5.11. The cubic equation.

The general form of a cubic equation with binomial coefficients is

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \quad \cdots \quad \text{(i)}$$

Let us apply the transformation $x = y + h$ in order that the transformed equation may want the second term.

The transformed equation is

$$a_0(y+h)^3 + 3a_1(y+h)^2 + 3a_2(y+h) + a_3 = 0$$

or, $a_0y^3 + 3(a_0h + a_1)y^2 + 3(a_0h^2 + 2a_1h + a_2)y + (a_0h^3 + 3a_1h^2 + 3a_2h + a_3) = 0.$

Since the equation wants the second term, $h = -\frac{a_1}{a_0}$ and the equation reduces to

$$y^3 + 3\frac{(a_0a_2 - a_1{}^2)}{a_0{}^2}y + \frac{(a_0{}^2a_3 - 3a_0a_1a_2 + a_1{}^3)}{a_0{}^2} = 0.$$

Using the standard symbols $H = a_0a_2 - a_1{}^2, G = a_0{}^2a_3 - 3a_0a_1a_2 + 2a_1{}^3$, the equation takes the form

$$y^3 + \frac{3H}{a_0{}^2}y + \frac{G}{a_0{}^3} = 0 \ldots \text{(ii)}$$

The roots of the equation are $\alpha + \frac{a_1}{a_0}, \beta + \frac{a_1}{a_0}, \gamma + \frac{a_1}{a_0}$ where α, β, γ are the roots of the cubic equation (i).

Since $\alpha + \beta + \gamma = -\frac{3a_1}{a_0}$, the roots of the equation (ii) are $\frac{1}{3}(2\alpha - \beta - \gamma), \frac{1}{3}(2\beta - \gamma - \alpha), \frac{1}{3}(2\gamma - \alpha - \beta)$.

Multiplying the roots of this equation by a_0, the transformed equation becomes $z^3 + 3Hz + G = 0$.

This is called the standard form of a cubic equation.

The roots of this equation are $a_0\alpha + a_1, a_0\beta + a_1, a_0\gamma + a_1$

i.e., $\frac{a_0}{3}(2\alpha - \beta - \gamma), \frac{a_0}{3}(2\beta - \gamma - \alpha), \frac{a_0}{3}(2\gamma - \alpha - \beta)$.

Note. $\Sigma(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha) = \frac{27H}{a_0{}^2}$ and

$(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta) = -\frac{27G}{a_0{}^3}$.

5.11.1. The equation whose roots are squares of the differences of the roots of a cubic equation.

Let α, β, γ be the roots of the cubic equation $x^3 + qx + r = 0$.

To find the equation whose roots are $(\beta - \gamma)^2, (\gamma - \alpha)^2, (\alpha - \beta)^2$.

Let $y = (\beta - \gamma)^2$.

Then $y = (\beta + \gamma)^2 - 4\beta\gamma = \alpha^2 + \frac{4r}{\alpha}$, since $\alpha + \beta + \gamma = 0, \quad \alpha\beta\gamma = -r$.

Therefore $\alpha^3 - \alpha y + 4r = 0$.

Since $\alpha^3 + q\alpha + r = 0$, we have $(q + y)\alpha - 3r = 0$, or $\alpha = \frac{3r}{y+q}$.

Hence $(\frac{3r}{y+q})^3 + q(\frac{3r}{y+q}) + r = 0$

or, $(y+q)^3 + 3q(y+q)^2 + 27r^2 = 0$

or, $y^3 + 6qy^2 + 9q^2y + 27r^2 + 4q^3 = 0.$ ... (A)

This is the required equation.

If it is proposed to form an equation whose roots are squares of the differences of the roots of the cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \quad \cdots \quad \text{(i)}$$

we first remove the second term. The transformed equation is

$$y^3 + \frac{3H}{a_0^2}y + \frac{G}{a_0^3} = 0 \quad \cdots \quad \text{(ii)}$$

and its roots are $\alpha + \frac{a_1}{a_0}, \beta + \frac{a_1}{a_0}, \gamma + \frac{a_1}{a_0}$ where α, β, γ are the roots of the equation (i).

Let $\alpha' = \alpha + \frac{a_1}{a_0}, \beta' = \beta + \frac{a_1}{a_0}, \gamma' = \gamma + \frac{a_1}{a_0}$.

