1.10. Mapping.

Let A and B be two non-empty sets. If f is a relation between A and B then an element x of A may be related to one element or no element or many elements of B by the relation f.

A relation f with the property that each element x of A is related to exactly one element y of B is said to be a *mapping* from A to B.

Since a relation f between A and B is a subset of $A \times B$, f is a mapping from A to B if each element x of A appears as the first component exactly in one of the ordered pairs of f. Therefore no two distinct ordered pairs of f have the same first component. If $(x, y) \in f$, f is said to assign y to the element x of A.

Definition. Let A and B be two non-empty sets. A mapping f from A to B is a *rule* that assigns to *each* element x of A a *definite* element y in B.

A is said to be the *domain* of f and B is said to be the *co-domain* of f. The mapping f with the domain A and co-domain B is displayed symbolically by $f : A \to B$.

We can imagine f as a kind of agent that carries (or transforms, or maps) *each* element x of A to a *unique* element y in B.

A mapping f is also called a *function*, or a *transformation*, or a *map*.

Let $f : A \to B$ be a mapping and $x \in A$. Then the unique element y of B that corresponds to (is associated with) x by the mapping f is called the *f-image* of x (or the image of x under f) and is denoted by $f(x)$. If $f(x) = y$, we often say that 'f maps x to y'.

The set of all f-images, i.e., the set $\{f(x) : x \in A\}$ is denoted by $f(A)$ and is said to be the *image set* of f (denoted by *im* f) or the *range set* of f.

In some texts the domain of f is denoted by $D(f)$ and the range of f is denoted by $R(f)$.

Examples.

1. Let $S = \{1, 2, 3, 4\}, T = \{a, b, c, d\}$. Let us examine the following relations f_1, f_2, f_3, f_4 between S and T.

(i) $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$.

(ii) $f_2 = \{(1, a), (2, b), (3, c)\}$.

(iii) $f_3 = \{(1, b), (2, b), (3, c), (4, d)\}$.

f_1 is not a mapping from S to T since the element 1 in S is related to two different elements of T by the relation.

f_2 is not a mapping from S to T since the element 4 in S is not related to any element of T by the relation.

f_3 is a mapping from S to T. Here the image set is $\{b, c, d\}$ and it is a proper subset of the co-domain set T.

2. Let $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{1}{x}\}$. Let us examine if f is a mapping from $\mathbb{R}$ to $\mathbb{R}$.

The element 0 in the domain set $\mathbb{R}$ is not related to an element of the codomain set. Therefore f is not a mapping from $\mathbb{R}$ to $\mathbb{R}$.

Let $S = \mathbb{R} - \{0\}$. Then $f = \{(x, y) \in S \times \mathbb{R} : y = \frac{1}{x}\}$ is a mapping from S to $\mathbb{R}$. It is displayed symbolically as

"$f : S \to \mathbb{R}$ is defined by $f(x) = \frac{1}{x}$, $x \in S$".

Definitions.

1. A mapping $f : A \to B$ is said to be an *into mapping* if $f(A)$ is a proper subset of B. In this case we say that f maps A into B.

2. A mapping $f : A \to B$ is said to be an *onto mapping* if $f(A) = B$. In this case we say that f maps A onto B.

Examples (continued).

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$. Then f is an into mapping because $f(\mathbb{Z})$ (the set of all even integers) is a proper subset of the co-domain set $\mathbb{Z}$.

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = |x|$, $x \in \mathbb{Z}$. Then f is an into mapping because $f(\mathbb{Z})$ (the set of all non-negative integers) is a proper subset of the co-domain set $\mathbb{Z}$.

5. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 1$, $x \in \mathbb{Z}$. Then every element y in the co-domain set $\mathbb{Z}$ has a pre-image $y - 1$ in the domain set $\mathbb{Z}$. Therefore $f(\mathbb{Z}) = \mathbb{Z}$ and f is an onto mapping.

If $f : A \to B$ be a mapping and $x \in A$, then $f(x)$ is a definite element in B. The element x is said to be a *pre-image* (or *inverse image*) of $f(x)$. It may happen that an element y in the co-domain set B has only *one* pre-image, *no* pre-image or *many* pre-images in A.

