types of Matrices

1. Idempotent matrix

A square matrix A is said to be an idempotent matrix if $A^2 = A$.

For example, $A = \begin{bmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is an idempotent matrix.

2. Periodic matrix

A square matrix A is said to periodic with period k (where k is a least positive integer such that $A^{k+1} = A$, i.e. if $A^3 = A, A^5 = A, A^7 = A$, it is a periodic matrix and $A^{2+1} = A$, so its period = 2.

3. Nilpotent matrix

A square matrix A is called a nilpotent matrix if there exists $k \in N$ such that $A^k = 0$, where k is called the index of the nilpotent of matrix A.

For example, $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is a nilpotent matrix.

4. Involutory matrix

A square matrix A is called an involutory matrix, if $A^2 = I$, i.e. $A^{-1} = A$.

For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an involutory matrix.

5. Orthogonal matrix

A square matrix A is said to be orthogonal matrix, if $AA^T = I$, where A^T is the transpose of matrix A and I is an identity matrix.

For example, $A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}$ is an orthogonal matrix.

> ***Note***
> (i) If $AA^T = I$, then $A^{-1} = A^T$.
> (ii) Every orthogonal matrix is non-singular.
> (iii) I is an orthogonal matrix.
> (iv) If A and B are orthogonal, then AB is also orthogonal
> (v) If A is orthogonal, then A^{-1} and A^T are also orthogonal.

6. Conjugate of a matrix

Let A be any matrix containing complex number as its elements, then a matrix is obtained from A on replacing its elements by the corresponding conjugate complex numbers, is called the conjugate of the matrix A and it is denoted by $\bar{A}$.

For example, If $A = \begin{bmatrix} 1+2i & 2+3i \\ 1-i & 3+4i \end{bmatrix}$,

then $\bar{A} = \begin{bmatrix} 1-2i & 2-3i \\ 1+i & 3-4i \end{bmatrix}$.

7. Complex conjugate transpose of a matrix

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is denoted by A^θ.

If $A = \begin{pmatrix} 2+4i & 3 \\ 4 & \alpha+i\beta \end{pmatrix}$, then $A^\theta = \begin{pmatrix} 2-4i & 3 \\ 4 & \alpha-i\beta \end{pmatrix}$.

8. Hermitian matrix

A square matrix A is called a hermitian matrix, if $A^\theta = A$.

For example, $A = \begin{pmatrix} 2 & \alpha+i\beta \\ \alpha-i\beta & 3 \end{pmatrix}$.

9. Skew hermitian matrix

A square matrix A is called a skew-hermitian matrix, if $A^\theta = -A$.

For example, $A = \begin{pmatrix} 2i & -\alpha-i\beta \\ \alpha+i\beta & -i \end{pmatrix}$

10. Unitary matrix

A square matrix A is called unitary, if $AA^{\theta} = I$.

For example, $A = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}$

> ***Note*** If A and B are unitary, then AB is also a unitary.

11. Equivalent matrices

Let A and B are two matrices. If B is obtained from A by elementary transformation, then A and B are called equivalent matrices.

12. Rank of a matrix

Rank of a matrix represents the non-zero rows of an equivalent matrix.

Rule to find out the rank of a matrix

Let A be any type of matrix

Case I: When A is a null matrix, then the rank of a matrix is zero.

Case II: When A is a square matrix, then we shall first find the determinant of A.

(i) If A is non-singular (i.e. $|A| \neq 0$), then rank of the matrix = order of the matrix

(ii) If A is singular (i.e. $|A| = 0$), then we shall find the minor along rows:

(a) If at-least one minor is zero and rest are non-zero, then rank of the matrix = order of the matrix –1.

(b) If all minor is non-zero, then rank of the matrix = 0.

Case III: When A is a rectangular matrix of order $m \times n$, then we shall find an equivalent matrix of A.

(i) If any one row is zero, then rank of the matrix = Minimum of $\{m-1, n-1\}$

(ii) If any two row is zero, then rank of the matrix = Minimum of $\{m-2, n-2\}$

(iii) If all rows are non-zero, then rank of the matrix = Minimum of $\{m, n\}$.

EXERCISES

LEVEL I

(Problems based on Fundamentals)

ORDER OF MATRICES

1. Find the number of all possible matrices of order 2×2 with each entry 0 or 1.
2. Find the number of all possible matrices of order 3×3 with each entry either 1 or 2.

ADDITION OF MATRICES

3. If $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$, find the additive inverse of A.
4. Find a matrix X, if $X + \begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 2 & 7 \end{pmatrix}$.
5. Find X and Y, if $X + Y = \begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix}$.

   and $X - Y = \begin{pmatrix} 4 & 2 \\ 8 & -2 \end{pmatrix}$.
6. Find a matrix X such that

   $A + 2B + X = \mathbf{O}$,

   where $A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$.
7. Find x and y, if

   $$\begin{pmatrix} |x| & 2 \\ 5 & |y-2| \end{pmatrix} = \begin{pmatrix} <3 & 2 \\ 5 & <4 \end{pmatrix}$$
8. Find $\Sigma(x+y)$, if

   $$\begin{pmatrix} x^3 - 3x + 2 & 2 \\ 3 & y^3 + 7y^2 - 35 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$$
9. Find x, y, z and t satisfying the equations

   $$2\begin{pmatrix} x & y \\ z & t \end{pmatrix} + 3\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = 4\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$$
10. Find the matrices X and Y, if

    $$2X + 3Y = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix} \text{ and } 3X + 2Y = \begin{pmatrix} -1 & 2 \\ 1 & -5 \end{pmatrix}$$
11. If $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

    and $f(x) = 1 + x + x^2 + \ldots +$ to ∞, find $f(A)$.

MULTIPLICATION OF MATRICES

12. Let $A = [1\ 2]$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

    Find AB and BA.
13. Let $A = [a\ b\ c]$ and $B = \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}$.

    Find AB and BA.
14. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix}$.

    Find AB and BA.

15. If A be a 2×3 matrix and AB a 2×5 matrix, find the order of the matrix B.

16. If A be a $(A)_{2\times3}$ matrix and B a $(B)_{5\times3}$ matrix and C a $(C)_{3\times7}$ matrix, find $2015ABC$.

17 If $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$,

prove that

$$A^2 + 2A + I_2 = \mathbf{O}$$

18. If $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $B = (2 \quad 3 \quad 4)$, AB and BA.

19. If $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$, find A^2, A^3, and A^4.

20 Find a 2×2 matrix X such that

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} . X = \begin{pmatrix} 3 & 5 \\ 0 & 6 \end{pmatrix}$$

21. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,

prove that $A^2 - 4A - 5I_3 = \mathbf{O}$.

22. Find x, if $(1 \quad x \quad 1)\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ x \end{pmatrix} = 0$.

23. If $A = \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$, the value of a for which $A^2 = B$ is

(a) 1 (b) −1
(c) 4 (d) No real values

24. If $A = \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$ and $|A^3| = 125$, the value of α is

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 5

25. If A and B be two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$, find $(A + B)^7$.

26. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are two matrices such that $AB = BA$ and $c \neq 0$, find the value of $\dfrac{a-d}{3b-c}$.

27. If A be a square matrix of order 2 such that $A^2 = \mathbf{O}$, find $A(I + A)^{2009}$.

28. If A be a square matrix and I be an identity matrix of the same order such that $A^2 = I$, $(I - A)(I + A)$.

29. If A be a square matrix such that $A^2 = A$, find $(I + A)^3 - 7A$.

30. If $A = \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$ and $f(x) = \sum_{n=0}^{16} x^n$, find $f(A)$.

31. If A be a square matrix such that $A^2 = A + I$, where I is the identity matrix of the same order, find A^5.

32. If A be a square matrix such that $A^2 = A - I$, where I is the identity matrix of the same order, find A^n.

33. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $n \in N$, find A^n.

34. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$.

If U_1 and U_2 are column matrices such that

$AU_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $AU_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, find $U_1 + U_2$.

SYMMETRIC AND SKEW-SYMMETERIC MATRICES

35. Express the matrix $A = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ as a sum of a symmetric and a skew-symmetric matrices.

36. Express the matrix $B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 5 \\ 4 & 3 & 6 \end{pmatrix}$ as a sum of a symmetric and a skew-symmetric matrices.

DETERMINANT

37. If A be a skew-symmetric matrix of odd order, find det (A).

38. If A be a skew-symmetric matrix of even order, find $\det(A)$.

39. If $A = (a_{ij})$ be a square matrix of order n such that

$a_{ij} = \begin{cases} 0: & i = j \\ 1: & i \neq j \end{cases}$, find $\det(A)$.

40. If $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ be a square matrix of order 2, find $\det(A^{2010} + 2A^{2009})$.

41. If A be a square matrix such that $A^2 = A$, find $\det(A)$.

42. If A be a square matrix of order 3 such that $\det(A) = 8$, find $\det(3A)$.

43. If A be a square matrix of order n such that $|A| = 2$, find $|An|$ for $n \in I^+$.

44. Find the maximum value of the determinant

$$\begin{vmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{vmatrix}$$

45. Let P and Q be two square matrices of order 3 such that $P^3 = Q^3$ and $P^2Q = Q^2P$ and $P \neq Q$, find

$\det(P^2 + Q^2)$.

46. Expand the determinant

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

47. Evaluate: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

48. Evaluate: $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$.

49. Evaluate: $\begin{vmatrix} \sin\alpha & \cos\beta & \cos(\alpha+\theta) \\ \sin\beta & \cos\beta & \cos(\beta+\theta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\theta) \end{vmatrix}$.

50. Prove that the value of $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(a+c) \\ 1 & ab & c(a+b) \end{vmatrix}$ is independent of a, b, c.

51. Prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$$

52. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$

53. Prove that $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$

54. Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3a \end{vmatrix} = a^3.$

55. Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

$= (1+a^2+b^2)^3.$

56. Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & cb & c^2+1 \end{vmatrix} = (1+a^2+b^2+c^2).$

57. Prove that $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix} = \frac{xyz}{12}(x-y)(y-z)(z-x).$

58. Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab.$$

59. Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

60. Prove that $\begin{vmatrix} (a+1)(a+2) & (a+2) & 1 \\ (a+2)(a+3) & (a+3) & 1 \\ (a+3)(a+4) & (a+4) & 1 \end{vmatrix} = -2.$

61. Prove that $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}.$

62. Solve the following system of equations:

$2x + 3y = 4$
$3x - 2y = 5.$

63. Solve the following system of equations:

$x + 3y = 4$
$2x + 6y = 10.$

64. Solve the following system of equations:

$2x + 5y = 6$
$6x + 15y = 18.$

65. Find the number of triplets of a, b and c for which the system of equations

$ax - by = 2a - b$
$(c+1)x + cy = 10 - a + 3b.$

has infinitely many solutions.

66. Solve for x, y, z:

$x + y + z = 1$
$ax + by + cz = d$
$a^2x + b^2y + c^2z = d$

67. Solve for x, y, z:

$$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=\frac{1}{4}$$
$$\frac{2}{x}-\frac{1}{y}+\frac{3}{z}=\frac{9}{4}$$
$$-\frac{1}{x}-\frac{2}{y}+\frac{4}{z}=1.$$

68. Find the equation of the parabola $y = ax^2 + bx + c$, which passes through the points (2, 4), (–1, 1) and (–2, 5).

69. Find the value of k, for which the system of equations

$2x + ky = 5$
$3x - 4y = 7$

has a unique solution.

70. Find the value of λ, for which the system of equations

$$3x+4y=5$$
$$\lambda x+8y=10$$

give infinitely many solutions.

71. If the system of equations

$$x+2y-3z=1$$
$$(p+2)z=3$$
$$(2p+1)y+z=2$$

is inconsistent, find the value of p.

72. If the system of equations

$$2x-y+2z=2$$
$$x-2y+z=-4$$
$$x+2y+\lambda z=4$$

has no solutions, find λ.

HOMOGENEOUS EQUATIONS

73. Solve the system of equations:

$$2x+3y=0$$
$$4x+6y=0.$$

74. Find the number of values of t for which the system of equations

$$(a-t)x+by+cz=0$$
$$bx+(c-t)y+az=0$$
$$cx+ay+(b-t)z=0$$

has non-trivial solution.

75. Find the value of λ, if the system of equations

$$6x+5y+\lambda z=0$$
$$3x-y+4z=0$$
$$x+2y-3z=0$$

has a unique solution.