Then $\beta' - \gamma' = \beta - \gamma, \gamma' - \alpha' = \gamma - \alpha, \alpha' - \beta' = \alpha - \beta$.

Therefore the equation whose roots are squares of the differences of the roots of the cubic equation (i) is same as the equation whose roots are $(\beta' - \gamma')^2, (\gamma' - \alpha')^2, (\gamma' - \beta')^2$, and the equation can be obtained by putting $q = \frac{3H}{a_0^2}, r = \frac{G}{a_0^3}$ in (A).

Therefore the required equation is

$$x^3 + \frac{18H}{a_0^2}x^2 + \frac{81H^2}{a_0^4}x + \frac{27}{a_0^6}(G^2 + 4H^3) = 0. \quad \cdots \quad \text{(B)}$$

Note 1. It follows that $(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = -\frac{27}{a_0^6}(G^2 + 4H^3)$.

2. Nature of the roots.

Assuming that the coefficients are all real, we discuss the nature of the roots of the equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$.

Case I. $G^2 + 4H^3 > 0$.

In this case $(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 < 0$. The cubic has two imaginary roots, because otherwise, if all the roots be real then each of $(\alpha - \beta)^2, (\beta-\gamma)^2, (\gamma-\alpha)^2$ is non-negative and therefore their product cannot be negative.

Case II. $G^2 + 4H^3 < 0$ and $H < 0$.

In this case the signs of the coefficients of the equation (B) are alternately positive and negative and therefore, by Descartes' rule of signs, the equation (B) has no negative root and consequently, the given equation has all its roots real. Because otherwise, if $\lambda + \mu i$ be a root of the given cubic then $\lambda - \mu i$ will be another root and the square of their difference which is a root of the equation (B) is obviously a negative real number, a contradiction.

Case III. $G^2 + 4H^3 = 0$.

In this case one of $(\alpha - \beta), (\beta - \gamma), (\gamma - \alpha)$ is zero and this proves the existence of a multiple root of the given cubic.

Case IV. $G^2 + 4H^3 = 0$ and $H = 0$.

In this case the equation (B) reduces to $x^3 = 0$ and this proves that $(\alpha - \beta)^2 = 0, (\beta - \gamma)^2 = 0, (\gamma - \alpha)^2 = 0$. Therefore the given cubic has three equal roots.

Worked Examples.

1. Find the equation whose roots are squares of the differences of the roots of the equation $x^3 + 9x^2 + 24x + 20 = 0$. What conclusion do you draw about the nature of the roots of the given equation?

Let α, β, γ be the roots of the given equation.

Let us apply the transformation $x = y + h$ in order to remove the second term.

The equation transforms to

$(y+h)^3 + 9(y+h)^2 + 24(y+h) + 20 = 0$

or, $y^3 + (3h+9)y^2 + (3h^2+18h+24)y + (h^3+9h^2+24h+20) = 0$.

Since the second term of this equation is to be absent, $h = -3$.

The transformed equation is $y^3 - 3y + 2 = 0$.

The roots of the transformed equation are $\alpha+3, \beta+3, \gamma+3$.

Let $\alpha' = \alpha + 3, \beta' = \beta + 3, \gamma' = \gamma + 3$.

Therefore the equation whose roots are $(\alpha-\beta)^2, (\beta-\gamma)^2, (\gamma-\alpha)^2$ is same as the equation whose roots are $(\alpha'-\beta')^2, (\beta'-\gamma')^2, (\gamma'-\alpha')^2$.

Let $z = (\alpha' - \beta')^2$.

Then $z = (\alpha'+\beta')^2 - 4\alpha'\beta' = \gamma'^2 + \frac{8}{\gamma'}$, since $\alpha'+\beta'+\gamma' = 0$, $\alpha'\beta'\gamma' = -2$.

Therefore $\gamma'^3 - \gamma' z + 8 = 0$.

Since $\gamma'^3 - 3\gamma' + 2 = 0$, we have $\gamma'(z-3) = 6$, or $\gamma' = \frac{6}{z-3}$.

Hence $(\frac{6}{z-3})^3 - 3(\frac{6}{z-3}) + 2 = 0$

or, $2(z-3)^3 - 18(z-3)^2 + 216 = 0$

or, $z^3 - 18z^2 + 81z = 0$.