In Example 4, 0 in the co-domain set $\mathbb{Z}$ has only one pre-image in the domain set; 1 in the co-domain set $\mathbb{Z}$ has two pre-images in the domain set; -2 in the co-domain set $\mathbb{Z}$ has no pre-image.

Thus the pre-images of an element y in B form a subset of A, which may be the null set, or a singleton set (a set containing one element only), or a set containing more than one elements. The pre-image set (or the inverse image set) of y is denoted by $f^{-1}(y)$.

Examples (continued).

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$, $x \in \mathbb{R}$. Here the image set of f is $\{x \in \mathbb{R} : -1 \leq x \leq 1\}$. Every element in the image set has infinite number of pre-images. For example, the pre-image set of 1 is $\{(4n + 1)\frac{\pi}{2} : n \text{ is an integer }\}$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$, $x \in \mathbb{R}$. For an element y in the co-domain set $\mathbb{R}$, $f^{-1}(y) = \{\frac{1}{2}y\}$, a one-element subset of the domain set $\mathbb{R}$.

Definitions.

3. A mapping $f : A \to B$ is said to be *injective* (or *one-to-one*) if for each pair of distinct elements of A, their f-images are distinct.

4. A mapping $f : A \to B$ is said to be *surjective* (or *onto*) if $f(A) = B$.

5. A mapping $f : A \to B$ is said to be *bijective* if f is both injective and surjective.

Thus $f : A \to B$ is injective if $x_1 \neq x_2$ in A implies $f(x_1) \neq f(x_2)$ in B. In this case , each element of B has *at most* one pre-image.

If f is surjective, each element of B has *at least* one pre-image.

If f is bijective, each element of B has *exactly one* pre-image.

An injective mapping is called an *injection*, a surjective mapping a *surjection* and a bijective mapping a *bijection*.

In Example 3, f is an injective mapping since for two distinct elements x_1, x_2 in the domain set $\mathbb{Z}$, $f(x_1) \neq f(x_2)$. f is not surjective because $f(\mathbb{Z})$ is a proper subset of the codomain set $\mathbb{Z}$.

In Example 4, f is not injective because two distinct elements 1 and -1 in the domain set $\mathbb{Z}$ have the same f-image. f is not surjective because $f(\mathbb{Z})$ is a proper subset of the codomain set $\mathbb{Z}$.

In Example 5, f is injective since for two distinct elements x_1, x_2 in the domain set $\mathbb{Z}$, $f(x_1) \neq f(x_2)$. f is surjective because $f(\mathbb{Z})$ = the codomain set $\mathbb{Z}$. Therefore f is a bijection.

When f is a bijection from A to B, f sets up *a one-to-one correspondence* between the elements of A and the elements of B. Each element x of A is put in correspondence with the single element $f(x)$ of B and each element y of B is put in correspondence with the single element $f^{-1}(y)$ of A.

Definitions.

6. A mapping $f : A \to B$ is said to be a *constant mapping* (a constant function) if f maps each element of A to one and the same element of B, i.e., $f(A)$ is a singleton set.

For example, the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2$, $x \in \mathbb{R}$ is a constant mapping. Here $f(\mathbb{R}) = \{2\}$.

A mapping $f : A \to A$ is said to be the *identity mapping on* A if $f(x) = x, x \in A$. The identity mapping on A is denoted by i_A.

The identity mapping on A is clearly a bijection on A.

Equality of mappings.

Two mappings $f : A \to B$ and $g : A \to C$ are said to be *equal* if $f(x) = g(x)$ for all $x \in A$. For the equality of two mappings f and g the following conditions must hold:

(i) f and g have the same domain D; and

(ii) for all $x \in D$, $f(x) = g(x)$.

Examples (continued).

8. Let $S = \{x \in \mathbb{R} : x > 0\}$. Let $f : S \to \mathbb{R}$ be defined by $f(x) = \frac{|x|}{x}$, $x \in S$ and $g : S \to \mathbb{R}$ be defined by $g(x) = 1$, $x \in S$. Then $f = g$.