76. If the system of equations $x+ay=0$, $y+az=0$ and $z+ax=0$ has infinite solutions, find a.

MULTIPLICATION OF DETERMINANTS

77. Prove that

$$\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}^2 = \begin{vmatrix} c^2+a^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix}.$$

78. Prove that

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0$$

79. Prove that

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0$$

80. If $\alpha, \beta \neq 0$ and $f(n)=\alpha^n+\beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2,$$

find k.

DIFFERENTIATION OF DETERMINANT

81. If $F(x)=\begin{vmatrix} 1 & a & a^2 \\ x & x^2 & x^3 \\ e^{x-a} & e^{x^2-a^2} & e^{x^3-a^3} \end{vmatrix}$,

find the value of $F'(a)$.

82. Let $f(x)=\begin{vmatrix} 3 & 2 & 1 \\ 6x^2 & 2x^3 & x^4 \\ 1 & b & b^2 \end{vmatrix}$,

find $f''(b)$

INTEGRATION OF DETERMINANTS

83. If $f(x)=\begin{vmatrix} \sin^2 x & \log(\sin x) & \dfrac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} \\ n & \sum_{k=1}^{n}(k) & \prod_{k=1}^{n}(k) \\ \dfrac{8}{15} & \dfrac{\pi}{2}\log\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4} \end{vmatrix}$,

find the value of $\int_0^{\pi/2} f(x)\,dx$.

84. If $\Delta_r=\begin{vmatrix} r & 2012 & \dfrac{n(n+1)}{2} \\ 2r-1 & 2013 & n^2 \\ 3r-2 & 2014 & \dfrac{n(3n-1)}{2} \end{vmatrix}$,

find $\sum_{r=1}^{n}\Delta_r$

85. If $D_r=\begin{vmatrix} 2^{r-1} & 101 & (2^n-1) \\ 3^{r-1} & 102 & \left(\dfrac{3^n-1}{2}\right) \\ 5^{r-1} & 103 & \left(\dfrac{5^n-1}{4}\right) \end{vmatrix}$,

find the value of $\sum_{r=1}^{n} D_r$

ADJOINT OF MATRICES

86. If A be a square matrix of order n, find $\text{adj}(A') - \text{adj } A)'$.

87. If A be a non-singular matrix of order 3, find $\det(\text{adj } A^3)$.

88. If A be a square matrix of order 2 such that $A \cdot (\text{adj } A) = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$, find $\det(A)$.

89. If A be a square matrix of order 3 such that $\det(A) = 4$, find $\det(\text{adj } A)$.

90. If A be a square matrix of order n such that $|\text{adj}(\text{adj}A)| = |A|^{16}$, find n.

91. If the adjoint of a matrix P of order 3 is $\begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{pmatrix}$, the possible values of the determinant of P is (are)
(a) -2 (b) -1 (c) 1 (d) 2

92. Let $P = |a_{ij}|$ be a 3×3 matrix and let $Q = |b_{ij}|$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$.
If the determinant of P is 2, the determinant of the matrix Q is
(a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13}

93. If $P = \begin{pmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{pmatrix}$ is the adjoint of a matrix A of order 3×3 and $|A| = 4$, find α.

94. Find the inverse of a skew symmetric matrix of odd order.

95. If B be a non-singular matrix and A is a square matrix of the same order, find $|B^{-1} AB|$.

96. If $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, find the inverse of a matrix R, such that $R = (P \cos \theta + Q \sin \theta)$.

97. If A and B be two non-singular square matrices such that $B^{-1}AB = A^2$, find $B^{-3}AB^3$.

98. If A be a non-singular matrix satisfying
$I + A + A^2 + A^3 + \ldots + A^k = \mathbf{O}$,
find A^{-1}.

99. If A be a non-singular matrix satisfying $A^2 - A + I = \mathbf{O}$, find A^{-1}.

100. If A be a non-singular square matrix of order 3×3 such that $\det(A) = 5$, find $\det(\text{adj}(A^{-1}))$.

101. If A be a non-singular square matrix of order 3×3, find $|A^{-1} \text{ adj}(A)|$.

102. If $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, find the multiplicative inverse of A.

103. If X be a non-singular square matrix of order 2 such that $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} . X = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$,
find X.

104. Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 7 & 3 \end{pmatrix}$, find the value of det $(2A^9B^{-1})$.

105. Let P and Q be two square matrices such that $|P| = 1 = Q$. If A and B be two square matrices of the same order such that $(\text{adj}B) = A$, find $\text{adj}(QBP)$.

106. If A is an 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, find BB'. **[JEE Main, 2014]**

107. Find the inverse of the matrix, $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix}$.

108. If $A = \begin{pmatrix} 0 & -\tan\left(\dfrac{\alpha}{2}\right) \\ \tan\left(\dfrac{\alpha}{2}\right) & 0 \end{pmatrix}$,

prove that $(I + A) = (I - A)\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

109. Solve the following system of equations by matrix method.
$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$

110. Compute A^{-1} for the matrix
$$A = \begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$
and hence solve the following system of equations
$$-x + 2y + 5z = 2$$
$$2x - 3y + z = 15$$
$$-x + y + z = -3$$

111. Solve the following system of equations by matrix (inverse) method.
$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

112. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system
$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
has exactly two solutions, is
(a) 0 (b) $2^9 - 1$ (c) 168 (d) 12

113. Let a, b, c be positive real numbers. Prove that the system of equations
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
in x, y, z have a unique solutions.

114. By using elementary row operation, find the inverse of $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$.

RANK OF A MATRIX

115. Find the rank of $A=\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 7\\ 3 & 6 & 10\end{pmatrix}$.

116. Find the rank of $A=\begin{pmatrix}2 & 4 & 3\\ 1 & 2 & -1\\ -1 & -2 & 6\end{pmatrix}$.

117. Find the rank of $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 4 & 5 & 6\end{pmatrix}$.

118. Find the rank of $A=\begin{pmatrix}1 & -2 & 1 & -1\\ 1 & 1 & -2 & 3\\ 4 & 1 & -5 & 6\end{pmatrix}$.

ADVANCE MATRICES

119. Prove that the matrix $A=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}$ is idempotent

120. Prove that the matrix $A=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -1\end{bmatrix}$ is periodic.

121. Show that $\begin{bmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{bmatrix}$ is nilpotent matrix of order 3.

122. Show that the matrix $A=\begin{bmatrix}-5 & -8 & 0\\ 3 & 5 & 0\\ 1 & 2 & -1\end{bmatrix}$ is involuntary.

123. Prove that the matrix $A=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}$ is orthogonal matrix.

124. Prove that $A=\begin{bmatrix}3 & 3-4i & 5+2i\\ 3+4i & 5 & -2+i\\ 5-2i & -2-i & 2\end{bmatrix}$ is an Hermitian matrix.

125. Prove that $A=\begin{bmatrix}2i & -2-3i & 2+i\\ 2-3i & -i & 3i\\ 2+i & 3i & 0\end{bmatrix}$ is an skew-hermitian matrix.

126. Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix}$ is unitary.

Level II

(Mixed Problems)

1. If a, b and c are non-zero real numbers, then $\Delta=\begin{vmatrix}b^2c^2 & bc & b+c\\ c^2a^2 & ca & c+a\\ a^2b^2 & ab & a+b\end{vmatrix}=$
   (a) abc (b) $a^2b^2c^2$
   (c) $ab+bc+ca$ (d) none of these

2. The value of the determinant $\begin{vmatrix}1/a & a^2 & bc\\ 1/b & b^2 & ca\\ 1/c & c^2 & ab\end{vmatrix}$ is
   (a) abc (b) $1/abc$
   (c) $ab+bc+ca$ (d) 0

3. The value of the determinant
   $$\begin{vmatrix}b^2+c^2 & a^2 & a^2\\ b^2 & c^2+a^2 & b^2\\ c^2 & c^2 & a^2+b^2\end{vmatrix}=$$
   (a) abc (b) $4abc$
   (c) $4a^2b^2c^2$ (d) $a^2b^2c^2$

4. The value of the determinant
   $$\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}=$$
   (a) $(a+b+c)^2$
   (b) $(a+b+c)^3$
   (c) $(a+b+c)(ab+bc+ca)$
   (d) None of these

5. The value of the determinant
   $$\begin{vmatrix}a-b & b-c & c-a\\ x-y & y-z & z-x\\ p-q & q-r & r-p\end{vmatrix}=$$
   (a) $a(x+y+z)+b(p+q+r)+c$
   (b) 0
   (c) $abc+xyz+qr$
   (d) none of these

6. The value of the determinant $\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}=$
   (a) $3abc+a^3+b^3+c^3$
   (b) $3abc-a^3-b^3-c^3$
   (c) $abc-a^3+b^3+c^3$
   (d) $abc+a^3-b^3-c^3$

7. For non-zero a, b, c, if

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$$

the value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=$

(a) abc (b) $\frac{1}{abc}$

(c) $-(a+b+c)$ (d) none of these

8. The value of the determinant

$$\begin{vmatrix} a_1 & la_1+mb_1 & b_1 \\ a_2 & la_2+mb_2 & b_2 \\ a_3 & la_3+mb_3 & b_3 \end{vmatrix} \text{ is}$$

(a) 0 (b) l (c) m (d) lm

9. If T_p, T_q, T_r are respectively, the pth, qth and rth terms of an AP, then $\begin{vmatrix} T_p & T_q & T_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ is equal to

(a) 1 (b) −1 (c) 0 (d) $p+q+r$

10. The value of the determinant

$$\begin{vmatrix} 2ac-b^2 & a^2 & c^2 \\ a^2 & 2ab-c^2 & b^2 \\ c^2 & b^2 & 2bc-a^2 \end{vmatrix} =$$

(a) $4abc$ (b) $-4abc$

(c) 0 (d) $(a^3+b^3+c^3-3abc)^2$

11. If $a \neq b \neq c$ and $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = 0$,

the value of $(a+b+c)$ is

(a) 1 (b) 0 (c) 2 (d) $-a$

12. Let $\Delta = \begin{vmatrix} 1 & \sin\alpha & 1 \\ -\sin\alpha & 1 & \sin\alpha \\ -1 & -\sin\alpha & 1 \end{vmatrix}$,

the value of Δ lies in the interval

(a) [2, 3] (b) [3, 4]

(c) [1, 4] (d) [2, 4]

13. If A, B, C are the angles of triangle, the value of

$$\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is}$$

(a) $\cos A \cos B \cos C$ (b) $\sin A \sin B \sin C$

(c) 0 (d) none of these

14. If a, b, c are in AP, the value of

$$\begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix} \text{ is}$$

(a) $x-(a+b+c)$ (b) $9x^2+a+b+c$

(c) $a+b+c$ (d) 0

15. The value of the determinant

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} =$$

(a) $9a^2(a+b)$ (b) $9b^2(a+b)$

(c) $a^2(a+b)$ (d) $b^2(a+b)$

16. The value of the determinant

$$\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix} =$$

(a) 0 (b) $2abc$

(c) $a^2b^2c^2$ (d) none of these

17. If $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = k(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$,

the value of k is

(a) 1 (b) 2 (c) −1 (d) −2

18. If the determinant

$$\begin{vmatrix} a & a+d & a+2d \\ a^2 & (a+d)^2 & (a+2d)^2 \\ 2a+3d & 2(a+d) & 2a+d \end{vmatrix} = 0,$$

(a) $d=0$ (b) $a+d=0$

(c) $d=0$ or $a+d=0$ (d) none

19. The value of the determinant

$$\begin{vmatrix} a+b & a+2b & a+3b \\ a+2b & a+3b & a+4b \\ a+4b & a+5b & a+6b \end{vmatrix} =$$

(a) $a^2+b^2+c^2-3abc$ (b) $3ab$

(c) $3a+5b$ (d) 0

20. If $0<\theta<\frac{\pi}{2}$ and

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0,$$

θ is equal to

(a) $\frac{\pi}{24}, \frac{5\pi}{24}$ (b) $\frac{5\pi}{24}, \frac{7\pi}{24}$

(c) $\frac{7\pi}{24}, \frac{11\pi}{24}$ (d) none of these

21. The value of the determinant $\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\alpha-\beta) & 1 & \cos(\beta-\gamma) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$ is

(a) $\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}^2$

(b) $\begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}^2$

(c) $\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\beta & 0 & \cos\beta \\ 0 & \cos\gamma & \sin\gamma \end{vmatrix}^2$

(d) none of these.

22. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$,

$\sin^2 A + \sin^2 B + \sin^2 C =$

(a) $\frac{9}{4}$ (b) $\frac{4}{9}$ (c) 1 (d) $3\sqrt{3}$

23. $a \neq p$, $b \neq q$, $c \neq r$ and

$$\begin{vmatrix} p & b & c \\ p+a & q+b & 2c \\ a & b & r \end{vmatrix} = 0,$$

then $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} =$

(a) 3 (b) 2 (c) 1 (d) 0

24. The value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$, where a, b, c are respectively, the pth, qth, rth terms of an HP is

(a) $ap + bq + cr$ (b) $(a + b + c)(p + q + r)$

(c) 0 (d) none of these

25. If $\sqrt{-1} = i$ and ω be non-real cube root of unity, the value of $\begin{vmatrix} 1 & \omega^2 & 1+i+\omega^2 \\ -i & -1 & -1-i+\omega \\ 1-i & \omega^2-1 & -1 \end{vmatrix} =$

(a) 1 (b) i (c) ω (d) 0

26. Let $f(n) = \begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & n+1C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$

where the symbols have their usual meanings. Then $f(n)$ is divisible by

(a) $n^2 + n + 1$ (b) $(n + 1)!$

(c) $n!$ (d) none of these

27. If $a + b + c = 0$, one root of

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0 \text{ is}$$

(a) $x = 1$ (b) $x = 2$

(c) $x = a^2 + b^2 + c^2$ (d) $x = 0$

28. $\begin{vmatrix} x+1 & \omega & \omega^2 \\ x+\omega & \omega^2 & 1 \\ x+\omega^2 & 1 & \omega \end{vmatrix} = 3$

is an equation of x, where ω, ω^2 are the complex cube roots of unity, the value of x is

(a) 0 (b) 1 (c) −1 (d) 2

29. If $ab + bc + ca = 0$ and

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0,$$

one of the value of x is

(a) $(a^2+b^2+c^2)^{\frac{1}{2}}$ (b) $\left[\frac{3}{2}(a^2+b^2+c^2)\right]^{\frac{1}{2}}$

(c) $\left[\frac{1}{2}(a^2+b^2+c^2)\right]^{\frac{1}{2}}$ (d) none of these

30. Let $\begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix}$

$= ax^5 + bx^4 + cx^3 + dx^2 + \lambda x + \mu$

be an identity in x, where a, b, c, d, λ, μ are independent of x. Then the value of x is

(a) 3 (b) 2

(c) 4 (d) none of these

31. If the entries in a 3×3 determinant are either 0 or 1, the greatest value of this determinant is

(a) 1 (b) 2 (c) 3 (d) 9

32. If $\Delta = \begin{vmatrix} a & c & b \\ b & b & a \\ c & a & c \end{vmatrix}$ and $\Delta' = \begin{vmatrix} a^2 & c^2 & b^2 \\ b^2 & b^2 & a^2 \\ c^2 & a^2 & c^2 \end{vmatrix}$,

then

(a) $\Delta = a^2b^2c^2\Delta'$ (b) $\Delta' = a^2b^2c^2\Delta$

(c) $\Delta = abc\Delta b$ (d) None of these

33. If $\Delta = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$,

then $\begin{vmatrix} 2x+4p & p+6a & a \\ 2y+4q & q+6b & b \\ 2z+4r & r+6c & c \end{vmatrix} =$

(a) 2Δ (b) 4Δ (c) 6Δ (d) Δ

34. If $\Delta_1 = \begin{vmatrix} 1 & 0 \\ a & b \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 1 & 0 \\ c & d \end{vmatrix}$,

then $\Delta_2\Delta_1 =$

(a) ac (b) bd

(c) $(b-a)(d-c)$ (d) none of these

35. If $D_r = \begin{vmatrix} 1 & n & n \\ 2r & n^2+n+1 & n^2+n \\ 2r-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{r=1}^{n} D_r = 56$,

the value of n is

(a) 4 (b) 6 (c) 7 (d) 8

36. If $f(x) = \begin{vmatrix} \cos x & 1 & 0 \\ 1 & \cos x & 1 \\ 0 & 1 & \cos x \end{vmatrix}$,

then $f'\left(\frac{\pi}{3}\right)$ equals

(a) $\frac{11\sqrt{3}}{8}$ (b) $\frac{5\sqrt{3}}{8}$

(c) $\frac{-5\sqrt{3}}{8}$ (d) none of these

37. If $f(x) = \begin{vmatrix} \sec x & \cos & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$,

then $\int_0^{\pi/2} f(x)\,dx =$

(a) $\frac{\pi}{4} + \frac{8}{15}$ (b) $\left(-\frac{\pi}{4} + \frac{8}{15}\right)$

(c) $-\left(\frac{\pi}{4} + \frac{8}{15}\right)$ (d) none of these

38. Equations $x + y = 2$, $2x + 2y = 3$ will have
(a) only one solution
(b) infinitely many solutions
(c) no solution
(d) none of these

39. The system of equations $x + y + z = 2$, $3x - y + 2z = 6$ and $3x + y + z = -18$ has
(a) a unique solution
(b) no solution
(c) an infinite number of solutions
(d) zero solution as the only solution

40. $x + y + z = 6$, $x - y + z = 2$ and $2x + y - z = 1$, then x, y, z are respectively

(a) 3, 2, 1 (b) 1, 2, 3

(c) 2, 1, 3 (d) none of these

41. The value of k for which the set of equations $3x + ky - 2z = 0$, $x + ky + 3z = 0$ and $2x + 3y - 4z = 0$ has a non-trivial solution is

(a) 15 (b) 16 (c) 31/2 (d) 33/2

42. If the system of equations $3x - y + 4z - 3 = 0$, $x + 2y - 3z + 2 = 0$, $6x + 5y + \lambda z + 3 = 0$ has infinite number of solutions, then $\lambda =$

(a) 7 (b) –7 (c) 5 (d) –5

43. If the system of following equations $2x + 3y + 5 = 0$, $x + ky + 5 = 0$, and $kx - 12y - 14 = 0$ be constant, then $k =$

(a) $2, \frac{12}{5}$ (b) $-1, \frac{1}{5}$

(c) $-6, \frac{17}{5}$ (d) $6, -\frac{12}{5}$

44. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $n \in N$, then A^{4n} equals

(a) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

45. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$,

then A^n is equal to

(a) $\begin{bmatrix} a^n & 0 & 0 \\ 0 & a^n & 0 \\ 0 & 0 & a \end{bmatrix}$ (b) $\begin{bmatrix} a^n & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

(c) $\begin{bmatrix} a^n & 0 & 0 \\ 0 & a^n & 0 \\ 0 & 0 & a^n \end{bmatrix}$ (d) $\begin{bmatrix} na & 0 & 0 \\ 0 & na & 0 \\ 0 & 0 & na \end{bmatrix}$

46. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, values of a and b are

(a) $a = 4, b = 1$ (b) $a = 1, b = 4$

(c) $a = 0, b = 4$ (d) $a = 2, b = 4$

47. The matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is

(a) orthogonal (b) involutory

(c) idempotent (d) nilpotent

48. If $A = \begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,

then $(B^{-1}A^{-1})^{-1} =$

(a) $\begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix}$

(c) $\frac{1}{10}\begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix}$ (d) $\frac{1}{10}\begin{bmatrix} 3 & 2 \\ -2 & 2 \end{bmatrix}$

49. If A be a non-singular matrix of order 3, then adj (adj A) is equal to
(a) $|A|A$ (b) $|A|^2A$
(c) $|A|^{-1}A$ (d) none of these

50. Let the matrix $A=\begin{bmatrix} i & 1-2i \\ -1-2i & 0 \end{bmatrix}$,
then A matrix is
(a) symmetric (b) skew-symmetric
(c) hermitian (d) skew-hermitian

51. If $E(\theta)=\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$, then $E(\theta)E(\varphi)$ is a
(a) null matrix (b) unit matrix
(c) diagonal matrix (d) none of these

52. The inverse of a symmetric matrix is
(a) symmetric (b) skew-symmetric
(c) diagonal matrix (d) none of these

53. The inverse of a diagonal matrix is
(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a diagonal matrix
(d) none of these

54. If A be a symmetric matrix and $n \in N$, then A^n is
(a) symmetric (b) skew-symmetric
(c) a diagonal matrix (d) none of these

55. If A be a skew-symmetric matrix and n a positive integer, then A^n is
(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a diagonal matrix
(d) none of these

56. if A be a skew-symmetric matrix and n odd positive integer, then A^n is
(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a diagonal matrix
(d) none of these

57. If A be a square matrix of order $n \times n$ and k a scalar, then adj (kA) is equal to
(a) k adj A (b) k^n adj A
(c) k^{n-1}adj A (d) k^{n+1}adj A

58. If A and B are square matrices of order 3 such that $|A|=-1$, $|B|=3$, then $|3AB|$ equals
(a) –9 (b) –81 (c) –27 (d) 81

59. The number of all possible matrices of order 3×3 with each entry with 1 or 0 is
(a) 27 (b) 18 (c) 81 (d) 512.

60. If A be a square matrix such that $A^2 = A$, then $(1 + A)^3 - 7A$ is
(a) A (b) $I-A$ (c) I (d) $3A$

61. If A be a square matrix of order 3 such that $|A| = 2$, then $|\text{adj } A^{-1}|$ is
(a) 11 (b) 13 (c) 17 (d) 19.

62. If $A=\begin{pmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $AB = I$, then $x + y$ is
(a) 0 (b) –1 (c) 2 (d) 5

63. If the value of $D=\begin{vmatrix} ^{10}C_4 & ^{10}C_5 & ^{11}C_m \\ ^{11}C_6 & ^{11}C_7 & ^{12}C_{m+2} \\ ^{12}C_8 & ^{12}C_9 & ^{13}C_{m+4} \end{vmatrix}$ is zero, then m is
(a) 6 (b) 4 (c) 5 (d) 10

64. If $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = Ax^4 + Bx^3 + Cx^2 + Dx + E$,
the value of $5A + 4B + 3C + 2D + E$ is
(a) 0 (b) –16 (c) 16 (d) none

65. The number of values of t for which the system of equations
$(a-t)x + by + cz = 0$
$bx + (c-t)y + az = 0$
$cx + ay + (b-t)z = 0$
has non-trivial solution is
(a) 3 (b) 4 (c) 5 (d) 6

LEVEL III

(Problems for JEE-Advanced)

1. Prove that
$$\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$$ is independent of a, b, c.

2. Prove that
$$\begin{vmatrix} b+c & a & a \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc.$$

3. Prove that
$$\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & a^2+c^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$$

4. We have,

$$X+\begin{pmatrix}2 & 5\\ 3 & -2\end{pmatrix}=\begin{pmatrix}3 & 6\\ 2 & 7\end{pmatrix}$$

$$\Rightarrow \quad X=\begin{pmatrix}3 & 6\\ 2 & 7\end{pmatrix}-\begin{pmatrix}2 & 5\\ 3 & -2\end{pmatrix}$$

$$\Rightarrow \quad X=\begin{pmatrix}1 & 1\\ -1 & 9\end{pmatrix}$$

5. Clearly, $X=\frac{1}{2}\left(\begin{pmatrix}2 & 5\\ 3 & -2\end{pmatrix}+\begin{pmatrix}4 & 2\\ 8 & -2\end{pmatrix}\right)$

$$\Rightarrow \quad X=\frac{1}{2}\begin{pmatrix}6 & 7\\ 11 & -4\end{pmatrix}=\begin{pmatrix}3 & 7/2\\ 11/2 & -2\end{pmatrix}$$

and $Y=\frac{1}{2}\left(\begin{pmatrix}2 & 5\\ 3 & -2\end{pmatrix}-\begin{pmatrix}4 & 2\\ 8 & -2\end{pmatrix}\right)$

$$\Rightarrow \quad Y=\frac{1}{2}\begin{pmatrix}-2 & 3\\ -5 & 0\end{pmatrix}=\begin{pmatrix}-1 & 3/2\\ -5/2 & 0\end{pmatrix}$$

6. Given $A+2B+X=\mathbf{O}$

$\Rightarrow \quad X=-(A+2B)$

$$=-\left(\begin{pmatrix}2 & -1\\ 3 & 5\end{pmatrix}+2\begin{pmatrix}-1 & 1\\ 0 & 2\end{pmatrix}\right)$$

$$=-\left(\begin{pmatrix}2 & -1\\ 3 & 5\end{pmatrix}+\begin{pmatrix}-2 & 2\\ 0 & 4\end{pmatrix}\right)$$

$$=-\begin{pmatrix}0 & 1\\ 3 & 9\end{pmatrix}=\begin{pmatrix}0 & -1\\ -3 & -9\end{pmatrix}$$