This is the required equation.

One root of the transformed equation, say $(\alpha - \beta)^2$ is zero. That is, two roots of the given equation are equal.

2. Find the equation whose roots are squares of the differences of the roots of the equation $x^3 + x + 2 = 0$ and deduce from the resulting

equation the nature of the roots of the given cubic.

Let α, β, γ be the roots of the given equation.

Let $y = (\beta - \gamma)^2$.

Then $y = (\beta + \gamma)^2 - 4\beta\gamma = \alpha^2 + \frac{8}{\alpha}$ since $\alpha + \beta + \gamma = 0,\ \alpha\beta\gamma = -2$

Therefore $\alpha^3 - \alpha y + 8 = 0$.

Since $\alpha^3 + \alpha + 2 = 0$, we have $\alpha(y + 1) = 6$, or $\alpha = \frac{6}{y+1}$.

Hence $(\frac{6}{y+1})^3 + \frac{6}{y+1} + 2 = 0$

or, $(y + 1)^3 + 3(y + 1)^2 + 108 = 0$

or, $y^3 + 6y^2 + 9y + 112 = 0$. This is the required equation.

By Descartes' rule of signs, this equation has at least one negative root. Therefore the given cubic must have a pair of imaginary roots.

5.11.2. The general solution of a cubic (Cardan's method).

Let the cubic equation be $ax^3 + 3bx^2 + 3cx + d = 0 \quad \cdots \quad$ (i)

This can be put in the standard form $z^3 + 3Hz + G = 0$, where $z = ax + b,\ H = ac - b^2,\ G = a^2d - 3abc + 2b^3$.

To solve the equation, let us assume $z = u + v$.

Then $z^3 = u^3 + v^3 + 3uv(u + v) = u^3 + v^3 + 3uvz$

or, $z^3 - 3uvz - (u^3 + v^3) = 0$.

Comparing this with $z^3 + 3Hz + G = 0$, we have

$uv = -H \quad u^3 + v^3 = -G$.

Therefore $u^3 = \frac{1}{2}(-G + \sqrt{G^2 + 4H^3}),\ v^3 = \frac{1}{2}(-G - \sqrt{G^2 + 4H^3})$.

If p denotes any one of the three values of $\{\frac{1}{2}(-G + \sqrt{G^2 + 4H^3})\}^{1/3}$, then the three values of u are $p, \omega p, \omega^2 p$ where ω is an imaginary cube root of unity.

And since $uv = -H$, the three coresponding values of v are $\frac{-H}{p}, \frac{-\omega^2 H}{p}, \frac{-\omega H}{p}$.

Hence the values of z are $p - \frac{H}{p}, \omega p - \frac{\omega^2 H}{p}, \omega^2 p - \frac{\omega H}{p}$ and the three values of x are $\frac{1}{a}(p - \frac{H}{p} - b), \frac{1}{a}(\omega p - \frac{\omega^2 H}{p} - b), \frac{1}{a}(\omega^2 p - \frac{\omega H}{p} - b)$.

These give the complete solution of the equation (i).

The method of solution is called the Cardan's method of solution although the method owes its origin to Tartaglia.

Note. When $G^2 + 4H^3 < 0$, the roots of the cubic equation are all real but Cardan's solution give them in imaginary form.

In this case we use De Moivre's theorem to obtain the real roots in the following manner.

Let $G^2 + 4H^3 = -k^2$.

Then $u^3 = \frac{1}{2}(-G + ik), v^3 = \frac{1}{2}(-G - ik)$.

Let $\frac{-G}{2} = r(\cos\ \theta), \frac{k}{2} = r\sin\ \theta$, where $-\pi < \theta \leq \pi$.

Then $u^3 = r(\cos\theta + i\ \sin\theta)$ and $r^2 = -H^3$.

Therefore the three values of u are $\sqrt[3]{r}(\cos\frac{\theta}{3} + i\ \sin\frac{\theta}{3})$,

$\sqrt[3]{r}(\cos\frac{2\pi+\theta}{3} + i\ \sin\frac{2\pi+\theta}{3}), \sqrt[3]{r}(\cos\frac{4\pi+\theta}{3} + i\ \sin\frac{4\pi+\theta}{3})$.