9. Let $S = \{x \in \mathbb{R} : 1 \leq x < 2\}$. Let $f : S \to \mathbb{R}$ be defined by $f(x) = [x] - x$, $x \in S$ and $g : S \to \mathbb{R}$ be defined by $g(x) = 1 - x$, $x \in S$. Then $f = g$.

Restriction of a mapping.

Let $f : A \to B$ be a mapping and let D be a non-empty subset of A. Then the mapping $g : D \to B$ defined by $g(x) = f(x), x \in D$ is said to be the *restriction* of f to D. This is denoted by f/D.

In this case, f is said to be an *extension* of g to A. An extension $f : A \to B$ of the mapping $g : D \to B$ is not unique, since the f-images of the elements of $A - D$ may be arbitrarily chosen.

When a non-bijective mapping f is given, sometimes it is possible to have a bijective restriction of f.

Examples (continued).

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1$, if x be rational
$= 0$, if x be irrational.

Then the restriction mapping $f/\mathbb{Q} : \mathbb{Q} \to \mathbb{R}$ is given by $f/\mathbb{Q}(x) = 1$ for all $x \in \mathbb{Q}$.

The restriction mapping $f/(\mathbb{R} - \mathbb{Q}) : (\mathbb{R} - \mathbb{Q}) \to \mathbb{R}$ is given by $f/(\mathbb{R} - \mathbb{Q})(x) = 0$ for all $x \in (\mathbb{R} - \mathbb{Q})$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, $x \in \mathbb{R}$.

Let $S = \mathbb{R}^+$(the set of all positive rel numbers). The restriction mapping $f/S : S \to \mathbb{R}$ is defined by $f/S(x) = x^2$, $x \in S$.

Here f is not injective, but the restriction mapping f/S is injective.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$, $x \in \mathbb{R}$. This mapping f is neither surjective nor injective.

If we reduce the codomain to $T = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$, then the mapping $f : \mathbb{R} \to T$ defined by $f(x) = \sin x$, $x \in \mathbb{R}$ is surjective, but not injective.

Let $S = \{x \in \mathbb{R} : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$. Then the restriction mapping $g : S \to T$ defined by $g(x) = \sin x$, $x \in S$ is a bijection.

Worked Examples.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 1, x \in \mathbb{R}$. Examine if f is (i) injective, (ii) surjective.

(i) Let us take two distinct elements x_1, x_2 in the domain set $\mathbb{R}$.

$f(x_1) = 3x_1 + 1, f(x_2) = 3x_2 + 1$.

$f(x_1) - f(x_2) = 3(x_1 - x_2) \neq 0$ since $x_1 \neq x_2$.

Since $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, f is injective.

(ii) Let us take an arbitrary element y in the co-domain set $\mathbb{R}$ and let us examine if there is a pre-image x of the element y under f.

Then $f(x) = y$. Therefore $3x + 1 = y$ or, $x = \frac{y-1}{3}$.

Since $y \in \mathbb{R}$, $\frac{y-1}{3} \in \mathbb{R}$. Therefore y has a pre-image $\frac{y-1}{3}$ in the domain set $\mathbb{R}$. Since y is arbitrary, each element in the co-domain set $\mathbb{R}$ has a pre-image under f. Therefore f is surjective.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, $x \in \mathbb{R}$. Examine if f is (i) injective, (ii) surjective.

(i) $f(2) = 4, f(-2) = 4$. f is not injective since two distinct elements 2 and -2 in the domain set $\mathbb{R}$ have the same image.

(ii) f is not surjective since -1 in the co-domain set $\mathbb{R}$ has no pre-image in the domain set $\mathbb{R}$.

1.11. Composition of mappings.

Let $f : A \to B$ and $g : C \to D$ be two mappings such that $f(A)$ is a subset of C.

Let $x \in A$. Then f maps x to an element y in $f(A) \subset B$ and since $y \in f(A) \subset C$, g maps y to an element z in D.

