KHARAGPUR COLLEGE

DEPARTMENT OF MATHEMATICS

STUDY MATERIALS

SUBJECT: MATHEMATICS GENERIC

CLASS: B. Sc. Hons.

SEMESTER: 1 ST

PAPER: GE 1

UNIT: IV [Differential Equations]

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UG CBCS FIRST SEMESTER / SUBJECT – GE1.

UNIT – IV : Differential Equation [Marks : 09].

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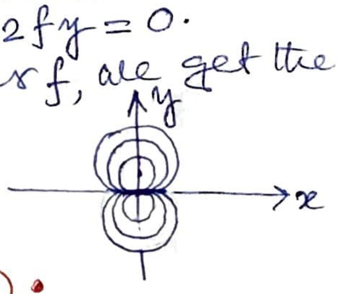
Introduction →

A differential equation is a concise form of expressing the properties of a family of functions. Many general laws of nature – in physics, chemistry, Biology, Economics, space science, and, in fact, in any branch of human knowledge – find their most general expression in the language of differential equations.

Examples:

1. The equation of motion of a particle of mass m which is acted by a force F is given by $F = ma \Rightarrow \frac{d^2x}{dt^2} = \frac{F}{m}$.
2. For a certain substance the rate of change of vapour pressure P w.r.t. temperature T is proportional to P and inversely proportional to T^2. i.e, $\frac{dP}{dT} = K\frac{P}{T^2}$ [$K \equiv$ Constant of proportionality]
3. Find the DE of all circles each of which touches the x-axis at the origin. $x^2 + y^2 + 2fy = 0$. Eliminating the arbitrary parameter f, we get the DE $(x^2 - y^2)y' = 2xy$.



Definition :

1. Ordinary Differential Equation (ODE) :

An ODE is an equation involving an independent variable and a dependent variable with its derivatives or differentials.

Examples: 1. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = xe^x$.

2. $(2xy + e^x)y\,dx - e^x dy = 0$.

The general form of an ODE is $F(x, y, y', \cdots, y^{(n)}) = 0$.

2. Partial Differential Equation (PDE) :

A PDE is an equation involving more than one independent variables and one dependent variable with its partial derivatives. eg. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

ORDER, DEGREE AND LINEARITY OF A DE.

- <u>ORDER</u> → The order of a DE is the order of the highest ordered derivative involved in the DE.
- <u>DEGREE</u> → The degree of a DE is the power of the highest ordered derivative involved in the DE after it has been made rational and integral as far as derivatives are concerned.
- <u>Linearity</u> → If a DE is of first degree in the dependent variable y and its derivative, then it is called a LINEAR DE.

  Otherwise it is called NON-LINEAR DE.

The general form of a linear DE is

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x).$$

<u>Examples</u>: Determine ORDER, DEGREE AND LINEARITY.

1. $\left\{1+\frac{d^2y}{dx^2}\right\}^{3/2} = a\frac{d^2y}{dx^2}$ → Order-2, Degree-3, Nonlinear.
2. $y = x\frac{dy}{dx} + c\frac{dx}{dy}$ → Order-1, Degree-2, Non-linear.
3. $\sqrt[3]{\frac{d^2y}{dx^2}} = x\sqrt{\frac{dy}{dx}}$ → Order-2, Degree-2, Non-linear
4. $x^3\frac{d^2y}{dx^2} + \cos x\frac{dy}{dx} + (\sin x)y = 0$ → Order-2, Degree-1, Linear.
5. $x\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - (\cos x)\sqrt{y} = 2x^2$ → Order-2, No degree, Non-linear (Because of $\sqrt{y}$).

<u>NOTE</u>: If an ODE cannot be written as a polynomial in the unknown function and its derivatives, then ODE has no degree.

<u>REMARK</u>: A DE can be constructed by eliminating the parameters of a family of curves or a group of motions.

The order of the equation is equal to the number of parameters of the family.

Solution of an ODE.