Also since $uv = -H$, the corresponding values of v are

$\sqrt[3]{r}(\cos\frac{\theta}{3} - i\ \sin\frac{\theta}{3}), \sqrt[3]{r}(\cos\frac{2\pi+\theta}{3} + i\ \sin\frac{2\pi+\theta}{3}), \sqrt[3]{r}(\cos\frac{4\pi+\theta}{3} + i\ \sin\frac{4\pi+\theta}{3})$.

Hence the values of z are $2\sqrt[3]{r}\cos\frac{\theta}{3}, 2\sqrt[3]{r}\cos\frac{2\pi+\theta}{3}, 2\sqrt[3]{r}\cos\frac{4\pi+\theta}{3}$

i.e., $2\sqrt{-H}\cos\frac{\theta}{3}, 2\sqrt{-H}\cos\frac{2\pi+\theta}{3}, 2\sqrt{-H}\cos\frac{4\pi+\theta}{3}$.

Worked Examples.

1. Solve the equation $x^3 - 18x - 35 = 0$.

Let $x = u + v$.

Then $x^3 = u^3 + v^3 + 3uvx$

or, $x^3 - 3uvx - (u^3 + v^3) = 0$.

Comparing with the given cubic, we have $uv = 6$ and $u^3 + v^3 = 35$.

Therefore $u^3 = \frac{1}{2}(35 + \sqrt{35^2 - 864}) = 27$ and

$$v^3 = \frac{1}{2}(35 - \sqrt{35^2 - 864}) = 8\ .$$

The three values of u are $3, 3\omega, 3\omega^2$ and the three values of v are $2, 2\omega, 2\omega^2$.

Since $uv = 6$, we have $u + v = 3 + 2, 3\omega + 2\omega^2, 3\omega^2 + \omega$.

Hence the roots of the given equation are $5, \frac{-5+\sqrt{3}i}{2}, \frac{-5-\sqrt{3}i}{2}$.

2. Solve the equation $x^3 - 15x^2 - 33x + 847 = 0$.

Let us apply the transformation $x = y + h$ in order to remove the second term.

The transformed equation is

$(y + h)^3 - 15(y + h)^2 - 33(y + h) + 847 = 0$

or, $y^3 + (3h - 15)y^2 + (3h^2 - 30h - 33)y + (h^3 - 15h^2 - 33h + 847) = 0$.

So $h = 5$ and the equation reduces to $y^3 - 108y + 432 = 0 \quad \cdots \quad$ (i)

Let $y = u + v$.

Then $y^3 = u^3 + v^3 + 3uvy$

or, $y^3 - 3uvy - (u^3 + v^3) = 0$.

Comparing with the equation (i), we have $uv = 36$ and $u^3 + v^3 = -432$. Therefore $u^3 = v^3 = -216$.

The three values of u are $-6, -6\omega, -6\omega^2$.

Since $uv = 36$, the corresponding values of v are $-6, -6\omega^2, -6\omega$.

Then $y = -12, 6, 6$ and the roots of the given equation are $-7, 11, 11$.

3. Solve the equation $x^3 - 3x - 1 = 0$.

Let $x = u + v$.

Then $x^3 = u^3 + v^3 + 3uvx$

or, $x^3 - 3uvx - (u^3 + v^3) = 0$.

Comparing with the given cubic, we have $uv = 1$ and $u^3 + v^3 = 1$.

Therefore $u^3 = \frac{1}{2}(1 + \sqrt{3}i), v^3 = \frac{1}{2}(1 - \sqrt{3}i)$.

or, $u = (\cos\frac{\pi}{3} + i\ \sin\frac{\pi}{3})^{\frac{1}{3}}, v = (\cos\frac{\pi}{3} - i\ \sin\frac{\pi}{3})^{\frac{1}{3}}$.

The three values of u are $\cos\frac{\pi}{9} + i\ \sin\frac{\pi}{9}, \cos\frac{7\pi}{9} + i\ \sin\frac{7\pi}{9}, \cos\frac{13\pi}{9} + i\ \sin\frac{13\pi}{9}$

and the three values of v are $\cos\frac{\pi}{9} - i\ \sin\frac{\pi}{9}, \cos\frac{7\pi}{9} - i\ \sin\frac{7\pi}{9}, \cos\frac{13\pi}{9} - i\ \sin\frac{13\pi}{9}$.

