2. Cauchy-Schwarz inequality.

If $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be all real numbers, then $({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$, the equality occurs when either

(i) $a_i = 0$ for $i = 1, 2, \ldots, n$; or $b_i = 0$ for $i = 1, 2, \ldots, n$; or both $a_i = 0$ and $b_i = 0$ for $i = 1, 2, \ldots, n$;

or (ii) $a_i = kb_i$ for some non-zero real $k, i = 1, 2, \ldots, n$.

Proof. **Case I.** If $a_i = 0$ for $i = 1, 2, \ldots, n$; or $b_i = 0$ for $i = 1, 2, \ldots, n$; or both $a_i = 0$ and $b_i = 0$ for $i = 1, 2, \ldots, n$; then the equality holds, each side being zero.

Case II. Let not all of a_i and not all of b_i be zero.

Sub-case (i). Let $a_i = kb_i$ for some non zero real $k, i = 1, 2, \ldots, n$.

Then $({a_1}^2+{a_2}^2+\cdots+{a_n}^2)({b_1}^2+{b_2}^2+\cdots+{b_n}^2) = k^2({b_1}^2+{b_2}^2+\cdots+{b_n}^2)^2$ and $(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 = k^2({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2)^2$.

Therefore the equality holds in this sub-case.

Sub-case (ii). Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be not proportional. Let us consider the expression

$(a_1 - \lambda b_1)^2 + (a_2 - \lambda b_2)^2 + \cdots + (a_n - \lambda b_n)^2$, where λ is real.

For all real λ, the expression ≥ 0. The equality occurs only when

$a_1 - \lambda b_1 = 0, a_2 - \lambda b_2 = 0, \cdots, a_n - \lambda b_n = 0$

i.e., when $(a_1, a_2, \ldots, a_n) = \lambda(b_1, b_2, \ldots, b_n)$

i.e., when $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are proportional.

Therefore, in this sub-case $(a_1 - \lambda b_1)^2 + (a_2 - \lambda b_2)^2 + \cdots + (a_n - \lambda b_n)^2 > 0$ for all real λ.

or, $({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2) - 2\lambda(a_1b_1 + a_2b_2 + \cdots + a_nb_n) + \lambda^2({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2) > 0$

or, $B\lambda^2 - 2C\lambda + A > 0$, where $A = {a_1}^2 + {a_2}^2 + \cdots + {a_n}^2$, $B = {b_1}^2 + {b_2}^2 + \cdots + {b_n}^2$, $C = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

The roots of the equation $Bx^2 - 2Cx + A = 0$ must be imaginary, because otherwise, there would exist some real λ for which the equality $B\lambda^2 - 2C\lambda + A = 0$ would hold, a contradiction.

Therefore $AB > C^2$

or, $({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2) > (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$.

This completes the proof.

Note. In particular, if $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be all positive real numbers, then

$$({a_1}^2+{a_2}^2+\cdots+{a_n}^2)({b_1}^2+{b_2}^2+\cdots+{b_n}^2) \geq (a_1b_1+a_2b_2+\cdots+a_nb_n)^2,$$

the equality occurs if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Worked Examples (continued).

2. For all real x, y, prove that $-\frac{1}{2} \leq \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \leq \frac{1}{2}$.

Let us consider ordered pairs $(2x, 1 - x^2)$ and $(1 - y^2, 2y)$.

By Cauchy-Schwarz inequality,

$[2x(1-y^2)+(1-x^2)2y]^2 \le [(2x)^2+(1-x^2)^2][(1-y^2)^2+(2y)^2]$

or, $[2(x+y)(1-xy)]^2 \le (1+x^2)^2(1+y^2)^2$.

or, $[\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}]^2 \le \frac{1}{4}$

or, $-\frac{1}{2} \le \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \le \frac{1}{2}$.

3. If $a_i > -\frac{1}{3}$, $(i = 1, 2, 3)$ and $a+b+c = 1$, prove that $\sqrt{3a+1}+\sqrt{3b+1}+\sqrt{3c+1} \le 3\sqrt{2}$.