7. Given $\begin{pmatrix}|x| & 2\\ 5 & |y-2|\end{pmatrix}=\begin{pmatrix}<3 & 2\\ 5 & <4\end{pmatrix}$

$\Rightarrow \quad |x|<2, |y-2|<3$

$\Rightarrow \quad -2<x<2, -3<(y-2)<3$

$\Rightarrow \quad -2<x<2, -1<y<5$

8. We have

$$\begin{pmatrix}x^3-3x+2 & 2\\ 3 & y^3+7y^2-35\end{pmatrix}=\begin{pmatrix}0 & 2\\ 3 & 1\end{pmatrix}$$

$x^3-3x+2=0, \quad y^3+7y^2-35=1$

$x^3-3x+2=0, \quad y^3+7y^2-36=1$

Now, $x^3-3x+2=0$

$\Rightarrow \quad x^3-x^2+x^2-x-2x+2=1$

$\Rightarrow \quad x^2(x-1)+x(x-1)-2(x-1)=0$

$\Rightarrow \quad (x-1)(x^2+x-2)=0$

$\Rightarrow \quad (x-1)(x+2)(x-1)=0$

$\Rightarrow \quad (x-1)^2(x+2)=0$

$\Rightarrow \quad x=1,-2$

Also, $y^3+7y^2-36=0$

$\Rightarrow \quad y^3-2y^2+9y^2-18y+18y-36=1$

$\Rightarrow \quad y^2(y-2)+9y(y-2)+18(y-2)=0$

$\Rightarrow \quad (y-2)(y^2+9y+18)=0$

$\Rightarrow \quad (y-2)(y+3)(y+6)=0$

$\Rightarrow \quad y=2,-3,-6$

Thus,

$\Sigma(x+y)=(1-2+2-3-6)=-8$

9. We have

$$2\begin{pmatrix}x & y\\ z & t\end{pmatrix}+3\begin{pmatrix}1 & -2\\ 0 & 4\end{pmatrix}=4\begin{pmatrix}3 & 5\\ 4 & 6\end{pmatrix}$$

$$\Rightarrow \quad 2\begin{pmatrix}x & y\\ z & t\end{pmatrix}=4\begin{pmatrix}3 & 5\\ 4 & 6\end{pmatrix}-3\begin{pmatrix}1 & -2\\ 0 & 4\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}2x & 2y\\ 2z & 2t\end{pmatrix}=\begin{pmatrix}12 & 20\\ 16 & 24\end{pmatrix}-\begin{pmatrix}3 & -6\\ 0 & 12\end{pmatrix}$$

$$=\begin{pmatrix}9 & 26\\ 16 & 12\end{pmatrix}$$

$\Rightarrow \quad x=9/2, y=13, z=8, t=6$

10. Let $A=\begin{pmatrix}2 & 3\\ 4 & 0\end{pmatrix}$ and $B=\begin{pmatrix}-1 & 2\\ 1 & -5\end{pmatrix}$

Solving, we get

$X=\frac{1}{5}(3B-2A)$ and $Y=\frac{1}{5}(3A-2B)$

Thus, $X=\frac{1}{5}\left(\begin{pmatrix}-3 & 6\\ 3 & -15\end{pmatrix}-\begin{pmatrix}4 & 6\\ 8 & 0\end{pmatrix}\right)$

$$\Rightarrow \quad X=\frac{1}{5}\begin{pmatrix}-7 & 0\\ -5 & -15\end{pmatrix}=\begin{pmatrix}-7/5 & 0\\ -1 & -3\end{pmatrix}$$

and $Y=\frac{1}{5}\left(\begin{pmatrix}6 & 9\\ 12 & 0\end{pmatrix}-\begin{pmatrix}-2 & 4\\ 2 & -10\end{pmatrix}\right)$

$$\Rightarrow \quad Y=\frac{1}{5}\begin{pmatrix}8 & 5\\ 10 & 10\end{pmatrix}=\begin{pmatrix}8/5 & 1\\ 2 & 2\end{pmatrix}$$

11. Given $A=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$

$$=2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=2I$$

We have,

$f(x)=1+x+x^2+\ldots$ to ∞

$$=\frac{1}{1-x}$$

$$\Rightarrow \quad f(A)=\frac{I}{I-A}$$

$$=\frac{I}{I-2I}$$

$$=-\frac{I}{I}$$

$$=-\frac{I^2}{I}$$

$$=-I$$

12. Then $AB = [1 \times 2 + 2 \times 3] = [8]$

13. Then $AB = [a \times a^2 + b \times b^2 + c \times c^2]$
$= [a^3 + b^3 + c^3]$

14. Then $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix}$

$$= \begin{pmatrix} 1.2 + 2.5 & 1.4 + 2.7 \\ 3.2 + 4.5 & 3.4 + 4.7 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 18 \\ 26 & 40 \end{pmatrix}$$

Also,

$$\begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2.1 + 4.3 & 2.2 + 4.7 \\ 5.1 + 7.3 & 5.2 + 7.4 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 32 \\ 26 & 38 \end{pmatrix}$$

Thus, $AB \neq BA$
Clearly, the matrix multiplication is not commutative.

15. Given A is a 2×3 matrix and AB is a 2×5 matrix. Thus, B is a matrix of 3×5.

16. Clearly, the matrix multiplication is not defined.

17. Given $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

Now, $A^2 = A.A$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Also

$$A^2 - 2A + I_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= 0$$

Hence, the result

18. We have,

$$AB = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (2 \quad 3 \quad 4)$$

$$= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{pmatrix}$$

Also,

$$BA = (2 \quad 3 \quad 4) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= (2 + 6 + 12) = (20)$$

19. We have,

$$A^2 = A.A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 + 12 & 6 + 3 \\ 8 + 4 & 12 + 1 \end{pmatrix} = \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix}$$

Now, $A^3 = A^2.A$

$$= \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 32 + 36 & 48 + 9 \\ 24 + 52 & 36 + 39 \end{pmatrix} = \begin{pmatrix} 68 & 57 \\ 76 & 69 \end{pmatrix}$$

Also, $A^4 = A^3.A$

$$= \begin{pmatrix} 68 & 57 \\ 76 & 69 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 136 + 228 & 204 + 57 \\ 152 + 276 & 228 + 69 \end{pmatrix} = \begin{pmatrix} 364 & 261 \\ 428 & 297 \end{pmatrix}$$

20. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

So, $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 6 \end{pmatrix}$

$\Rightarrow \quad \begin{pmatrix} 2a + c & 2b + d \\ a + 4c & b + 4d \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 6 \end{pmatrix}$

$2a + c = 3 \quad 2b + d = 5$

$\Rightarrow \quad a + 4c = 0 \quad b + 4d = 6$

$\Rightarrow \quad -8c + c = 3, 2(6 - 4d) + d = 5$

$\Rightarrow \quad c = -\frac{3}{7}, -7d = 5 - 12 = -7$

$\Rightarrow \quad c = -\frac{3}{7}, d = 1$

Therefore, $a = \frac{12}{7}, b = 2$

Thus, the matrix $X = \begin{pmatrix} 12/7 & 2 \\ -3/7 & 1 \end{pmatrix}$.

21. We have,

$$A^2 = A.A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

Now, $A^2 - 4A - 5I_3$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

22. We have

$$(1 \quad x \quad 1)\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ x \end{pmatrix} = 0$$

$$\Rightarrow \quad (1 \quad x \quad 1)\begin{pmatrix} 1+6+2x \\ 2+10+x \\ 15+6+2x \end{pmatrix} = 0$$

$$\Rightarrow \quad (1 \quad x \quad 1)\begin{pmatrix} 7+2x \\ 12+x \\ 21+2x \end{pmatrix} = 0$$

$\Rightarrow \quad (7+2x) + x(12+x) + (21+x) = 0$

$\Rightarrow \quad (7+2x) + 12x + x^2 + (21+x) = 0$

$\Rightarrow \quad x^2 + 15x + 28 = 0$

$$\Rightarrow \quad x = \frac{-15 \pm \sqrt{225-112}}{2} = \frac{-15 \pm \sqrt{113}}{2}$$

Hence, the solution set is

$$\left\{\frac{-15+\sqrt{113}}{2}, \frac{-15+\sqrt{113}}{2}\right\}.$$

23. We have

$$A^2 = A \cdot A = \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ a+1 & 1 \end{pmatrix}$$

Given relation is

$A^2 = B$

$$\Rightarrow \quad \begin{pmatrix} a^2 & 0 \\ a+1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$$

$\Rightarrow \quad a^2 = 1, a + 1 = 5$

$\Rightarrow \quad a = \pm 1, a = 4$

24. We have $A^2 = A.A$

$$= \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix} \cdot \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^2+4 & 4\alpha \\ 4\alpha & \alpha^2+4 \end{pmatrix}$$

Now, $A^3 = A^2.A$

$$= \begin{pmatrix} \alpha^2+4 & 4\alpha \\ 4\alpha & \alpha^2+4 \end{pmatrix} \cdot \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^3+12\alpha & 6\alpha^2+8 \\ 6\alpha^2+8 & \alpha^3+12\alpha \end{pmatrix}$$

Given $|A^3| = 125$

$$\Rightarrow \quad \begin{vmatrix} \alpha^3+12\alpha & 6\alpha^2+8 \\ 6\alpha^2+8 & \alpha^3+12\alpha \end{vmatrix} = 125$$

$\Rightarrow \quad (\alpha^3 + 12\alpha)^2 - (6\alpha^2 + 8)^2 = 125$

$\Rightarrow \quad (\alpha^3 + 12\alpha + 6\alpha^2 + 8)(\alpha^3 + 12\alpha - 6\alpha^2 - 8) = 125$

$\Rightarrow \quad (\alpha^3 + 6\alpha^2 + 12\alpha + 8)(\alpha^3 + 6\alpha^2 + 12\alpha - 8) = 125$

$\Rightarrow \quad (\alpha+2)^3(\alpha-2)^3 = 125$

$\Rightarrow \quad \{(\alpha+2)(\alpha-2)\}^3 = (5)^3$

$\Rightarrow \quad (\alpha+2)(\alpha-2) = 5$

$\Rightarrow \quad \alpha^2 - 4 = 5$

$\Rightarrow \quad \alpha^2 = 9$

$\Rightarrow \quad \alpha = \pm 3$

25. Given $AB = A$ and $BA = B$

Now $A = AB$

$\Rightarrow \quad A^2 = ABA = AB = A$

Similarly $B^2 = B$

We have,

$(A + B)^2 = (A^2 + AB + BA + B^2)$

$= (A + A + B + B)$

$= 2(A + B)$

$\Rightarrow \quad (A + B)^4 = 4(A + B)^2 = 8(A + B)$

and $(A + B)^3 = (A + B)^2.(A + B)$

$= 2(A + B).(A + B) = 2(A + B)^2$

$= 4(A + B)$

Thus,

$(A + B)^7 = (A + B)^4.(A + B)^3$

$= 32(A + B)^2 = 64(A + B)$

26. Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\text{Now, } AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$$

Also, $BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$

It is given that

$AB = BA$

$\Rightarrow \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} = \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$

$a + 2c = a + 3b \quad \Rightarrow \quad 2C = 3b$

$b + 2d = 2a + 4b$

$3a + 4c = c + 3d$

$\Rightarrow \quad 3b + 4d = 2c + 4d$

$2c = 3b$

$\Rightarrow \quad 2a - 2d = -3b$

Now, $\dfrac{a-d}{3b-c} = \dfrac{-\frac{3}{2}b}{2c-c} = -\dfrac{\frac{3}{2}b}{c} = -\dfrac{\frac{3}{2}b}{\frac{3}{2}b} = -1$

27. Given $A^2 = \mathbf{O}$

$A^2 = \mathbf{O} = A^3 = A^4 = \ldots A^{2009}$

Now, $A(I+A)^{2009}$

$= A\left({}^{2009}C_0 \cdot I^n + {}^{2009}C_1 \cdot I^{n-1} \cdot A + {}^{2009}C_2 \cdot I^{n-2} \cdot A^2 + \ldots + {}^{2009}C_{2009} \cdot I^{n-n} A^{2008} \right)$

$= A(I + 2009A + \mathbf{O} + \mathbf{O} + \ldots + \mathbf{O})$

$= A(I + 2009A)$

$= A.I + 2009\, A^2$

$= A + \mathbf{O}$

$= A$

28. Given $A^2 = I$.

Now,

$(I - A)(I + A) = I^2 + IA - AI - A^2$

$= I + A - A - A^2$

$= I + A - A - I$

$= \mathbf{O}$

29. We have,

$(I + A)^3 - 7A = (I^3 + 3I^2A + 3IA^2 + A^3) - 7A$

$= (I + 3A + 3A^2 + A^3) - 7A$

$= (I + 3A + 3A + A) - 7A$

$= I + 7A - 7A$

$= I$

30. Here, $A^2 = A.A$.

$= \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{O}$

Thus, $A^2 = \mathbf{O} = A^3 = A^4 = \ldots = A^{16}$.