- A relation between the dependent and independent variables by means of which and the derivatives derived from it, the DE is identically satisfied, is said to represent a solution of an ODE.
- GENERAL SOLUTION → A solution of a DE which contains as many independent arbitrary constants as the order of the DE is called the general solution (g.s.). It is also called a complete primitive or a complete integral.

eg. $c_1 \cos x + c_2 \sin x$ is the general solution of $\frac{d^2y}{dx^2} + y = 0$.

- PARTICULAR SOLUTION → Any solution of a DE obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution.

eg. If we put $c_1 = 1$, $c_2 = 0$ in the g.s. $c_1 \cos x + c_2 \sin x$ of the DE $\frac{d^2y}{dx^2} + y = 0$, then $y = \cos x$ is a particular soln.

- SINGULAR SOLUTION → A solution of a DE which is not deducible from the g.s. by giving any particular values to the arbitrary constants is called a singular solution (s.s.).

eg. $y = cx + \frac{a}{c}$ is the g.s. of the DE $y = px + \frac{a}{p}$.
Now $y^2 = 4ax$ is also a soln of the given DE which is not obtained from the g.s. Hence $y^2 = 4ax$ is a s.s. of the DE.

Explicit and Implicit Solutions of an ODE :-

- A solution of the form $y = \phi(x)$ is called an explicit solution of an ODE $F(x, y, y', \dots, y^{(n)}) = 0$ on the interval I.

Many DEs do not have an explicit solution. The best we can do for these DEs is to eliminate the derivatives that appear and to simplify the equation into a derivative free equation of the form $G(x, y) = 0$ → implicit solution.
However, explicit solutions are preferable.

Implicit Solution of an ODE :-

A relation of the form $G(x,y)=0$ is said to be an implicit solution of an ODE $F(x,y,y',\cdots,y^{(n)})=0$ on an interval I, provided there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

$F(x,y,y',\cdots,y^{(n)})=0$; $G(x,y)=0$

ODE Implicit Solution

$F(x,\phi(x),\phi'(x),\cdots,\phi^{(n)}(x))=0$; $G(x,\phi(x))=0$

$\boxed{y=\phi(x)}$

Explicit solution

At times it is impossible to try and solve an implicit solution for y explicitly in terms of x. The best we can do is to numerically generate the graph of the solution and determine an appropriate interval of definition.

For example,

1. Verify that the following function is an implicit solution of the given ODE. Find an appropriate interval I of definition of/for each solution.

$$x^2+y^2=9 \rightarrow (1) \qquad y'=-\frac{x}{y} \longrightarrow (2)$$

(Implicit solution) (ODE)

Taking the derivative of (1) implicitly, we obtain

$$2x+2yy'=0$$

or, $y'=-\dfrac{x}{y}$.

Verify L.H.S. = R.H.S. of (2)

$$-\frac{x}{y}=-\frac{x}{y} \checkmark$$

$\therefore$ The given function (1) is an implicit solution of (2).

Next to find appropriate interval I of this solution.

Implicit solution : $x^2+y^2=9$

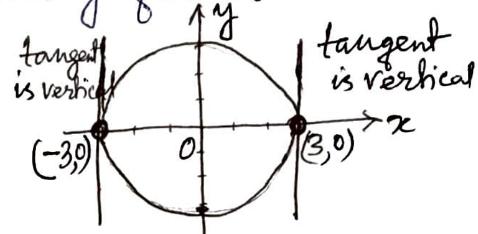
ODE : $y' = -\frac{x}{y}$.

Now at $(-3,0)$ and at $(3,0)$, y' does not exist.

Implicit function is not differentiable at $x=-3, x=3$.

$\therefore$ This solution is broken up into two semi circles.

It turns out that we can actually find an explicit solution to this ODE.



$y = \pm\sqrt{9-x^2}$
(Explicit solutions)

$y = \phi_1 = \sqrt{9-x^2}$; $y = \phi_2 = -\sqrt{9-x^2}$

Each explicit solution represents a segment of the implicit solution: ϕ_1 forms the top semi circle, ϕ_2 forms the bottom one

$x^2+\phi_1^2 = 9$; $x^2+\phi_2^2 = 9$.