We can conceive of a mapping $h : A \to D$ defined by $h(x) = g(f(x))$, $x \in A$. The mapping $h : A \to D$ is said to be the *composite* (or the *product*) of f and g and is denoted by $g \circ f$ (or by gf). The composite $g \circ f : A \to D$ is defined only if $f(A)$ is a subset of the domain of g.

For the mappings $f : A \to B$ and $g : B \to C$ the composite $g \circ f : A \to C$ is defined. For the mappings $f : A \to B$ and $g : B \to A$ both the composites $g \circ f : A \to A$ and $f \circ g : B \to B$ are defined.

Examples.

1. Let $A = \{1, 2, 3\}, B = \{p, q, r\}$, and let
$f : A \to B$ be defined by $f(1) = p, f(2) = q, f(3) = r$;
$g : B \to A$ be defined by $g(p) = 3, g(q) = 2, g(r) = 1$;
$h : B \to A$ be defined by $h(p) = 1, h(q) = 2, h(r) = 3$.

$g \circ f : A \to A$ is defined by $g \circ f(1) = 3, g \circ f(2) = 2, g \circ f(3) = 1$.

$f \circ g : B \to B$ is defined by $f \circ g(p) = r, f \circ g(q) = q, f \circ g(r) = p$.

$h \circ f : A \to A$ is defined by $h \circ f(1) = 1, h \circ f(2) = 2, h \circ f(3) = 3$.

$f \circ h : B \to B$ is defined by $f \circ h(p) = p, f \circ h(q) = q, f \circ h(r) = r$.

Here $g \circ f \neq f \circ g$, $h \circ f = i_A$, $f \circ h = i_B$.

2. Let $f : \mathbb{Z} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{Q}$ be defined by $f(x) = \frac{1}{2}x$, $x \in \mathbb{Z}$ and $g(x) = x^2$, $x \in \mathbb{Q}$.

$g \circ f : \mathbb{Z} \to \mathbb{Q}$ is defined by $g \circ f(x) = g(\frac{1}{2}x) = \frac{1}{4}x^2$, $x \in \mathbb{Z}$.

For example, $g \circ f(1) = g(f(1)) = g(\frac{1}{2}) = \frac{1}{4}$. $g \circ f(2) = g(f(2)) = g(1) = 1$.

Here $f \circ g$ is not defined since the range of g is not a subset of the domain of f.

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$, $x \in \mathbb{R}$ and $g(x) = 3x$, $x \in \mathbb{R}$.

Here $g \circ f$ and $f \circ g$ are both defined.

$g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by $g \circ f(x) = g(x + 1) = 3x + 3$, $x \in \mathbb{R}$;
$f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by $f \circ g(x) = f(3x) = 3x + 1$, $x \in \mathbb{R}$.

Both the mappings $g \circ f$ and $f \circ g$ have the same domain but $g \circ f(x) \neq f \circ g(x), x \in \mathbb{R}$. Therefore $g \circ f \neq f \circ g$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$, $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x + 5$, $x \in \mathbb{R}$.

Here both $g \circ f$ and $f \circ g$ are defined.

$fog : \mathbb{R} \to \mathbb{R}$ is defined by $fog(x) = f(x+5) = x+6,\ x \in \mathbb{R}$;
$gof : \mathbb{R} \to \mathbb{R}$ is defined by $gof(x) = g(x+1) = x+6,\ x \in \mathbb{R}$.

Since $gof(x) = fog(x)$ for all $x \in \mathbb{R}$, we have $gof = fog$.

It is quite clear from the examples that the composition of mappings is not commutative. That is, for two mappings f and g their composites gof and fog are, in general, not equal. In fact, when one of them is defined,the other may not be defined at all. In particular cases, however, the equality holds.

Although the composition of mappings is not commutative, it is associative. That is, for three mappings f, g and h, $ho(gof) = (hog)of$, when both sides are defined mappings.

Theorem 1.11.1. Let $f : A \to B,\ g : B \to C,\ h : C \to D$ be three mappings. Then $ho(gof) = (hog)of$.

Proof. Here the composite mappings gof, hog are defined because the range $f \subset$ dom g and the range $g \subset$ dom h. The composite mappings $ho(gof), (hog)of$ are defined because the range $gof \subset$ dom h and the range $f \subset$ dom hog.