We have $f(x) = \sum_{n=0}^{16} x^n$

$= 1 + x + x^2 + x^3 + \ldots + x^{16}$

Now, $f(A) = I + A + A^2 + A^3 + \ldots + A^{16}$

$= I + A + \mathbf{O} + \mathbf{O} + \ldots + \mathbf{O}$

$= I + A$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

31. Given,

$A^2 = A + I$

$\Rightarrow \quad A^3 = A^2 + A = A + I + A = 2A + I$

$\Rightarrow \quad A^4 = 2A^2 + A = 2(A + I) + A = 3A + 2I$

$\Rightarrow \quad A^5 = 3A^2 + 2A = 3(A + I) + 2A = 5A + 3I$

32. Given,

$A^2 = 2A - I$

$\Rightarrow \quad A^3 = 2A^2 - A = 2(2A - I) - A = 3A - 2I$

$\Rightarrow \quad A^4 = 3A^2 - 2A = 3(2A - I) - 2A = 4A - 3I$

$\Rightarrow \quad A^5 = 4A^2 - 3A = 4(2A - I) - 3A = 5A - 4I$

Thus, by symmetry, we can say that

$A^n = nA - (n-1)I$

33. Given,

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow \quad A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2A$

$\Rightarrow \quad A^3 = 2A^2 = 2(2A) = 4A = 2^2A$

$\Rightarrow \quad A^4 = 2^2A^2 = 2^2(2A) = 2^3A$

Thus, by symmetry, we can say that

$A^n = 2^{n-1} A$

34. Let $U_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $U_2 = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

Given $AU_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a \\ 2a+b \\ 3a+2b+c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad U_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Similarly, we can easily find that

$U_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

Thus, $U_1 + U_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

37. Let $A = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}$

Then $|A| = \begin{vmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & 4 \\ -4 & 0 \end{vmatrix} - 2\begin{vmatrix} -2 & 4 \\ -3 & 0 \end{vmatrix} + 3\begin{vmatrix} -2 & 0 \\ -3 & -4 \end{vmatrix}$

$= 0 - 2(0 + 12) + 3(8 - 0)$

$= -24 + 24 = 0$

Thus, the determinant of skew-symmetric matrix of odd order is zero.

38. Let $A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$

Then $|A| = \begin{vmatrix} 0 & b \\ -b & 0 \end{vmatrix} = b^2$

Thus, the determinant of skew-symmetric matrix of even order is a perfect square.

39. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Then $|A| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$.

$= (2 - 1)(-1)^{2-1}$

Again, let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Thus, $|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= 0 - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2$

$= (3 - 1)(-1)^{3-1}$

Therefore, in general

$|A| = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 0 \end{vmatrix}$

$= (n - 1)(-1)^{n-1}$

40. We have,

$(A^{2016} + 2A^{2015}) = A^{2015}(A + 2I_2)$

$= A^{2015}\left[\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]$

$= A^{2015}\begin{pmatrix} 4 & 5 \\ 1 & 5 \end{pmatrix}$

Thus,

$|A^{2016} + 2A^{2015}| = \left|A^{2015}\begin{pmatrix} 4 & 5 \\ 1 & 5 \end{pmatrix}\right|$

$= |A^{2015}|\begin{vmatrix} 4 & 5 \\ 1 & 5 \end{vmatrix}$

$= |A|^{2015} \times \begin{vmatrix} 4 & 5 \\ 1 & 5 \end{vmatrix}$

$= (1)^{2015} \times (20 - 5)$

$= 15$

41. Given,

$A = A^2$

$\Rightarrow \quad |A| = |A^2|$

$\Rightarrow \quad |A| = |A|^2$

$\Rightarrow \quad |A|(|A| - 1) = 0$

$\Rightarrow \quad |A| = 0$ or $|A| = 1$

42. We have,

$\det(3A) = 3^3 \times \det(3A)$

$= 3^3 \times 8 = 216$

43. We have,

$|A^n| = |A|^n = 2^n$

44. We have,

$\begin{vmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{vmatrix} = \cos^4\theta - \sin^2\theta$

$= (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$

$= (\cos^2\theta - \sin^2\theta)$

$= \cos(2\theta)$

Thus, the maximum value is 1.

45. Given $P^3 = Q^3$ and $P^2Q = Q^2P$

We have,

$P^3 - P^2Q = Q^3 - Q^2P$

$\Rightarrow \quad P^2(P - Q) = Q^2(Q - P)$

$\Rightarrow \quad P^2(P - Q) = -Q^2(P - Q)$

$\Rightarrow \quad (P^2 + Q^2)(P - Q) = \mathbf{O}$

$\Rightarrow \quad (P^2 + Q^2) = \mathbf{O}$, since $(P - Q) \neq \mathbf{O}$

Thus, $\det(P^2 + Q^2) = \det(\mathbf{O}) = 0$

46. We have,

$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix}$

$= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$

$= abc - a^3 - b^3 + abc + abc - c^3$

$= 3abc - a^3 - b^3 - c^3$

$= -(a^3 + b^3 + c^3 - 3abc)$

47 The given determinant $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$= (b-a)(c-a)(c+a-b-a)$
$= (b-a)(c-a)(c-b)$
$= (a-b)(b-c)(c-a)$

48. The given determinant $= \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & b+c+a \\ 1 & c & c+a+b \end{vmatrix} \quad (C_3 \to C_2 + C_3)$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$= (a+b+c) \times 0$
$= 0$

49. The given determinant is

$$\begin{vmatrix} \sin\alpha & \cos\beta & \cos(\alpha+\theta) \\ \sin\beta & \cos\beta & \cos(\beta+\theta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\theta) \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}$$

$[C_3 \to C_3 - (C_1 \cos\theta + C_2 \sin\theta)]$

$= 0$

50. The given determinant $= \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(a+c) \\ 1 & ab & c(a+b) \end{vmatrix}$

$$= \begin{vmatrix} 1 & bc & bc+ab+ac \\ 1 & ca & ac+ab+bc \\ 1 & ab & ab+ac+bc \end{vmatrix} \quad (C_3 \to C_2 + C_3)$$

$$= (ab+bc+ca) \times \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$= (ab+bc+ca) \times 0$
$= 0$

51. The given determinant

$$= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$(C_1 \to C_1 + C_2 + C_3)$

$$= 2(a+b+c) \times \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \times \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$

$$= 2(a+b+c)^3 \times \begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= 2(a+b+c)^3 \times 1$
$= 2(a+b+c)^3$

52. The given determinant

$$= \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ b & c+a & b \\ c & c & a+\text{b} \end{vmatrix}$$

$(R_1 \to R_1 + R_2 + R_3)$

$$= 2\begin{vmatrix} (b+c) & (c+a) & (a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= 2\begin{vmatrix} b+c & (c+a) & (a+b) \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$

$$= 2\begin{vmatrix} 0 & \text{c} & \text{b} \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix} \quad (R_1 \to R_1 + R_2 + R_3)$$

$= 2[-c(-ab-0) + b(ca-0)]$

$= 2(abc + abc)$

$= 4abc$

53. The given determinant

$$= \begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 2(b^2+c^2) & 2(c^2+a^2) & 2(a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$(R_1 \to R_1 + R_2 + R_3)$

$$= 2\begin{vmatrix} (b^2+c^2) & (c^2+a^2) & (a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} b^2+c^2 & (c^2+a^2) & (a^2+b^2) \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$

$$= 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix} \quad (R_1 \to R_1 + R_2 + R_3)$$

$= 2(-c^2(-a^2b^2 - 0) + b^2(c^2a^2 - 0))$

$= 2(a^2b^2c^2 + a^2b^2c^2)$

$= 4a^2b^2c^2$

54. The given determinant

$$= \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$$

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix} \quad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$= a\begin{vmatrix} a & 2a+b \\ 3a & 7a+3b \end{vmatrix}$$

$= a(7a^2 + 3ab - 6a^2 - 3ab)$

$= a(a^2) = a^3$

55. The given determinant

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2-b^2+2b^2 & 2ab-2ab & -2b \\ 2ab-2ab & 1-a^2+b^2+2a^2 & 2a \\ 2b-b(1-a^2-b^2) & -2a+a(1-a^2-b^2) & 1-a^2-b^2 \end{vmatrix}$$

$\begin{pmatrix} C_1 \to C_1 - bC_3 \\ C_2 \to C_2 + aC_3 \end{pmatrix}$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

$(R_3 \to R_3 - bR_1)$

$$(1+a^2+b^2)^2\begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

$= (1 + a^2 + b^2)^2\,(1 - a^2 + b^2 + 2a^2)$

$= (1 + a^2 + b^2)^3$

56. The given determinant

$$= \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & cb & c^2+1 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ ac^2 & c^2b & c(c^2+1) \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} (a^2+1) & a^2 & a^2 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$$= \begin{vmatrix} (1+a^2+b^2+c^2) & (1+a^2+b^2+c^2) & (1+a^2+b^2+c^2) \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$(R_1 \to R_1 + R_2 + R_3)$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix} \quad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$$= (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= (1 + a^2 + b^2 + c^2)$$

57. The given determinant

$$\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$$

$$= \begin{vmatrix} x & \frac{x(x-1)}{2} & \frac{x(x-1)(x-2)}{6} \\ y & \frac{y(y-1)}{2} & \frac{y(y-1)(y-2)}{6} \\ z & \frac{z(z-1)}{2} & \frac{z(z-1)(z-2)}{6} \end{vmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & (x-1) & (x-1)(x-2) \\ 1 & (y-1) & (y-1)(y-2) \\ 1 & (z-1) & (z-1)(z-2) \end{vmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & (x-1) & (x-1)(x-2) \\ 0 & (y-x) & (y^2-x^2)-3(y-x) \\ 0 & (x-x) & (z^2-x^2)-3(z-x) \end{vmatrix}$$

$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} (y-x) & (y^2-x^2)-3(y-x) \\ (z-x) & (z^2-x^2)-3(z-x) \end{vmatrix}$$

$$= \frac{xyz(y-x)(z-x)}{12}\begin{vmatrix} 1 & y+x-3 \\ 1 & z+x-3 \end{vmatrix}$$

$$= \frac{xyz(y-x)(z-x)(z-y)}{12}$$

$$= \frac{xyz(x-y)(y-z)(z-x)}{12}$$

58. The given determinant

$$= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= abc\begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

59. The given determinant is

$$= \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$\begin{pmatrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{pmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$$R_3 \to R_3 - (R_1 + R_2)$$

$[= 2ab(c+a)\,(b+c-a) + 2a^2b(c+a-b)$

$= 2ab(c+a)\,(b+c-a) + (c+a-b)$

$= 2ab[(b\,(c+a) + (c^2-a^2) - ac + a^2 - ab$

$= 2ab(bc + c^2 + ac)$

$= 2abc(a+b+c)$

$= 2abc(a+b+c)^3$

60. The given determinant is

$$\begin{vmatrix} (a+1)(a+2) & (a+2) & 1 \\ (a+2)(a+3) & (a+3) & 1 \\ (a+3)(a+4) & (a+4) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+3a+2 & (a+2) & 1 \\ a^2+5a+6 & (a+3) & 1 \\ a^2+7a+12 & (a+4) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+3a+2 & (a+2) & 1 \\ 2a+4 & 1 & 0 \\ 4a+10 & 2 & 0 \end{vmatrix} \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \begin{vmatrix} 2a+4 & 1 \\ 4a+10 & 2 \end{vmatrix}$$

$= (4a+8) - (4a+10)$
$= -2$

61. The given determinant is

$$\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & a+b & b+c \end{vmatrix} \quad (C_1 \to C_1 + C_2 + C_3)$$

$$= 2\begin{vmatrix} (a+b+c) & c+a & a+b \\ (a+b+c) & b+c & c+a \\ (a+b+c) & a+b & b+c \end{vmatrix}$$

$$= 2\begin{vmatrix} (a+b+c) & -b & -c \\ (a+b+c) & -a & -b \\ (a+b+c) & -c & -a \end{vmatrix} \quad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$$= 2\begin{vmatrix} a & -b & -c \\ c & -a & -b \\ b & -c & -a \end{vmatrix} \quad (C_1 \to C_1 + C_2 + C_3)$$

$$= 2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

62 Here, $D = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -4 - 9 = -13$

$D_1 = \begin{vmatrix} 4 & 3 \\ 5 & -2 \end{vmatrix} = -8 - 15 = -23$

$D_2 = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 10 - 12 = -2$

Thus, $x = \dfrac{D_1}{D} = \dfrac{-23}{-13} = \dfrac{23}{13}$

and $y = \dfrac{D_2}{D} = \dfrac{-2}{-13} = \dfrac{2}{13}$

$D = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$

63. Here, $D_1 = \begin{vmatrix} 4 & 3 \\ 10 & 6 \end{vmatrix} = 24 - 30 = -6 \neq 0$

We know that, if $D = 0$ and any one of D_1 and D_2 is non-zero, it has no solution.
So the system of equation is inconsistent.