Since $uv = 1$,

$u = \cos\frac{\pi}{9} + i\ \sin\frac{\pi}{9}$ corresponds to $v = \cos\frac{\pi}{9} - i\ \sin\frac{\pi}{9}$;

$u = \cos\frac{7\pi}{9} + i\ \sin\frac{7\pi}{9}$ corresponds to $v = \cos\frac{7\pi}{9} - i\ \sin\frac{7\pi}{9}$;

$u = \cos\frac{13\pi}{9} + i\ \sin\frac{13\pi}{9}$ corresponds to $v = \cos\frac{13\pi}{9} - i\ ;\sin\frac{13\pi}{9}$.

Taking $x = u + v$, the roots of the given equation are $2\cos\frac{\pi}{9}, 2\cos\frac{7\pi}{9}, 2\cos\frac{13\pi}{9}$.

10. Solve by Cardan's method

(i) $x^3 - 27x - 54 = 0$,

(ii) $x^3 - 9x + 28 = 0$,

(iii) $x^3 - 12x + 8 = 0$,

(iv) $x^3 - 3x - 2\cos A = 0 \; (-\pi < A \leq \pi)$,

(v) $x^3 - 6x + 4 = 0$,

(vi) $9x^3 - 9x - 4 = 0$,

(vii) $2x^3 - 3x + 1 = 0$,

(viii) $x^3 + 9x^2 + 15x - 25 = 0$,

(ix) $x^3 - 6x^2 - 6x - 7 = 0$,

(x) $x^3 + 3x^2 - 3 = 0$.

5.12. The biquadratic equation.

The general form of the biquadratic equation with binomial coefficients is

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0. \quad \cdots \quad \text{(i)}$$

Let us apply the transformation $x = y + h$ in order that the transformed equation may want the second term.

The transformed equation is

$$a_0(y+h)^4 + 4a_1(y+h)^3 + 6a_2(y+h)^2 + 4a_3(y+h) + a_4 = 0$$

or, $a_0y^4 + 4(a_0h + a_1)y^3 + 6(a_0h^2 + 2a_1h + a_2)y^2 + 4(a_0h^3 + 3a_1h^2 + 3a_2h + a_3)y + (a_0h^4 + 4a_1h^3 + 6a_2h^2 + 4a_3h + a_4) = 0$

Since the transformed equation wants the second term $h = -\frac{a_1}{a_0}$ and the equation reduces to

$a_0y^4 + \frac{6}{a_0}(a_0a_2 - a_1{}^2)y^2 + \frac{4}{a_0{}^2}(a_0{}^2a_3 - 3a_0a_1a_2 + 2a_1{}^3)y + \frac{1}{a_0{}^3}(a_0{}^3a_4 - 4a_0{}^2a_1a_3 + 6a_0a_1{}^2a_2 - 3a_1{}^4) = 0$

Using the standard symbols $H = a_0a_2 - a_1{}^2, G = a_0{}^2a_3 - 3a_0a_1a_2 + 2a_1{}^3$ and $I = a_0a_4 - 4a_1a_3 + 3a_2{}^2$, the equation takes the form

$$a_0y^4 + \frac{6H}{a_0}y^2 + \frac{4G}{a_0{}^2}y + \frac{1}{a_0{}^3}(a_0{}^2I - 3H^2) = 0.$$

The roots of the equation are $\alpha + \frac{a_1}{a_0}, \beta + \frac{a_1}{a_0}, \gamma + \frac{a_1}{a_0}, \delta + \frac{a_1}{a_0}$ where $\alpha, \beta, \gamma, \delta$ are the roots of the equation (i).

Since $\alpha + \beta + \gamma + \delta = -\frac{4a_1}{a_0}$, the roots are $\frac{1}{4}(3\alpha - \beta - \gamma - \delta), \frac{1}{4}(3\beta - \gamma - \delta - \alpha), \frac{1}{4}(3\gamma - \delta - \alpha - \beta), \frac{1}{4}(3\delta - \alpha - \beta - \gamma)$.