Let us consider ordered triplets $(1, 1, 1)$ and $(\sqrt{3a+1}, \sqrt{3b+1}, \sqrt{3c+1})$.

By Cauchy-Schwarz inequality,

$(\sqrt{3a+1}+\sqrt{3b+1}+\sqrt{3c+1})^2 \le (1+1+1)[(3a+1)+(3b+1)+(3c+1)]$.

or, $\sqrt{3a+1}+\sqrt{3b+1}+\sqrt{3c+1} \le 3\sqrt{2}$.

The equality occurs if $3a+1 = 3b+1 = 3c+1$, i.e., if $a = b = c$.

4. If $a, b, c, d > 0$ and $a+b+c+d = 1$, prove that $\frac{a}{1+b+c+d}+\frac{b}{1+a+c+d}+\frac{c}{1+a+b+d}+\frac{d}{1+a+b+c} \ge \frac{4}{7}$.

$\frac{a}{1+b+c+d}+\frac{b}{1+a+c+d}+\frac{c}{1+a+b+d}+\frac{d}{1+a+b+c}$

$=\frac{a}{2-a}+\frac{b}{2-b}+\frac{c}{2-c}+\frac{d}{2-d} = \frac{2}{2-a}+\frac{2}{2-b}+\frac{2}{2-c}+\frac{2}{2-d}-4$.

$=2[\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}+\frac{1}{2-d}]-4$

Let us consider positive numbers $\frac{1}{\sqrt{2-a}}, \frac{1}{\sqrt{2-b}}, \frac{1}{\sqrt{2-c}}, \frac{1}{\sqrt{2-d}}$ and $\sqrt{2-a}, \sqrt{2-b}, \sqrt{2-c}, \sqrt{2-d}$.

By Cauchy-Schwarz inequality,

$(1+1+1+1)^2 \le [\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}+\frac{1}{2-d}][8-(a+b+c+d)]$

or, $\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c}+\frac{1}{2-d} \ge \frac{16}{7}$.

Therefore $\frac{a}{1+b+c+d}+\frac{b}{1+a+c+d}+\frac{c}{1+a+b+d}+\frac{d}{1+a+b+c} \ge \frac{32}{7}-4$, i.e., $\ge \frac{4}{7}$.

The equality occurs when $2-a = 2-b = 2-c = 2-d$, i.e., when $a = b = c = d$.

5. If $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$; $c_1, c_2, \ldots, c_n$ be all positive real numbers prove that

$(a_1b_1c_1 + a_2b_2c_2 + \cdots + a_nb_nc_n)^2 < ({a_1}^2 + {a_2}^2 + \cdots + {a_n}^2)({b_1}^2 + {b_2}^2 + \cdots + {b_n}^2)({c_1}^2 + {c_2}^2 + \cdots + {c_n}^2)$.

Let $d_i = b_i c_i, i = 1, 2, ..., n$.

By Cauchy-Schwarz inequality, $(a_1 d_1 + a_2 d_2 + \cdots + a_n d_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(d_1^2 + d_2^2 + \cdots + d_n^2)$.

Again, $(d_1^2 + d_2^2 + \cdots + d_n^2) = b_1^2 c_1^2 + b_2^2 c_2^2 + \cdots + b_n^2 c_n^2$
$< (b_1^2 + b_2^2 + \cdots + b_n^2)(c_1^2 + c_2^2 + \cdots + c_n^2)$,
since b_i, c_i are all positive.

Therefore $(a_1 b_1 c_1 + a_2 b_2 c_2 + \cdots + a_n b_n c_n)^2 < (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)(c_1^2 + c_2^2 + \cdots + c_n^2)$.

Theorem 1.3.1. If $a_1, a_2, \ldots, a_n$ be n positive real numbers, not all equal, and p, q are rational numbers, then

$$\frac{a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}}{n} > \text{ or } < \frac{a_1^p + a_2^p + \cdots + a_n^p}{n} \cdot \frac{a_1^q + a_2^q + \cdots + a_n^q}{n}$$

according as p and q have the same sign or opposite signs.