64. Here, $D = \begin{vmatrix} 2 & 5 \\ 6 & 15 \end{vmatrix} = 30 - 30 = 0$

$D_1 = \begin{vmatrix} 6 & 5 \\ 18 & 15 \end{vmatrix} = 90 - 90 = 0$

$D_2 = \begin{vmatrix} 2 & 6 \\ 6 & 18 \end{vmatrix} = 36 - 36 = 0$

As we know that, if $D = 0 + D_1 = D_2$, the system of equations has infinitely many solutions.
Let $y = k$

Then $x = \dfrac{6-5k}{2}, k \in R.$

65. Since the system of equations has infinitely many solutions, so $D = 0 + D_1 = D_2$

Thus, $\begin{vmatrix} a & -b \\ (c+1) & c \end{vmatrix} = 0, \begin{vmatrix} 2a-b & -b \\ 10-a+3b & c \end{vmatrix} = 0$

and $\begin{vmatrix} a & 2a-b \\ c+1 & 10-a+3b \end{vmatrix} = 0$

66. We have,

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ d & b & c \\ d^2 & b^2 & c^2 \end{vmatrix} = (d-b)(b-c)(c-d)$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & d & c \\ a^2 & d^2 & c^2 \end{vmatrix} = (a-d)(d-c)(c-a)$$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & d \\ a^2 & b^2 & d^2 \end{vmatrix} = (a-b)(b-d)(d-a)$$

Thus,

$$x=\frac{D_1}{D}=\frac{(d-b)(b-c)(c-d)}{(a-b)(b-c)(c-a)}=\frac{(d-b)(c-d)}{(a-b)(c-a)}$$

$$y=\frac{D_2}{D}=\frac{(a-d)(d-c)(c-a)}{(a-b)(b-c)(c-a)}=\frac{(a-d)(d-c)}{(a-b)(b-c)}$$

$$z=\frac{D_3}{D}=\frac{(a-b)(b-d)(d-a)}{(a-b)(b-c)(c-a)}=\frac{(b-d)(d-a)}{(b-c)(c-a)}$$

67. Let $u=\frac{1}{x}, v=\frac{1}{y}, w=\frac{1}{z}$

The given system of equations reduces to

$$u+v-w=\frac{1}{4},$$

$$2u-v+3w=\frac{9}{4}$$

and $-u-2v+4w=1$

Here, $D=\begin{vmatrix}1 & 1 & -1\\ 2 & -1 & 3\\ -1 & -2 & 4\end{vmatrix}$

$$=1(-4+6)-1(8+3)-1(-4-1)$$
$$=2-11+5=-4$$

$$D_1=\begin{vmatrix}1/4 & 1 & -1\\ 9/4 & -1 & 3\\ 1 & -2 & 4\end{vmatrix}$$

$$=\frac{1}{4}\cdot 2-(9-3)-\left(-\frac{9}{2}+1\right)$$

$$=\frac{1}{2}+\frac{7}{2}-6=4-6=-2$$

Also, $D_2=\begin{vmatrix}1 & 1/4 & -1\\ 2 & 9/4 & 3\\ -1 & 1 & 4\end{vmatrix}$

$$=(9-3)-\frac{11}{4}-\left(2+\frac{9}{4}\right)$$

$$=4-5=-1$$

Again, $D_3=\begin{vmatrix}1 & 1 & 1/4\\ 2 & -1 & 9/4\\ -1 & -2 & 1\end{vmatrix}$

$$=\left(-1+\frac{9}{2}\right)-1\left(2+\frac{9}{4}\right)-\frac{5}{4}$$

$$=\frac{7}{2}-2-\frac{14}{4}=-2$$

Now, $u=\frac{D_1}{D}=\frac{-4}{-2}=2\Rightarrow x=\frac{1}{2}$

$$v=\frac{D_2}{D}=\frac{-1}{-2}=\frac{1}{2}\Rightarrow y=2$$

and $w=\frac{D_3}{D}=\frac{-2}{-2}=1\Rightarrow z=1$

68. Given parabola is $y=ax^2+bx+c$, which is passing through (2, 4), (–1, 1) and (–2, 5), so,

$$4a+2b+c=4$$
$$a-b+c=1$$
$$4a-2b+c=5$$

From Cramers rule,

$$a=\frac{D_1}{D}=\frac{15}{12}=\frac{5}{4}$$

$$b=\frac{D_2}{D}=\frac{1}{12}\text{ and}$$

$$c=\frac{D_3}{D}=\frac{2}{12}=\frac{1}{6}$$

where $D=\begin{vmatrix}4 & 2 & 1\\ 1 & -1 & 1\\ 4 & -2 & 1\end{vmatrix}=12$

$$D_1=\begin{vmatrix}4 & 2 & 1\\ 1 & -1 & 1\\ 5 & -2 & 1\end{vmatrix}=15$$

$$D_2=\begin{vmatrix}4 & 4 & 1\\ 1 & 1 & 1\\ 4 & 5 & 1\end{vmatrix}=1$$

$$D_3=\begin{vmatrix}4 & 2 & 4\\ 1 & -1 & 1\\ 4 & -2 & 5\end{vmatrix}=2$$

Hence, the required equation of the parabola is

$$y=ax^2+bx+c$$

$$\Rightarrow\quad y=\frac{5}{4}x^2+\frac{x}{12}+\frac{1}{6}$$

69. Since the system of equations has a unique solution, so,

$$\frac{2}{3}\neq\frac{k}{-4}$$

$$\Rightarrow\quad k\neq-\frac{8}{3}$$

Therefore, the value of k is $k\in R-\left\{-\frac{8}{3}\right\}$.

70. Since the system of equations has infinitely many solutions, so

$$\frac{3}{\lambda}=\frac{4}{8}=\frac{5}{10}$$

$$\Rightarrow\quad \frac{3}{\lambda}=\frac{1}{2}$$

$$\Rightarrow\quad \lambda=6$$

Hence, the value of λ is 6.

71. We know that, the system of equations has no solution Only when, if $D = 0$, but any one of D_1, D_2, D_3 is non-zero.

Now, $D = 0$ gives

$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & (p+2) \\ 0 & (2p+1) & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (p+2)(2p+1) = 0$

$\Rightarrow \quad p = -2, -\frac{1}{2}$

Now, $D_1 \neq 0$ gives $p \neq 1, -1$

$D_2 \neq 0$ gives $p \neq -\frac{1}{2}$

and $D_3 \neq 0$ gives $p \neq -\frac{1}{2}$

Hence, the value of p is -2.

72. Since the given system of equations has no solution, so $D = 0$ and at-least any one of D_1, D_2, D_3 is non-zero.

Now $D = 0$, gives

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 2 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-\lambda - 2) + (\lambda - 1) + 2(2 + 2) = 0$

$\Rightarrow \quad -3\lambda - 4 - 1 + 8 = 0$

$\Rightarrow \quad 3\lambda = 3$

$\Rightarrow \quad \lambda = 1$

Also $D_1 \neq 0$ gives

$$\begin{vmatrix} 2 & -1 & 2 \\ -4 & -2 & 1 \\ 4 & 2 & \lambda \end{vmatrix} \neq 0$$

$\Rightarrow \quad 2(-2\lambda - 2) + 2(-4\lambda - 4) \neq 0$

$\Rightarrow \quad -4\lambda - 4 - 8\lambda - 8 \neq 0$

$\Rightarrow \quad \lambda + 1 \neq 0$

$\Rightarrow \quad \lambda \neq -1$

Again, $D_2 \neq 0$ gives $\lambda \neq 1$.

Hence, the value of λ is φ.

73. Here, $D = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$

So the system of equations has infinitely many solutions.

Let $y = k$

Then $x = -\frac{3k}{2}, k \in R$

74. Since the system of equations has non-trivial solution, so $D = 0$

$$\begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & a-t \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a+b+c-t & b & c \\ a+b+c-t & c-t & a \\ a+b+c-t & a & a-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t)\begin{vmatrix} 1 & b & c \\ 1 & c-t & a \\ 1 & a & a-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t)\begin{vmatrix} 1 & b & c \\ 0 & c-b-t & a-c \\ 0 & a-b & a-c-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t)\begin{vmatrix} c-b-t & a-c \\ a-b & a-c-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t) = 0, \begin{vmatrix} c-b-t & a-c \\ a-b & a-c-t \end{vmatrix} = 0$$

$\Rightarrow \quad t = (a + b + c), t^2 + (a + b)t + (2ac - c^2) = 0$

$\Rightarrow \quad t = (a + b + c)$

$$t = \frac{-(a+b) \pm \sqrt{(a+b)^2 - 4(2ac - c^2)}}{2}$$

Thus, the number of values of t is 3.

75. Since the system of equations has a unique solution, so $D_1 \neq 0$

$$\Rightarrow \begin{vmatrix} 6 & 5 & \lambda \\ 3 & -1 & 4 \\ 1 & 2 & -3 \end{vmatrix} \neq 0$$

$\Rightarrow \quad 6(3 - 8) - 5(-9 - 4) + \lambda(6 + 1) \neq 0$

$\Rightarrow \quad -30 + 65 + 7\lambda \neq 0$

$\Rightarrow \quad 7\lambda + 35 \neq 0$

$\Rightarrow \quad \lambda + 5 \neq 0$

$\Rightarrow \quad \lambda \neq -5$

Hence, the value of λ is $R - \{-5\}$.

76. Since the system of equations has infinite solutions, so

$$\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1 - a(0 - a^2) = 0$

$\Rightarrow \quad 1 - a^3 = 0$

$\Rightarrow \quad (a + 1)(a^2 - a + 1) = 0$

$\Rightarrow \quad (a + 1) = 0, (a^2 - a + 1) = 0$

$\Rightarrow \quad a = -1, a = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$

Hence, the values of a are $\left\{-1, \frac{1 \pm i\sqrt{3}}{2}\right\}$.

77. We have,

$$\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}^2 = \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix} \times \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}$$

$$= \begin{vmatrix} a.a + 0.0 + c.c & a.a + 0.0 + 0.0 & a.0 + 0.b + c.c \\ a.a + b.0 + 0.c & a.a + b.b + 0.0 & a.0 + b.b + 0.c \\ 0.a + b.0 + c.c & 0.a + b.b + c.0 & 0.0 + b.b + c.c \end{vmatrix}$$

$$= \begin{vmatrix} c^2 + a^2 & a^2 & c^2 \\ a^2 & a^2 + b^2 & b^2 \\ c^2 & b^2 & b^2 + c^2 \end{vmatrix}$$

78. We have,

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta + gd & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix}$$

$$= 0 \times 0$$
$$= 0$$

79. We have

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$$= 0 \times 0$$
$$= 0$$

80. We have,

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & \beta-1 \\ 0 & \alpha^2-1 & \beta^2-1 \end{vmatrix}^2$$

$$= \begin{vmatrix} \alpha-1 & \beta-1 \\ \alpha^2-1 & \beta^2-1 \end{vmatrix}^2$$

$$= (\alpha-1)^2(\beta-1)^2 \begin{vmatrix} 1 & 1 \\ \alpha+1 & \beta+1 \end{vmatrix}^2$$

$$= (\alpha-1)^2(\beta-1)^2(\beta-\alpha)^2$$
$$= (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$$

Hence, the value of k is 1.

