MATH GE-1 / LINEAR D.E.

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Lecture # 5

General form of Linear DE of first order is:

$$\frac{dy}{dx} + P(x)\cdot y = Q(x) \longrightarrow (1),$$

[where $P(x), Q(x)$ are either functions of x or constants or zeros.]

In the D.E. (1), the dependent variable y and its derivative $\frac{dy}{dx}$ occur in the first degree only, hence it is linear.

The D.E. (1) can be written in the differential form as:

$$[P.y - Q]\,dx + 1.\,dy = 0 \longrightarrow (2), \text{ where } P \equiv P(x), Q \equiv Q(x).$$

To make the D.E. (2) exact, we multiply it by an I.F. $R(x) (\equiv R)$. Then

$[R.P.y - R.Q]\,dx + R.\,dy = 0$ will be exact if

$\frac{\partial}{\partial y}[R.P.y - R.Q] = \frac{\partial}{\partial x}(R)$; using the condition for exactness.

or, $R.P = \frac{dR}{dx}$, since P, Q, R are functions of x alone.

or, $\frac{dR}{R} = P\,dx \xrightarrow{\text{integrating}} \log R = \int P\,dx \Rightarrow R = e^{\int P dx}$.

$\therefore$ $e^{\int P dx}$ is an I.F. of the L.D.E. (1).

Note: We can find the I.F. of (1) by using Rule-2.

From (2), we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{1}\left[\frac{\partial}{\partial y}(P.y - Q) - \frac{\partial}{\partial x}(1)\right] = P$$

$= P(x)$ which is a function of x alone.

$\therefore$ I.F. of (1) is given by

$e^{\int P dx}$; using Rule-2.

Method of solution of L.D.E. ①: $\frac{dy}{dx} + P.y = Q$.

Multiplying bothsides of ① by the I.F. $e^{\int P dx}$, we obtain;

$$\left(\frac{dy}{dx} + P.y\right)e^{\int P dx} = Q.e^{\int P dx}.$$

a, $\frac{dy}{dx}.e^{\int P dx} + P.y e^{\int P dx} = Q.e^{\int P dx}$

a, $\frac{d}{dx}\left[y.e^{\int P dx}\right] = Q.e^{\int P dx}$

Integrating, $y.e^{\int P dx} = \int\{Q.e^{\int P dx}\}dx + C$.

∴ g.s. is: $y = e^{-\int P dx}\left\{\int\{Q.e^{\int P dx}\}dx + C\right\}$.

Ex: Show that the g.s. of the L.D.E. $\frac{dy}{dx} + P.y = Q \rightarrow ①$ is of the form $y = K(u-v)+v$, K being a constant, u & v are two particular solutions.

Since u & v are two solutions of ①, we have

$\frac{du}{dx} + P.u = Q \longrightarrow ②$, $\frac{dv}{dx} + P.v = Q \longrightarrow ③$.

Subtracting ③ from ②, $\frac{du}{dx} - \frac{dv}{dx} + P(u-v) = 0$

$$\Rightarrow \frac{d(u-v)}{dx} + P(u-v) = 0$$

$$\Rightarrow \frac{d(u-v)}{u-v} = -P dx \longrightarrow ④.$$

Again subtracting ③ from ①,

$$\frac{dy}{dx} - \frac{dv}{dx} + P(y-v) = 0 \Rightarrow \frac{d}{dx}(y-v) = -P(y-v).$$

a, $\frac{d(y-v)}{y-v} = -P dx \longrightarrow ⑤$.

Comparing ④ & ⑤, we have,

$$\frac{d(y-v)}{y-v} = \frac{d(u-v)}{u-v} = -P dx.$$

Integrating, $\log(y-v) = \log(u-v) + \log K$

a, $y - v = K(u-v)$

a, $\underline{y = K(u-v) + v}$. This is the g.s. of ①.

where K being an arbitrary constant.

Problems on L.D.E.

① Solve: $\frac{dx}{dt}+\frac{1}{t}x=t^2 \longrightarrow ①$

It is a L.D.E. in x, I.F. $= e^{\int \frac{1}{t}dt} = e^{\log t} = t$.

Multiplying ① by t (I.F.), we get

$$\frac{dx}{dt}.t + x.\frac{1}{t}.t = t^3$$

$$\Rightarrow \frac{d}{dt}(t.x) = t^3, \text{ integrating, } t.x = \int t^3 dt + C$$

or, $t.x = \frac{t^4}{4} + C$

or, $\boxed{x = \frac{t^3}{4} + \frac{C}{t}} \leftarrow$ g.s.

where C being an arbitrary constant.

② Solve: $\frac{dy}{dx} + y\cot x = 2\cos x \longrightarrow ①$.

It is a L.D.E. in y, I.F. $= e^{\int \cot x\, dx} = e^{\int \frac{\cos x}{\sin x}dx}$

$= e^{\log|\sin x|} = \sin x$

Multiplying ① by $\sin x$ (I.F.), we get

$$\frac{dy}{dx}.\sin x + y\cot x.\sin x = 2\cos x.\sin x$$

or, $\frac{dy}{dx}\sin x + y\cos x = \sin 2x$

or, $\frac{d}{dx}(y\sin x) = \sin 2x$

Integrating, $y\sin x = \int \sin 2x\, dx + C = -\frac{\cos 2x}{2} + C$

or, $y = -\frac{\operatorname{cosec} x.\cos 2x}{2} + C\operatorname{cosec} x \leftarrow$ g.s.