Multiplying the roots by a_0, the transformed equation is

$$z^4 + 6Hz^2 + 4Gz + (a_0{}^2I - 3H^2) = 0.$$

This is called the **standard form** of the biquadratic. The roots of this equation are $\frac{1}{4}a_0(3\alpha - \beta - \gamma - \delta), \frac{1}{4}a_0(3\beta - \gamma - \delta - \alpha), \frac{1}{4}a_0(3\gamma - \delta - \alpha - \beta), \frac{1}{4}a_0(3\delta - \alpha - \beta - \gamma)$.

Worked Example.

1. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, find the value of

(i) $(\alpha + \beta - \gamma - \delta)(\beta + \gamma - \alpha - \delta)(\gamma + \alpha - \beta - \delta)$,

(ii) $(3\alpha - \beta - \gamma - \delta)(3\beta - \gamma - \alpha - \delta)(3\gamma - \alpha - \beta - \delta)(3\delta - \alpha - \beta - \gamma)$.

Let us apply the transformation $x = y + h$ in order to remove the second term. Then

$$a_0(y+h)^4 + 4a_1(y+h)^3 + 6a_2(y+h)^2 + 4a_3(y+h) + a_4 = 0$$

or, $a_0y^4 + 4(a_0h + a_1)y^3 + 6(a_0h^2 + 2a_1h + a_2)y^2 + 4(a_0h^3 + 3a_1h^2 + 3a_2h + a_3)y + (a_0h^4 + 4a_1h^3 + 6a_2h^2 + 4a_3h + a_4) = 0$.

Therefore $h = -\frac{a_1}{a_0}$ and the equation reduces to

$$a_0y^4 + 6\left(\frac{a_0a_2 - a_1^2}{a_0}\right)y^2 + 4\left(\frac{a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3}{a_0^2}\right)y + \left(\frac{a_4a_0^3 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4}{a_0^3}\right) = 0.$$

Let $\alpha', \beta', \gamma', \delta'$ be the roots of the transformed equation.

Then $\alpha = \alpha' - \frac{a_1}{a_0}, \beta = \beta' - \frac{a_1}{a_0}, \gamma = \gamma' - \frac{a_1}{a_0}, \delta = \delta' - \frac{a_1}{a_0}$.

(i) $(\alpha + \beta - \gamma - \delta)(\beta + \gamma - \alpha - \delta)(\gamma + \alpha - \beta - \delta)$

$= (\alpha' + \beta' - \gamma' - \delta')(\beta' + \gamma' - \alpha' - \delta')(\gamma' + \alpha' - \beta' - \delta')$

$= -8(\gamma' + \delta')(\alpha' + \delta')(\beta' + \delta')$ since $\alpha' + \beta' + \gamma' + \delta' = 0$

$= -8[\delta'^3 + \delta'^2(\alpha' + \beta' + \gamma') + \delta'(\alpha'\beta' + \beta'\gamma' + \gamma'\alpha') + \alpha'\beta'\gamma']$

$= -8\Sigma\alpha'\beta'\gamma'$, since $\alpha' + \beta' + \gamma' = -\delta'$

$= 32\left(\frac{a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3}{a_0^3}\right) = \frac{32G}{a_0^3}$.

(ii) $(3\alpha - \beta - \gamma - \delta)(3\beta - \gamma - \delta - \alpha)(3\gamma - \delta - \alpha - \beta)(3\delta - \alpha - \beta - \gamma)$

$= (3\alpha' - \beta' - \gamma' - \delta')(3\beta' - \gamma' - \delta' - \alpha')(3\gamma' - \delta' - \alpha' - \beta')(3\delta' - \alpha' - \beta' - \gamma')$

$= 4\alpha'.4\beta'.4\gamma'.4\delta'$, since $\alpha' + \beta' + \gamma' + \delta' = 0$

$= 256\left(\frac{a_4a_0^3 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4}{a_0^4}\right) = 256\left(\frac{a_0^2I - 3H^2}{a_0^4}\right)$.

5.12.1. Ferrai's solution of a biquadratic equation.

Ferrari's method reduces the problem of solving a biquadratic equation to that of solving two quadratic equations. This is done by expressing the biquadratic as the difference of two perfect suqares.

Let the equation be $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$.