81. We have,

$$F(x) = \begin{vmatrix} 0 & 0 & 0 \\ x & x^2 & x^3 \\ e^{x-a} & e^{x^2-a^2} & e^{x^3-a^3} \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ 1 & 2x & 3x^2 \\ e^{x-a} & e^{x^2-a^2} & e^{x^3-a^3} \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & a & a^2 \\ x & x^2 & x^3 \\ e^{x-a} & 2x \times e^{x^2-a^2} & 3x^2 \times e^{x^3-a^3} \end{vmatrix}$$

$$\Rightarrow \quad F'(a) = \begin{vmatrix} 1 & a & a^2 \\ 1 & 2a & 3a^2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ a & a^2 & a^3 \\ 1 & 2a & 3a^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & 2a & 3a^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & 2a & 3a^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2a - 3a^2 - a + 3a^3 + a^2 - 2a^3$$
$$= a - 2a^2 + a^3.$$

82. Given, $f(x) = \begin{vmatrix} 3 & 2 & 1 \\ 6x^2 & 2x^3 & x^4 \\ 1 & b & b^2 \end{vmatrix}$

$$\Rightarrow \quad f'(x) = \begin{vmatrix} 3 & 2 & 1 \\ 12x & 6x^2 & 4x^3 \\ 1 & b & b^2 \end{vmatrix}$$

$$\Rightarrow \quad f''(x) = \begin{vmatrix} 3 & 2 & 1 \\ 12 & 12x & 12x^2 \\ 1 & b & b^2 \end{vmatrix}$$

$$\Rightarrow \quad f''(b) = \begin{vmatrix} 3 & 2 & 1 \\ 12 & 12b & 12b^2 \\ 1 & b & b^2 \end{vmatrix}$$

$$\Rightarrow \quad f''(b) = 12\begin{vmatrix} 3 & 2 & 1 \\ 1 & b & b^2 \\ 1 & b & b^2 \end{vmatrix} = 0$$

83. $\int_0^{\pi/2} f(x)\,dx$

$$= \begin{vmatrix} \int_0^{\pi/2} \sin^2 x\,dx & \int_0^{\pi/2} \log(\sin x)\,dx & \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right) dx \\ n & \sum_{k=1}^{n}(k) & \prod_{k=1}^{n}(k) \\ \frac{8}{15} & \frac{\pi}{2}\log\left(\frac{1}{2}\right) & \frac{\pi}{4} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{8}{15} & \frac{\pi}{2}\log\left(\frac{1}{2}\right) & \frac{\pi}{4} \\ n & \sum_{k=1}^{n}(k) & \prod_{k=1}^{n}(k) \\ \frac{8}{15} & \frac{\pi}{2}\log\left(\frac{1}{2}\right) & \frac{\pi}{4} \end{vmatrix}$$

$= 0$

84. We have,

$$\Delta_r = \begin{vmatrix} r & 2012 & \frac{n(n+1)}{2} \\ 2r-1 & 2013 & n^2 \\ 3r-2 & 2014 & \frac{n(3n-1)}{2} \end{vmatrix}$$

$$\Rightarrow \quad \sum_{r=1}^{n}\Delta_r = \begin{vmatrix} \sum_{r=1}^{n} r & 2012 & \frac{n(n+1)}{2} \\ \sum_{r=1}^{n}(2r-1) & 2013 & n^2 \\ \sum_{r=1}^{n}(3r-2) & 2014 & \frac{n(3n-1)}{2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n+1)}{2} & 2012 & \frac{n(n+1)}{2} \\ n^2 & 2013 & n^2 \\ \frac{n(3n-1)}{2} & 2014 & \frac{n(3n-1)}{2} \end{vmatrix}$$

$= 0$

85. We have $D_r = \begin{vmatrix} 2^{r-1} & 101 & (2^n-1) \\ 3^{r-1} & 102 & \left(\frac{3^n-1}{2}\right) \\ 5^{r-1} & 103 & \left(\frac{5^n-1}{4}\right) \end{vmatrix}$

Then $\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & 101 & (2^n-1) \\ \sum_{r=1}^{n} 3^{r-1} & 102 & \left(\frac{3^n-1}{2}\right) \\ \sum_{r=1}^{n} 5^{r-1} & 103 & \left(\frac{5^n-1}{4}\right) \end{vmatrix}$

$$= \begin{vmatrix} (2^n-1) & 101 & (2^n-1) \\ \left(\frac{3^n-1}{2}\right) & 102 & \left(\frac{3^n-1}{2}\right) \\ \left(\frac{5^n-1}{4}\right) & 103 & \left(\frac{5^n-1}{4}\right) \end{vmatrix}$$

$= 0$

Then $\sum_{r=1}^{n}\Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ p & q & r \end{vmatrix}$

where a, c, p, q, r are constants.

86. We have,

$\text{adj}(A') = (\text{adj } A)'$

$\Rightarrow \quad \text{adj}(A') - (\text{adj } A') = \mathbf{O}$

87. We know that,

$|\text{adj}(A)| = |A|^{n-1}$

Replace A by A^3 and n by 3, we get

$|\text{adj}(A^3)| = |A^3|^{3-1} = |A|^6$

88. We know that,

given $A \cdot \text{adj}(A) = |A| \cdot I_n$

$= \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$

$\Rightarrow \quad |A| \cdot I_2 = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$

$= 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= 10 \cdot I_2$

$\Rightarrow \quad |A| = 10$

89. We know that,

$|\text{adj}(A)| = |A|^{n-1}$

$\Rightarrow \quad |\text{adj}(A)| = |A|^{3-1} = |A|^2 = 16$

90. We know that,

$$|[\text{adj}\{\text{adj}(A)\}]| = |A|^{(n-1)^2}$$

$\Rightarrow$ $|A|^{(n-1)^2} = |A|^{16}$

$\Rightarrow$ $(n-1)^2 = 16$

$\Rightarrow$ $(n-1) = 4$

$\Rightarrow$ $n = 5$

91. Given $\text{adj}(P) = \begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{pmatrix}$

$\Rightarrow$ $|\text{adj}(P)| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = 4$

As we know that,

$$|\text{adj}(P)| = |P|^{3-1} = |P|^2$$

Thus, $|P|^2 = 4$

$\Rightarrow$ $|P| = 2, -2$

92. Let $P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where $|P| = 2$

Now, $Q = \begin{pmatrix} 2^2a_{11} & 2^3a_{12} & 2^4a_{13} \\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23} \\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33} \end{pmatrix}$

$\Rightarrow$ $|Q| = \begin{vmatrix} 2^2a_{11} & 2^3a_{12} & 2^4a_{13} \\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23} \\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33} \end{vmatrix}$

$$= 2^2 \cdot 2^3 \cdot 2^4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2^2a_{31} & 2^2a_{32} & 2^2a_{33} \end{vmatrix}$$

$$= 2^2 \cdot 2^3 \cdot 2^4 \cdot 2 \cdot 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} \cdot 2 = 2^{13}$$

93. We know that,

$$|\text{adj}(A)| = |A|^{n-1}$$

$\Rightarrow$ $|\text{adj}(A)| = |A|^2$

$= 16$

$\Rightarrow$ $1(12-12) - \alpha(4-6) + 3(4-6) = 16$

$\Rightarrow$ $2\alpha - 6 = 16$

$\Rightarrow$ $2\alpha = 16 + 6 = 22$

$\Rightarrow$ $\alpha = 1$

94. Since the determinant of a skew-symmetric matrix is zero, so the inverse does not exist

95. We know that,

$$BB^{-1} = 1$$

$\Rightarrow$ $|BB^{-1}| = |1|$

$\Rightarrow$ $|B||B^{-1}| = |I|$

$\Rightarrow$ $|B^{-1}| = \dfrac{1}{|B|}$

Now,

$$|B^{-1}AB| = |B^{-1}||A||B|$$

$$= \frac{1}{|B|}|A||B| = |A|$$

96. We have,

$$R = (P\cos\theta + Q\sin\theta)$$

$$= \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Now, $R^{-1} = \dfrac{\text{adj}(R)}{|R|} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

97. Given $B^{-1}AB = A^2$

Now,

$$B^{-3}AB^3 = B^{-2}(B^{-1}AB)B^2$$

$$= B^{-2}(A^2)B^2$$

$$= B^{-1}(B^{-1}(A^2)B)B$$

$$= B^{-1}(A^4)B$$

$$= (A^4)^2 = A^8$$

98. Given,

$$I + A + A^2 + A^3 + \ldots + A^k = \mathbf{O}$$

$\Rightarrow$ $A^{-1}(I + A + A^2 + A^3 + \ldots + A^k) = \mathbf{O}$

$\Rightarrow$ $(A^{-1} + I + A + A^2 + \ldots + A^{k-1}) = \mathbf{O}$

$\Rightarrow$ $A^{-1} + (-A^k) = \mathbf{O}$

$\Rightarrow$ $A^{-1} = A^k$

99. Given,

$$A^{-2} - A + I = \mathbf{O}$$

$\Rightarrow$ $A^{-1}(A^2 - A + I) = A^{-1}\mathbf{O} = \mathbf{O}$

$\Rightarrow$ $(A^{-1}A^2 - A^{-1}A + A^{-1}) = \mathbf{O}$

$\Rightarrow$ $(A - I + A^{-1}) = \mathbf{O}$

$\Rightarrow$ $A^{-1} = I - A$

100. We know that,

$$|\text{adj}(A)| = |A|^{n-1}$$

$$= |A|^{3-1} = |A|^2$$

$\Rightarrow$ $|\text{adj}(A^{-1})| = |A^{-1}|^2$

$\Rightarrow$ $|\text{adj}(A^{-1})| = \left(\dfrac{1}{|A|}\right)^2 = \dfrac{1}{|A|^2}$

$$= \frac{1}{|A|^2} = \frac{1}{25}$$

101. We have,

$$|A^{-1}\text{adj}(A)| = |A^{-1}||\text{adj}(A)|$$

$$= \frac{1}{|A|}|\text{adj}(A)|$$
$$= \frac{1}{|A|} \times |A|^{n-1}$$
$$= \frac{1}{|A|} \times |A|^2$$
$$= |A|.$$

102. Given,

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

Now, $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2$

Since $|A| \neq 0$, so its inverse exists.
Thus,

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$
$$= -\frac{1}{2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}$$

103. Given $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} . X = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

It can be written as $AX = B$, where

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Now, $|A| = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

So, its inverse exists.
Thus,

$$X = A^{-1}B$$
$$= \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}.$$

104. We have,

$$B^{-1} = \frac{\text{adj}(B)}{|B|} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -7 & 3 \end{pmatrix}$$

$$\Rightarrow \quad 2B^{-1} = \begin{pmatrix} 3 & -1 \\ -7 & 3 \end{pmatrix}$$

Now

$$\det(2A^9B^{-1}) = \det(A^9\, 2B^{-1})$$
$$= \det(A^9) \times \det(2B^{-1})$$
$$= |A|^9 \times |2B^{-1}|$$
$$= (-1)^9 \times (9-7)$$
$$= -2$$

105. We know that,

$$B^{-1} = \frac{\text{adj}(B)}{|B|} = \frac{A}{|B|}$$

$\Rightarrow \quad \text{adj}(B) = |B|B^{-1}$

Replacing B by QBP, we get

$\text{adj}(QBP) = |QBP|(QBP)^{-1}$

$\Rightarrow \quad \text{adj}(QBP) = |Q||B||P|P^{-1}B^{-1}Q^{-1}$

$\Rightarrow \quad \text{adj}(QBP) = |B|P^{-1}B^{-1}Q^{-1}$

$\Rightarrow \quad \text{adj}(QBP) = P^{-1}|B|B^{-1}Q^{-1}$

$\Rightarrow \quad \text{adj}(QBP) = P^{-1}(A)Q^{-1}$

106. We have,

$B = A^{-1}A'$

$\Rightarrow \quad AB = A'$

Now,

$ABB' = A'B'$

$\Rightarrow \quad ABB' = (BA)' = (A^{-1}AA)'$

$\Rightarrow \quad = (IA')' = (A')' = A$

$\Rightarrow \quad BB' = 1$

107. Now,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$
$$= 0 - 2(8-6) + 3(4-3)$$
$$= -4 - 3$$
$$= -7 \neq 0$$

So, its inverse exists.
We have,

$$\text{adj}(A) = \begin{pmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}^T$$
$$= \begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}^T$$
$$= \begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}$$