Also $\phi_1(x)$, $\phi_2(x)$ satisfy the ODE.

$\boxed{I: \ -3 < x < 3}$

Note: Any explicit solution that we find must be a real-valued function.

For example, if we take $x^2+y^2=-9$ (implicit soln.) $y' = -\frac{x}{y}$. (ODE)

$y = \pm\sqrt{-9-x^2}$.
(Explicit soln.)

Explicit solutions are here not real-valued function so we can not draw a graph for these. But they satisfy the implicit relation and the ODE too.

So the implicit relation $x^2+y^2=-9$ satisfies the ODE $y' = -\frac{x}{y}$, but an interval I of definition can not be determined here, because the explicit solutions are not real-valued function.

We say then that the implicit relation is a formal solution to the ODE.

Implicit relation is not a solution to the ODE since no interval of definition cannot be defined.

A solution of the form $y=\phi(x)$ is called an explicit solution of an ODE $F(x, y, y', \cdots, y^{(n)}) = 0$ on the interval I.

Fundamental Theorem:

A DE of order n has n, and cannot have more than n independent first integrals. Therefore, the equation cannot have more than n arbitrary and independent constants in the g.s.

Geometrical Interpretations of the Solution of a DE:

Geometrically,

1. the g.s. is the equation of a family of curves;
2. the particular solution is the equation of a particular curve of this family;
3. the s.s. (if exists) is the envelope of this family.

Family of Curves represented by ODE:

1. A first order DE: $f(x, y, \frac{dy}{dx}) = 0 \longrightarrow ①$

   g.s. $\longrightarrow$ $\phi(x, y, c) = 0 \longrightarrow ②$

   c is a parameter and for different values of c, we get different curves.

   g.s. ② represents one-parameter family of curves.

2. An n-th order and degree 1 DE:

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \cdots, y^{(n-1)}).$$

Its g.s. represents an n-parameter family of curves

Existence and Uniqueness Theorem:-

A first order and first degree DE:

$$\frac{dy}{dx} = f(x,y) \rightarrow ① \quad \text{for which if}$$

(i) $f(x,y)$ is continuous in a domain R containing the point (x_0, y_0), then there exists a function $y=\phi(x)$ satisfying the DE ① and $y_0 = \phi(x_0)$, representing a solution.

(ii) $\frac{\partial f}{\partial y}$ is continuous at the point (x_0, y_0) in R, the solution $y=\phi(x)$ of ① is unique.

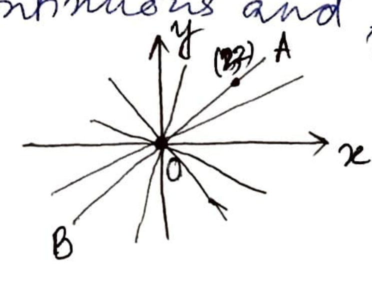
Geometrically, the theorem asserts the existence of a unique integral curve of the DE ① passing through (x_0, y_0).

Note: The condition $y = y_0$ for $x = x_0$ is known as the initial condition.

Examples:

① Consider the DE: $\frac{dy}{dx} = \frac{y}{x}$.

The g.s. is given by $y = mx$ (m being an arbitrary constant)
An infinite set of particular solution exists satisfying ~~Any~~ the initial condition $y=0$ when $x=0$.
Any straight line through the origin is an integral curve.
The uniqueness property does not hold here, because $f(x,y) = y/x$ is not continuous and $\frac{\partial f}{\partial y}$ does not exist at $x=0$.



② Consider the DE: $\frac{dy}{dx} = -\frac{x}{y}$.

The g.s. is given by $x^2+y^2=c^2$: family of concentric circles with centre at the origin.
(c^2 being an arbitrary constant)
No particular solution exists satisfying the initial condition $y=0$ when $x=0$.
Non-existence of a particular solution at $(0,0)$, because of the discontinuity of $f(x,y)$ and $\frac{\partial f}{\partial y}$ at $(0,0)$.