We shall now prove the equality of the mappings $ho(gof) : A \to D$ and $(hog)of : A \to D$.

Let x be an element of A and let $f(x) = y,\ g(y) = z,\ h(z) = w$.
Then $gof(x) = g(y) = z, \quad hog(y) = h(z) = w$.
$ho(gof) : A \to D$ is defined by $ho(gof)(x) = h(z) = w,\ x \in A$.
$(hog)of : A \to D$ is defined by $(hog)f(x) = hog(y) = w,\ x \in A$.

Since $ho(gof)(x) = (hog)of(x)$ for all $x \in A$, we have $ho(gof) = (hog)of$.

Theorem 1.11.2. If $f : A \to B$ and $g : B \to C$ be both injective mappings then the composite mapping $gof : A \to C$ is injective.

Proof. Let x_1, x_2 be two distinct elements of A.
Let $f(x_1) = y_1,\ f(x_2) = y_2$.
Since f is injective, y_1 and y_2 are distinct elements of B.
Let $g(y_1) = z_1,\ g(y_2) = z_2$.
Since g is injective z_1 and z_2 are distinct elements of C.
Now $gof(x_1) = g(y_1) = z_1,\ gof(x_2) = g(y_2) = z_2$ and $x_1 \neq x_2$ in A $\Rightarrow z_1 \neq z_2$ in C. Therefore gof is injective.

Note. The converse of the theorem is not true. However if gof is injective then f is injective (while g need not be).

Theorem 1.11.3. If $f : A \to B$ and $g : B \to C$ be two mappings such that $gof : A \to C$ is injective then f is injective.

Proof. Let $f(x_1) = f(x_2)$ for some x_1, x_2 in A. Then $g[f(x_1)] = g[f(x_2)]$, since g is a mapping. So $gof(x_1) = gof(x_2)$.

Because gof is injective, $gof(x_1) = gof(x_2)$ implies $x_1 = x_2$.

Thus $f(x_1) = f(x_2)$ implies $x_1 = x_2$ and therefore f is injective.

Note. In order that gof may be injective it is not necessary that g is injective.

For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$, $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$, $x \in \mathbb{R}$.

Here $gof : \mathbb{R} \to \mathbb{R}$ is defined by $gof(x) = e^{2x}$, $x \in \mathbb{R}$.

gof is injective but g is not injective.

Theorem 1.11.4. If $f : A \to B$ and $g : B \to C$ be both surjective then the composite mapping $gof : A \to C$ is surjective.

Proof. Let z be an element of C.

Since g is surjective, there is at least one pre-image of z in B. Let one such be y. Then $y \in B$ and $g(y) = z$.

Since f is surjective and $y \in B$, there is at least one pre-image of y in A. Let one such be x. Then $x \in A$ and $f(x) = y$.

$gof(x) = g(y) = z$. This implies that z has a pre-image x in A under the mapping gof. Since z is arbitrary, gof is surjective.

Note. The converse of the theorem is not true. However if gof is surjective then g is surjective (while f need not be).

Theorem 1.11.5. If $f : A \to B$ and $g : B \to C$ be two mappings such that $gof : A \to C$ is surjective then g is surjective.

Proof. Let z be an element of C. Since gof is surjective there is an element x in A such that $gof(x) = z$. Therefore $g(f(x)) = z$.

This shows that z has a pre-image $f(x)$ in B under the mapping g. Since z is arbitrary, g is surjective.

Note. In order that gof may be surjective it is not necessary that f is surjective.

For example, let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = [\frac{x}{2}]$, $x \in \mathbb{Z}$. $[x]$ denotes the greatest integer $\leq x$. Then $gof : \mathbb{Z} \to \mathbb{Z}$ is defined by $gof(x) = x, x \in \mathbb{Z}$.

$gof = i_{\mathbb{Z}}$ and is, therefore, surjective; but f is not surjective.

Theorem 1.11.6. If $f : A \to B$ and $g : B \to C$ be both bijective then the composite mapping $g \circ f : A \to C$ is bijective.