Multiplying by a, $a^2x^4 + 4abx^3 + 6acx^2 + 4adx + ae = 0$ $\cdots$ (i)

Let the left hand expression be expressed as the difference of two squares in the form

$$(ax^2 + 2bx + \lambda)^2 - (mx + n)^2.$$

Comparing with the left hand expression of (i) we have

$$6ac = 4b^2 + 2\lambda a - m^2,$$
$$4ad = 4b\lambda - 2mn,$$
$$ac = \lambda^2 - n^2.$$

Eliminating m, n from these we have

$$4(b\lambda - ad)^2 = (2\lambda a + 4b^2 - 6ac)(\lambda^2 - ac).$$

This is a cubic equation in λ, giving at least one real root λ_1.

Corresponding to $\lambda = \lambda_1$, we have the values of m^2 and n^2 and morever the relation $mn = 2b\lambda_1 - 2ad$ determines only one value of n corresponding to one value of m.

Thus the given equation is now put in the form

$(ax^2 + 2bx + \lambda_1)^2 - (m_1x + n_1)^2 = 0$, where m_1, n_1 are the values of m, n corresponding to λ_1.

The roots of the quadratic equations $ax^2 + 2bx + \lambda_1 \pm (m_1x + n_1) = 0$ give the solution of the given biquadratic equation.

Worked Examples.

1. Solve by Ferrari's method

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$$

The equation may be written as

$(x^2 - 5x + \lambda)^2 - (mx + n)^2 = 0$, where λ, m, n are constants.

Equating coefficients of like powers of x

$$35 = 25 + 2\lambda - m^2, \text{ or } m^2 = 2\lambda - 10$$
$$-50 = -10\lambda - 2mn, \text{ or } mn = -5\lambda + 25$$
$$24 = \lambda^2 - n^2, \text{ or } n^2 = \lambda^2 - 24.$$

Eliminating m, n we have

$$(\lambda^2 - 24)(2\lambda - 10) - (5\lambda - 25)^2 = 0$$
$$\text{or, } (\lambda - 5)[2\lambda^2 - 48 - 25\lambda + 125] = 0$$
$$\text{or, } (\lambda - 5)[2\lambda^2 - 25\lambda + 77] = 0.$$

Therefore $\lambda = 5, 7, \frac{11}{5}$.

Taking $\lambda = 5$, we have $m = 0, n = \pm 1$.

The equation takes the form

$$(x^2 - 5x + 5)^2 - 1 = 0$$
$$\text{or, } (x^2 - 5x + 6)(x^2 - 5x + 4) = 0$$
$$\text{or, } (x - 2)(x - 3)(x - 1)(x - 4) = 0.$$

Therefore $x = 2, 3, 1, 4$.

Hence the roots of the equation are $1, 2, 3, 4$.

2. If $f(x) = x^4 + 6x^2 + 14x^2 + 22x + 5$ find α, β, λ so that $f(x)$ may be expressed in the form $(x^2 + 3x + \lambda)^2 - (\alpha x + \beta)^2$.

Hence solve the equation $f(x) = 0$.

$x^4 + 6x^2 + 14x^2 + 22x + 5 = (x^2 + 3x + \lambda)^2 - (\alpha x + \beta)^2$.

Equating coefficients of like powers of x, we have

$$14 = 9 + 2\lambda - \alpha^2, \text{ or } \alpha^2 = 2\lambda - 5.$$
$$22 = 6\lambda - 2\alpha\beta, \text{ or } \alpha\beta = 3\lambda - 11.$$
$$5 = \lambda^2 - \beta^2, \text{ or } \beta^2 = \lambda^2 - 5.$$

Eliminating α, β we have

$$(\lambda^2 - 5)(2\lambda - 5) - (3\lambda - 11)^2 = 0$$
$$\text{or, } 2\lambda^3 - 14\lambda^2 + 56\lambda - 96 = 0$$
$$\text{or, } (\lambda - 3)(2\lambda^2 - 8\lambda + 32) = 0.$$

Therefore $\lambda = 3, 2 \pm 2\sqrt{3}i$.

Taking $\lambda = 3$, we have $\alpha = \pm 1, \beta = \pm 2$ and $\alpha\beta = -2$.

Therefore α and β are of opposite signs.

The equation can be expressed as

$$(x^2 + 3x + 3)^2 - (x - 2)^2 = 0$$
$$\text{or, } (x^2 + 4x + 1)(x^2 + 2x + 5) = 0.$$

Hence the roots of the equation are $-2 \pm \sqrt{3}, -1 \pm 2i$.