Thus, $A^{-1} = \frac{\text{adj}(A)}{|A|}$

$$= -\frac{1}{7}\begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}$$

108. We have,

$$(I+A)=\begin{pmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}$$

and

$$(I-A)=\begin{pmatrix} 1 & \tan\left(\frac{\alpha}{2}\right) \\ -\tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}$$

Let $I - A = B$

Now, $B^{-1}=\frac{\text{adj}(B)}{|B|}$

$$=\frac{1}{\sec^2\left(\frac{\alpha}{2}\right)}\begin{pmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}$$

Thus,

$(I-A)^{-1}(I+A) = B^{-1}(I+A)$

$$=\frac{1}{\sec^2\left(\frac{\alpha}{2}\right)}\begin{pmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}\begin{pmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}$$

$$=\frac{1}{\sec^2\left(\frac{\alpha}{2}\right)}\times\begin{pmatrix} 1-\tan^2\left(\frac{\alpha}{2}\right) & -2\tan\left(\frac{\alpha}{2}\right) \\ 2\tan\left(\frac{\alpha}{2}\right) & 1-\tan^2\left(\frac{\alpha}{2}\right) \end{pmatrix}$$

$$=\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

109. The given system of equations can be written in the matrix form as

$AX = B$, where

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}, X=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B=\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 6 \neq 0$$

Thus, A^{-1} exists

Now, $A^{-1}=\frac{\text{adj}(A)}{|A|}$

$$=\frac{1}{6}\begin{pmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{pmatrix}$$

Therefore, $X = A^{-1}B$

$$=\frac{1}{6}\begin{pmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$=\frac{1}{6}\begin{pmatrix} 6 \\ 12 \\ 18 \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\Rightarrow$ $x = 1, y = 2$ and $z = 3$.

110. Given matrix is

$$A=\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 11 \neq 0$$

$\Rightarrow$ A^{-1} exists.

Thus,

$$A^{-1}=\frac{\text{adj}(A)}{|A|}$$

$$=\frac{1}{11}\begin{pmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{pmatrix}$$

Also, the given system of equations can be written in matrix form as

$$\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 15 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 15 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\frac{1}{11}\begin{pmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 15 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\frac{1}{11}\begin{pmatrix} 44 \\ -33 \\ 11 \end{pmatrix}=\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

$\Rightarrow$ $x = 4, y = -3$ and $z = 1$.

111. The given system of equations can be written in matrix form as

$$\begin{pmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

$\Rightarrow \quad AX = B$, where

$$A = \begin{pmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

Now, $|A| = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{vmatrix} = 0$

So, inverse does not exist.

Also,

$$\text{adj}(A) = \begin{pmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{pmatrix}$$

Thus,

$$\text{adj}(A)B = \begin{pmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{pmatrix}\begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{O}$$

Therefore, the given system of equations have infinitely many solutions.

112. The given system of equations can be written in matrix form as

$$AX = B$$

It has either

(i) a unique solution, or
(ii) infinite solutions, or
(iii) no solution.

Thus, there cannot exist any matrix A such that

$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has two distinct solutions.

113. Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y$ and $\frac{z^2}{c^2} = Z$.

The given system of equations reduces to

$$X + Y - Z = 1$$
$$X - Y + Z = 1$$
$$X + Y + Z = 1$$

It can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Let $AX' = B$, where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, X' = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now,

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4 \neq 0$$

Thus, the system of equations have a unique solution.

114. We have,

$$A = I.A$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot A$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \cdot A$$

$(R_2 \to R_2 - 2R_1)$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix} \cdot A$$

$(R_1 \to R_1 + 2R_2)$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \cdot A$$

$\left[R_2 \to \left(\frac{-1}{2}\right) \times R_2\right]$

Thus, $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.

115. We have,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{vmatrix}$$

$$= 1(40 - 42) - 4(20 - 21) + 3(12 - 12)$$
$$= -2 + 2 + 0 = 0$$

Since the determinant of A is zero, so the rank of the given matrix is 2.

116. We have,

$$|A| = \begin{vmatrix} 2 & 4 & 3 \\ 1 & 2 & -1 \\ -1 & -2 & 6 \end{vmatrix}$$

$$= 2(12 - 2) - 4(6 - 1) + 3(-2 + 2)$$
$$= 20 - 20 + 0$$
$$= 0$$

Since the determinant of A is zero, so the rank of the given matrix is 2.

117. We have,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= 1(18 - 20) - 2(12 - 12) + 3(10 - 12)$$
$$= -2 - 6 = -8$$

Since the determinant of A is non-zero, so the rank of the given matrix is 3.

118. We have,

$$A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 6 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \end{pmatrix}$$

$$(R_2 \to R_2 - R_1 \cdot R_3 \to R_3 - 4R_1)$$

$$\Leftrightarrow \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(R_3 \to R_3 - 9R_2)$$

Since the number of non-zero rows is 2, so the rank of the given matrix is 2.

119. We have,

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3-5 & -6-12+15 & -10-15+20 \\ -2-4+5 & 3+16-15 & 5+20-20 \\ 2+3-4 & -3-12+12 & -5-15+16 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} = A$$

Thus, A is idempotent.

120. We have,

$$A^2 = A.A$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix} = A.$$

Hence, A is periodic.

121. Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$.

Now, $A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

Also, $A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= O$$

Hence A is nilpotent of order 3.

122. Let $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Now, $A^2 = A \cdot A$

$$= \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence A is involuntary.

123. Give, $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

For orthogonal matrix $AA^T = A^TA = I$

$$\therefore \quad AA^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence A is an orthogonal matrix.

124. We have,

$$A^T = \begin{bmatrix} 3 & 3+4i & 5-2i \\ 3-4i & 5 & -2-i \\ 5+2i & -2+i & 2 \end{bmatrix}$$

$$\Rightarrow \quad \overline{(A^T)} = \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$$

$$= A$$

Thus, A is a hermitian matrix.

125 We have,

$$\Rightarrow \quad A^T = \begin{bmatrix} 2i & -2-3i & 2+i \\ -2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix}$$

$$(\overline{A^T}) = \begin{bmatrix} -2i & 2+3i & 2-i \\ -2+3i & -i & -3i \\ -2-i & -3i & 0 \end{bmatrix}$$

$$\Rightarrow \quad (\overline{A^T}) = -\begin{bmatrix} 2i & -2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix} = -A$$

Hence A is an skew Hermitian matrix.

126. Let $A = \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-\text{i} & -1 \end{bmatrix}$

$$\Rightarrow \quad A^\theta = \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\Rightarrow \quad AA^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Hence A is unitary matrix.

Level III

1. We have,

$$\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a & abc & abc(b+c) \\ b & bca & bca(c+a) \\ c & cab & abc(a+b) \end{vmatrix}$$

$$= \frac{(abc)^2}{abc}\begin{vmatrix} a & 1 & (b+c) \\ b & 1 & (c+a) \\ c & 1 & (a+b) \end{vmatrix}$$

$$= (abc)\begin{vmatrix} a & 1 & (b+c) \\ b & 1 & (c+a) \\ c & 1 & (a+b) \end{vmatrix}$$

$$= (abc)\begin{vmatrix} a & 1 & (a+b+c) \\ b & 1 & (b+c+a) \\ c & 1 & (a+b+c) \end{vmatrix} \qquad (C_1 \to C_1 + C_3)$$

$$= (abc)(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$$

$$= 0$$

which is independent of a, b and c.

2. We have,

$$\begin{vmatrix} b+c & a & a \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = \begin{vmatrix} 0 & -2a & -2a \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$$

$[R_1 \to R_1 - (R_2 + R_3)]$

$$= -2a\begin{vmatrix} 0 & 1 & 1 \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$$

$$= -2a\begin{vmatrix} 0 & 0 & 1 \\ c & c & a \\ b & -b & a+b \end{vmatrix} \qquad (C_2 \to C_2 - C_3)$$

$$= -2a(-bc - bc)$$

$$= -2a \times -2bc$$

$$= 4abc$$

3. We have,

$$\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & a^2+c^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 2(b^2+c^2) & 2(a^2+c^2) & 2(a^2+b^2) \\ b^2 & a^2+c^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$(R_3 \to R_1 + R_2 + R_3)$

$$= 2\begin{vmatrix} (b^2+c^2) & (a^2+c^2) & (a^2+b^2) \\ b^2 & a^2+c^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} (b^2+c^2) & (a^2+c^2) & (a^2+b^2) \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$$= 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

$$= 2(a^2b^2c^2 + a^2b^2c^2)$$

$$= 4(a^2b^2c^2)$$

65. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if
    (a) the first column of M is the transpose of the second row of M
    (b) the second row of M is the transpose of the first column of M
    (c) M is a diagonal matrix with non-zero entries in the main diagonal
    (d) the product of entries in the main diagonal of M is not the square of an integer.

[IIT-JEE, 2014]

66. Let M and N be two 3×3 matrices such that $M \neq N^2$ and $M^2 = N^4$, then
    (a) determinant of $(M^2 + MN^2)$ is 0
    (b) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix
    (c) determinant of $(M^2 + MN^2) \geq 1$
    (d) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix.

[IIT-JEE, 2014]

67. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix.
    Which of the following matrices is (are) skew-symmetric?
    (a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$
    (c) $X^4Z^3 - Z^3X^4$ (d) $X^{23} + Y^{23}$

[IIT-JEE, 2015]

68. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha?$$

(a) –4 (b) 9 (c) –9 (d) 4

[IIT-JEE, 2015]

ANSWERS

LEVEL II

1. (d)	2. (d)	3. (c)	4. (b)	5. (b)
6. (b)	7. (d)	8. (a)	9. (c)	10. (d)
11. (b)	12. (d)	13. (c)	14. (d)	15. (b)
16. (a)	17. (c)	18. (c)	19. (d)	20. (c)
21. (b)	22. (a)	23. (b)	24. (c)	25. (d)
26. (c)	27. (d)	28. (b)	29. (a)	30. (a)
31. (b)	32. (d)	33. (a)	34. (b)	35. (c)
36. (b)	37. (d)	38. (c)	39. (a)	40. (b)
41. (d)	42. (d)	43. (c)	44. (c)	45. (c)
46. (b)	47. (a)	48. (a)	49. (a)	50. (d)
51. (a)	52. (a)	53. (c)	54. (a)	55. (d)
56. (b)	57. (c)	58. (a)	59. (d)	60. (c)
61. (d)	62. (a)	63. (c)	64. (d)	65. (a)

INTEGER TYPE QUESTIONS

1. 2	2. 3	3. 1	4. 2	5. 5
6. 6	7. 3	8. 8	9. 5	10. 2
11. 2	12. 3	13 7	14. 2	15. 3

COMPREHENSIVE LINK PASSAGES

Passgage I:	1. (d)	2. (b)	3. (a)
Passgage II:	1. (a)	2. (b)	3. (b)
Passgage III:	1. (c)	2. (a)	3. (b)
Passgage IV:	1. (d)	2. (c)	3. (d)

MATCH MATRICES

1. (A) → (T); (B) → (T); (C) → (Q); (D) → (S)
2. (A) → (Q); (B) → (R); (C) → (P); (D) → (P)
3. (A) → (Q); (B) → (R); (C) → (S); (D) → (P)
4. (A) → (R); (B) → (Q); (C) → (P); (D) → (S)

ASSERTION AND REASON

1. (b)	2. (b)	3. (a)	4. (d)
5. (c)	6. (d)	7. (b)	8. (a)

HINTS AND SOLUTIONS

LEVEL I

1. Each element of the given matrices of order 2×2 can be filled in 2 ways, i.e. either 1 or 0.
   Thus, the number of possible matrices of order 2×2
   $= 2 \times 2 \times 2 \times 2 = 16$
2. Each element of the given matrices of order 3×3 can be filled in 2 ways, i.e. either 1 or 0.
   Thus, the number of possible matrices of order 3×3 is
   $= 2 \times 2 \times 2 \times \ldots$ up to 9 times
   $= 2^9 = 512$
3. As we know that $A + (-A) = \mathbf{O}$, then $-A$ is the additive inverse of A.
   Therefore, the additive inverse of A,

$$-A = \begin{pmatrix} -2 & -4 \\ -3 & -5 \end{pmatrix}$$