This is a combination of the Theorems 1.11.2 and 1.11.4.

Note. The converse of the theorem is not true. However if $g \circ f$ is bijective then f is injective and g is surjective.

This follows from the Theorems 1.11.3 and 1.11.5.

In order that $g \circ f$ may be bijective, it is neither necessary that f is surjective nor necessary that g is injective.

The example given in the note of the previous theorem establishes the assertion.

1.12. Inverse mapping.

Definition. Let $f : A \to B$ be a mapping. If there exists a mapping $g : B \to A$ such that $g \circ f = i_A$ then g is said to be a *left inverse* of f. If there exists a mapping $h : B \to A$ such that $f \circ h = i_B$ then h is said to be a *right inverse* of f.

Definition. Let $f : A \to B$ be a mapping. f is said to be *invertible* if there exists a mapping $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$. In this case g is said to be *an inverse* of f.

Theorem 1.12.1. If $f : A \to B$ be an invertible mapping then its inverse is unique.

Proof. Since $f : A \to B$ is invertible, there exists a mapping $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.

If possible, let there exist another mapping $h : B \to A$ such that $h \circ f = i_A$ and $f \circ h = i_B$.

$h \circ (f \circ g) = (h \circ f) \circ g$, since composition of mappings is associative. Therefore $h \circ i_B = i_A \circ g$, i.e., $h = g$. This proves that g is unique.

Note 1. g is said to be *the* inverse of f and is denoted by f^{-1}. Therefore $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

2. If $f : A \to A$ be an invertible mapping then $f^{-1} \circ f = f \circ f^{-1} = i$, where i is the identity mapping on A.

Examples.

1. Let $f : \mathbb{R} \to \mathbb{Z}$ be defined by $f(x) = [x]$, $x \in \mathbb{R}$ and $g : \mathbb{Z} \to \mathbb{R}$ be defined by $g(x) = x + \frac{1}{2}$, $x \in \mathbb{Z}$.

$f \circ g : \mathbb{Z} \to \mathbb{Z}$ is defined by $f \circ g(x) = f(x + \frac{1}{2}) = [x + \frac{1}{2}] = x$, $x \in \mathbb{Z}$.
$g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by $g \circ f(x) = g([x]) = [x] + \frac{1}{2}$, $x \in \mathbb{R}$.

Here $fog = i_{\mathbb{Z}}$, $gof \neq i_{\mathbb{R}}$.

Therefore g is a right inverse of f, but not a left inverse of f.

Let $h : \mathbb{Z} \to \mathbb{R}$ be defined by $h(x) = x + \frac{1}{3}$, $x \in \mathbb{Z}$.

Then $foh = i_{\mathbb{Z}}$ and therefore h is a right inverse of f.

Note. There are many right inverses of f. Since f is not a bijection, f is not invertible.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x$, $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \frac{x}{3}$, $x \in \mathbb{R}$.

$gof : \mathbb{R} \to \mathbb{R}$ is defined by $gof(x) = g(3x) = x$, $x \in \mathbb{R}$

$fog : \mathbb{R} \to \mathbb{R}$ is defined by $fog(x) = f(\frac{x}{3}) = x$, $x \in \mathbb{R}$.

Here $gof = fog = i_R$. Threfore g is the inverse of f.

Theorem 1.12.2. A mapping $f : A \to B$ is invertible if and only if f is a bijection.

Proof. Let $f : A \to B$ be invertible. Then there exists a mapping $g : B \to A$ such that $gof = i_A$ and $fog = i_B$.

Since i_A is injective and $gof = i_A$, f is an injection.

Since i_B is surjective and $fog = i_B$, f is a surjection.

Therefore f is a bijection.

Conversely, let $f : A \to B$ be a bijection.

Let $y \in B$. Since f is a bijection, y has a unique pre-image x in A.

Define a mapping $g : B \to A$ by $g(y) = x$ (the pre-image of y under f), $y \in B$. Then $gof(x) = g(y) = x$, $x \in A$ and $fog(y) = f(x) = y$, $y \in B$.