Proof. **Case 1.** Let p, q have the same sign. Let i and j be any two of the numbers $1, 2, ..., n$. Since p, q are of the same sign, $a_i^p - a_j^p$ and $a_i^q - a_j^q$ are both positive or both negative or both zero.

Therefore $(a_i^p - a_j^p)(a_i^q - a_j^q) \geq 0$

or, $a_i^{p+q} + a_j^{p+q} \geq a_i^p a_j^q + a_i^q a_j^p$.

There are ${}^nC_2$ relations of this type, not of all them are equalities.

Adding, we have
$(n-1)(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) > \Sigma a_i^p a_j^q, i = 1, 2, ..., n; j = 1, 2, ..., n; i \neq j$

or, $n(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) > (a_1^p + a_2^p + \cdots + a_n^p)(a_1^q + a_2^q + \cdots + a_n^q)$

or, $$\frac{a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}}{n} > \frac{a_1^p + a_2^p + \cdots + a_n^p}{n} \cdot \frac{a_1^q + a_2^q + \cdots + a_n^q}{n}.$$

Case 2. Let p, q have opposite signs. Then $a_i^p - a_j^p$ and $a_i^q - a_j^q$ have opposite signs when $a_i \neq a_j$ and both are zero when $a_i = a_j$.

Therefore $(a_i^p - a_j^p)(a_i^q - a_j^q) \leq 0$.

Proceeding with similar arguments as in case 1, we can prove

$$\frac{a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}}{n} < \frac{a_1^p + a_2^p + \cdots + a_n^p}{n} \cdot \frac{a_1^q + a_2^q + \cdots + a_n^q}{n}.$$

This completes the proof.

Note. The theorem can be generalised. If $a_1, a_2, ..., a_n$ be n positive real numbers, not all equal, and $p_1, p_2, ..., p_m$ be m rational numbers, all positive or all negative, such that $s = p_1 + p_2 + \cdots + p_m$, then

$\frac{a_1{}^s+a_2{}^s+\cdots+a_n{}^s}{n} >$
$\frac{a_1{}^{p_1}+a_2{}^{p_1}+\cdots+a_n{}^{p_1}}{n} \cdot \frac{a_1{}^{p_2}+a_2{}^{p_2}+\cdots+a_n{}^{p_2}}{n} \cdots \frac{a_1{}^{p_m}+a_2{}^{p_m}+\cdots+a_n{}^{p_m}}{n}$.

Taking in particular, $p_1 = p_2 = \cdots = p_m = 1$, we have
$\frac{a_1{}^m+a_2{}^m+\cdots+a_n{}^m}{n} > \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m$.

Worked Example (continued).

6. If a, b, c, d be positive real numbers, not all equal, prove that

$$(a+b+c+d)\left(\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}+\tfrac{1}{d}\right) > 16.$$

We have $\frac{a^p+b^p+c^p+d^p}{4} \cdot \frac{a^q+b^q+c^q+d^q}{4} > \frac{a^{p+q}+b^{p+q}+c^{p+q}+d^{p+q}}{4}$, where p, q are rational numbers of opposite signs.

Let $p = 1, q = -1$. Then $\frac{a+b+c+d}{4} \cdot \frac{(a^{-1}+b^{-1}+c^{-1}+d^{-1})}{4} > 1$
or, $(a+b+c+d)(a^{-1}+b^{-1}+c^{-1}+d^{-1}) > 16$.

Another method.

Let us consider real numbers $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$ and $\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}$.

By Cauchy-Schwarz inequality,

$$(1+1+1+1)^2 \leq (a+b+c+d)\left(\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}+\tfrac{1}{d}\right).$$

The equality occurs when $a = b = c = d$.

Since here a, b, c, d are not all equal, $(a+b+c+d)(a^{-1}+b^{-1}+c^{-1}+d^{-1}) > 16$.