de-Broglie's hypothesis : wave-particle duality

Light behaves as wave when it undergoes interference, diffraction etc. and is completely described by Maxwell's equations. But then the wave nature of electromagnetic radiation is called into question when it is involved in black body radiation, photoelectric effect and such. Einstein forwarded his idea of photon, bundle of quantized radiant energy localized in a small vol^m, as a way to describe particle-like nature of light. The energy and momentum of such photon was proposed to be,

$$E = h\nu \ \& \ p = E/c = h/\lambda$$

de-Broglie (1924) made a great speculative hypothesis that just as radiation has particle-like properties, electrons and other material particle possess wave-like properties.

For free particles, de-Broglie assumed that the associated wave also has a frequency ν and wavelength λ related to its energy E & momentum p,

$$\nu = E/h \quad \text{and} \quad \lambda = h/p$$

For non-relativistic particles having mass m & moving with a velocity v and kinetic energy $E_K = mv^2/2$, the de-Broglie wavelength is

$$\lambda = h/mv = \frac{h}{\sqrt{2mE_K}}$$

For high energy particles, $E^2 = p^2c^2 + m_0^2c^4$, having $E_K = E - m_0c^2$ & the momentum is $pc = \sqrt{E_K(E_K + 2m_0c^2)}$

$$\therefore \lambda = h/p = \frac{hc}{\sqrt{E_K(E_K + 2m_0c^2)}}$$

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The de-Broglie hypothesis gives an interesting physical insight into Bohr's quantization rule,

$$mvr = pr = \frac{nh}{2\pi}$$

where 'p' is the linear momentum of an electron in an allowed orbit of radius r. As $p = h/\lambda$, so Bohr's quantization rule can be written as

$$\frac{hr}{\lambda} = \frac{nh}{2\pi} \Rightarrow 2\pi r = n\lambda \quad \text{where } n = 1, 2, 3, 4, \ldots$$

implying the allowed orbits are those in which the circumference of the orbit can contain exactly an integral number of de-Broglie wavelengths.

Group velocity & phase velocity :

Let us express the de-Broglie wave of a free micro particle by a plane wave of constant amplitude A, which will represent a particle of energy $E = h\nu = \hbar\omega$ and momentum $p = h/\lambda = \hbar k$,

$$\psi(\vec{r}, t) = A e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

in one dimension $\psi(x,t) = A e^{i(kx - \omega t)}$

In order to manufacture a wave train we revoke the idea of group or moving wave of classical wave motion. Suppose we have two waves of (ω, k) and $(\omega + d\omega, k + dk)$ and for simplicity let them be represented as

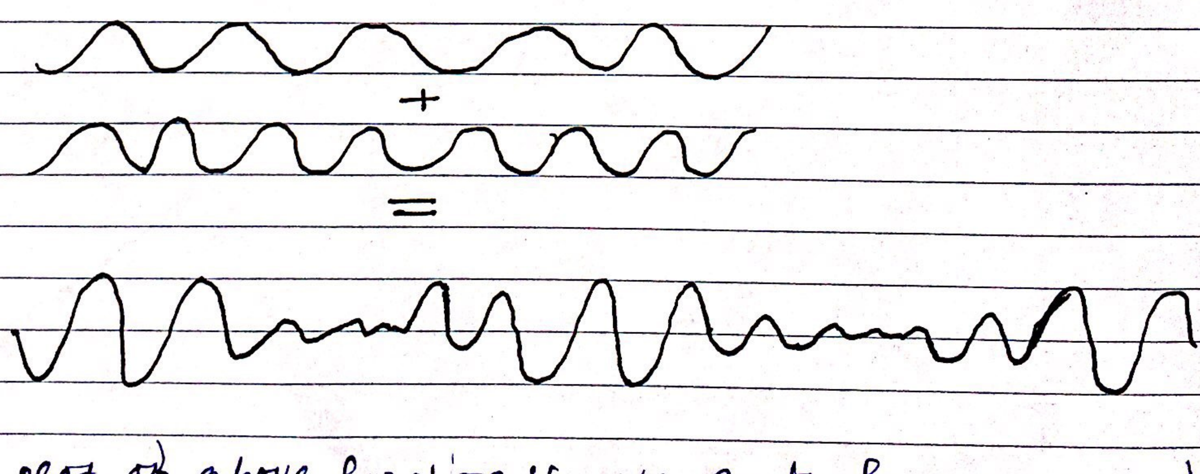
$$\psi_1 = A\cos[kx - \omega t] \quad \& \quad \psi_2 = A\cos[(k+dk)x - (\omega + d\omega)t]$$

We superpose these two waves, considering $d\omega << \omega$ & $dk << k$, to obtain

$$\psi = \psi_1 + \psi_2$$

$$\doteq 2A\cos\left[\frac{dk}{2}x - \frac{d\omega}{2}t\right]\cos\left[\frac{2k+dk}{2}x - \frac{2\omega + d\omega}{2}t\right]$$

$$\simeq 2A\cos\left[\frac{dk}{2}x - \frac{d\omega}{2}t\right]\cos[kx - \omega t]$$



From the plot of above function ψ, we see that two waves of slightly different frequency & wavelength, interfere & reinforce in such a way as to produce a series of groups. These groups and the individual waves they contain, are both moving in the same direction. The velocity of the group, called group velocity v_g and velocity of the individual waves, called phase velocity v_p, are given by

$$v_g = d\omega/dk \quad , \quad v_p = \omega/k$$

If the particle of mass m is moving with a velocity v, the kinetic energy $E = mv^2/2$ and momentum $p = mv$

$$\therefore \quad \omega = E/\hbar$$

$$\Rightarrow d\omega = dE/\hbar$$

$$\text{and } k = p/\hbar$$

$$\Rightarrow dk = dp/\hbar$$

$$\therefore \quad v_g = d\omega/dk = dE/dp = \frac{mv\,dv}{m\,dv} = v$$

$\therefore$ the velocity of the particle is equal to the velocity of the group of matter wave describing the particle. The same is true for relativistic particle.

Heisenberg uncertainty principle:

A non-trivial result follows from wave packet, the product of the finite extent of the wave packet Δx and the range of momentum $\Delta k \equiv \Delta p/\hbar$ chosen to construct the wave packet of the said extent is

$$\Delta x \Delta k = 4\pi$$

$$\Rightarrow \Delta x \Delta p = 4\pi\hbar$$

The upshot of the above equation is - if we try to get smaller wave train to better describe a localized particle, we have to superpose matter waves of wider range of Δk implying imprecise knowledge of the particle's momentum, while superposing too few matter waves in smaller Δk range will lead to greater uncertainty in the particle position Δx.

The above result is formally summarized in Heisenberg's Uncertainty Principle:

$$\Delta x \Delta p \geq \hbar/2$$

Heisenberg Uncertainty relation imposes restriction on accuracy of simultaneous measurement of position & momentum - the more precise our measurement of position is, the less accurate will be our momentum measurement ~~of position~~ and vice-versa. The physical origin of ~~the~~ ~~one~~ uncertainty principle is - with the quantum system, determination of position by performing measurement on the system disturbs it sufficiently to make the determination of momentum imprecise and vice-versa.