Here $gof = i_A$, $fog = i_B$ and therefore f is invertible.

Note. The theorem gives a clue how to determine f^{-1}. To each element y in B, f^{-1} assigns the pre-image of y under f.

Theorem 1.12.3. Let $f : A \to B$ be a bijective mapping. Then the mapping $f^{-1} : B \to A$ is also a bijection and $(f^{-1})^{-1} = f$.

Proof. Since $f : A \to B$ is a bijection, $f^{-1} : B \to A$ exists and $f^{-1}of = i_A$ and $fof^{-1} = i_B$.

Let y_1, y_2 be two distinct elements in B and $f^{-1}(y_1) = x_1$, $f^{-1}(y_2) = x_2$.

Then $f(x_1) = f[f^{-1}(y_1)] = fof^{-1}(y_1) = y_1$ and $f(x_2) = f[f^{-1}(y_2)] = fof^{-1}(y_2) = y_2$, since $fof^{-1} = i_B$.

Since f is a mapping, $y_1 \neq y_2 \Rightarrow x_1 \neq x_2$.

That is, $y_1 \neq y_2 \Rightarrow f^{-1}(y_1) \neq f^{-1}(y_2)$ and therefore f^{-1} is injective.

To establish that f^{-1} is surjective, let $x \in A$. Let $f(x) = y$.

Then $f^{-1}(y) = f^{-1}[f(x)] = f^{-1}of(x) = x$, since $f^{-1}of = i_A$.

This shows that y is a pre-image of x under the mapping f^{-1}. Therefore f^{-1} is surjective.

Since f^{-1} is injective as well as surjective, f^{-1} is a bijection.

Second part.

Since $f : A \to B$ is a bijection, $f^{-1} : B \to A$ exists and $f^{-1}of = i_A$ and $fof^{-1} = i_B$. Since $f^{-1} : B \to A$ is a bijection, $(f^{-1})^{-1} : A \to B$ exists and $f^{-1}o(f^{-1})^{-1} = i_A$ and $(f^{-1})^{-1}of^{-1} = i_B$.

Let $x \in A$ and $f(x) = y$.

$f^{-1}(y) = f^{-1}[f(x)] = f^{-1}of(x) = x$, since $f^{-1}of = i_A$.

$(f^{-1})^{-1}(x) = (f^{-1})^{-1}[f^{-1}(y)] = (f^{-1})^{-1}of^{-1}(y) = y$, since $(f^{-1})^{-1}of^{-1} = i_B$.

Therefore $f(x) = (f^{-1})^{-1}(x)$ for all $x \in A$.

Consequently, $f = (f^{-1})^{-1}$. This completes the proof.

Theorem 1.12.4. Let $f : A \to B$ and $g : B \to C$ be both bijective mappings. Then the mapping $gof : A \to C$ is invertible and $(gof)^{-1} = f^{-1}og^{-1}$.

Proof. Since $f : A \to B$ and $g : B \to C$ are both bijective, the composite mapping $gof : A \to C$ is a bijection, by Theorem 1.11.6.

Hence $(gof)^{-1} : C \to A$ exists, by Theorem 1.12.2.

Since $f : A \to B$ is a bijection, $f^{-1} : B \to A$ exists.

Since $g : B \to C$ is a bijection, $g^{-1} : C \to B$ exists.

Then the composite mapping $f^{-1}og^{-1} : C \to A$ exists.

Let $z \in C$. Let y be the pre-image of z under the bijective mapping g and x be the pre-image of y under the bijective mapping f.

Then $gof(x) = z$. Since gof is invertible, $(gof)^{-1}(z) = x$.

Since f is invertible and $f(x) = y$, $f^{-1}(y) = x$.

Since g is invertible and $g(y) = z$, $g^{-1}(z) = y$.

Therefore $f^{-1}og^{-1}(z) = f^{-1}[g^{-1}(z)] = f^{-1}(y) = x$.

Thus $(gof)^{-1}(z) = (f^{-1}og^{-1})(z)$ for all $z \in C$.

Consequently, $(gof)^{-1} = f^{-1}og^{-1}$. This completes the proof.

Worked Examples.

1. Let A, B be both finite sets of n elements and a mapping $f : A \to B$ is injective. Prove that f is a bijection.

Let $A = \{a_1, a_2, \ldots, a_n\}$. Then $f(a_1), f(a_2), \ldots, f(a_n)$ all belong to B. Since f is injective, $f(a_1), f(a_2), \ldots, f(a_n)$ are all distinct elements of B. As they are n in number, they are all the elements of B.

Let $b \in B$. Then $b = f(a_i)$, for some $a_i \in A$ and this shows that f is surjective. Since f is injective as well as surjective, f is a bijection.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 1$, $x \in \mathbb{R}$. Prove that f is invertible. Find f^{-1}.

As f is injective and surjective (worked Ex.1, 1.9), f is a bijection and therefore f is invertible.

Each y in the co-domain set $\mathbb{R}$ has a unique pre-image $\frac{y-1}{3}$.

$f^{-1} : \mathbb{R} \to \mathbb{R}$ is defined by $f^{-1}(y) = \frac{y-1}{3}$, $y \in \mathbb{R}$; or equivalently defined by $f^{-1}(x) = \frac{x-1}{3}$, $x \in \mathbb{R}$.

3. Let $S = \{x \in \mathbb{R} : -1 < x < 1\}$ and $f : \mathbb{R} \to S$ be defined by $f(x) = \frac{x}{1+|x|}$, $x \in \mathbb{R}$. Show that f is a bijection. Determine f^{-1}.

Let $x > 0$. Then $f(x) = \frac{x}{1+x}$ and $0 < f(x) < 1$.
Let $x = 0$. Then $f(x) = 0$.
Let $x < 0$. Then $f(x) = \frac{x}{1-x} = \frac{1}{1-x} - 1$ and $-1 < f(x) < 0$.
Let x_1, x_2 belong to $\mathbb{R}$ and $f(x_1) = f(x_2)$. Then $\frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|}$.

This implies x_1 and x_2 are either both positive, or both negative.
Let $x_1 > 0$ and $x_2 > 0$.
Then $f(x_1) = f(x_2) \Rightarrow \frac{x_1}{1+x_1} = \frac{x_2}{1+x_2} \Rightarrow x_1 = x_2$.
Let $x_1 < 0$ and $x_2 < 0$.
Then $f(x_1) = f(x_2) \Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1 = x_2$.
It follows that $x_1 \neq x_2$ in $\mathbb{R} \Rightarrow f(x_1) \neq f(x_2)$. So f is injective.

Let y be an element in S and let $0 < y < 1$. Let us examine if $x(> 0)$ in $\mathbb{R}$ be a pre-image of y. Then $\frac{x}{1+x} = y$
or, $x = \frac{y}{1-y} \in \mathbb{R}$, since $y \in \mathbb{R}$. Therefore $\frac{y}{1-y}$ is a pre-image of y.

Let y be an element in S and let $-1 < y < 0$. Let us examine if $x(< 0)$ in $\mathbb{R}$ be a pre-image of y. Then $\frac{x}{1-x} = y$
or, $x = \frac{y}{1+y} \in \mathbb{R}$, since $y \in \mathbb{R}$. Therefore $\frac{y}{1+y}$ is a pre-image of y.
Let $y = 0 \in S$. Then $x = 0$ is a pre-image of y.

It follows that each y in S has a pre-image in $\mathbb{R}$. So f is surjective.

Since f is injective as well as surjective, f is a bijection.
Since f is a bijection, each y in S has a unique pre-image.

For $y > 0$, the pre-image is $\frac{y}{1-y} = \frac{y}{1-|y|}$.
For $y < 0$, the pre-image is $\frac{y}{1+y} = \frac{y}{1-|y|}$.
For $y = 0$, the pre-image is $0 = \frac{y}{1-|y|}$.
It follows that for each y in S the pre-image is $\frac{y}{1-|y|}$.

Hence $f^{-1} : S \to \mathbb{R}$ is defined by $f^{-1}(x) = \frac{x}{1-|x|}$, $x \in S$.