③ Solve: $(1-x^2)\frac{dy}{dx} - xy = 1. \longrightarrow ①$.

$\frac{dy}{dx} + \frac{x}{x^2-1}.y = \frac{1}{1-x^2}. \longrightarrow ②$ It is a L.D.E. in y.

I.F. $= e^{\int \frac{x}{x^2-1}dx} = e^{\frac{1}{2}\int \frac{2x}{x^2-1}dx} = e^{\frac{1}{2}\log(x^2-1)} = \sqrt{x^2-1}$.

Multiplying ② by $\sqrt{x^2-1}$ (= I.F.), we get

$$\sqrt{x^2-1}\,\frac{dy}{dx} + \frac{x}{\sqrt{x^2-1}}.y = \frac{-1}{\sqrt{x^2-1}}.$$

$\Rightarrow \frac{d}{dx}\{\sqrt{x^2-1}.y\} = -\frac{1}{\sqrt{x^2-1}}$; Integrating, we get

$y.\sqrt{x^2-1} = C + \int \frac{(-1)dx}{\sqrt{x^2-1}} = C - \log|x+\sqrt{x^2-1}|$; This is the g.s..

④ Find $f(x)$, if $f'(x)+f(x)=0$; $f(0)=2$.
⑤ Find $f(x)$, if $f'(x)=xf(x)$; $f(0)=1$.

④ $f'(x)+f(x)=0$; $f(0)=2$

$\Rightarrow \frac{f'(x)}{f(x)} = -1$, integrating we get

$\int \frac{f'(x)}{f(x)}dx = -\int 1.dx \Rightarrow \log f(x) = -x+\log c$

$\Rightarrow f(x) = c.e^{-x}$ ← g.s. c is an arbitrary constant.

Now $f(0)=2 \Rightarrow 2 = c.e^0 \Rightarrow c=2$.

$\therefore$ $\underline{f(x) = 2.e^{-x}}$ ← Particular Soln.

⑤ $f'(x) = x f(x)$; $f(0)=1$.

$\Rightarrow \frac{f'(x)}{f(x)} = x \Rightarrow \int \frac{f'(x)}{f(x)}dx = \int x\,dx$.

$\Rightarrow \log f(x) = \frac{x^2}{2} + \log c$

$\Rightarrow f(x) = ce^{x^2/2}$. ← g.s.

$f(0)=1 \Rightarrow 1 = c.e^0 \Rightarrow c=1$.

$\therefore$ $\underline{f(x) = e^{x^2/2}}$, ← Particular Soln.

⑥ Solve: $(x+2y^3)\frac{dy}{dx} = y$. ⟶ ①.

a, $\frac{dx}{dy} = \frac{x}{y} + 2y^2$, or, $\frac{dx}{dy} - \frac{1}{y}.x = 2y^2$. (L.D.E. in x) ⟶ ②

I.F. $= e^{-\int \frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$.

Multiplying ② by the I.F. $(=\frac{1}{y})$, we get

$\frac{1}{y}.\frac{dx}{dy} - \frac{1}{y^2}.x = 2y$.

a, $\frac{d}{dy}\left(\frac{x}{y}\right) = 2y$; or, $d\left(\frac{x}{y}\right) = 2y\,dy$.

Integrating, $\int d\left(\frac{x}{y}\right) = \int 2y\,dy + c$, c being a constant

$\Rightarrow \frac{x}{y} = y^2 + c \Rightarrow \underline{x = y^3 + cy}$ ← g.s.

⑦ Solve: $(x+y+1)\frac{dy}{dx} = 1$ $\boxed{\text{Do yourself}}$

Ans: $x+y+2 = ce^y$ ← g.s.

$x = cy + y^2$.

Equations reducible to linear form:

Bernoulli's eqn: $\frac{dy}{dx} + P.y = Q.y^n$

where P & Q are functions of x or constants.

Dividing bothsides by y^n, we have

$$y^{-n}\frac{dy}{dx} + P.y^{1-n} = Q. \longrightarrow (1)$$

Let us substitute $y^{1-n} = v$.

So that $(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$.

Eqn (1) $\Rightarrow$ $\frac{dv}{dx} + (1-n).P.v = Q.(1-n)$

which is a l.d.e. in v.

eg. Solve: $\frac{dy}{dx}(x^2y^3 + xy) = 1$

$\Rightarrow \frac{dx}{dy} = x^2y^3 + xy$.

$\Rightarrow \frac{dx}{dy} - y.x = x^2y^3$, which is a Bernoulli's eqn.

$\Rightarrow x^{-2}\frac{dx}{dy} - y.\frac{1}{x} = y^3$.

Put $\left(-\frac{1}{x}\right) = v \Rightarrow \frac{1}{x^2}\frac{dx}{dy} = \frac{dv}{dy}$.

$\therefore \frac{dv}{dy} + y.v = y^3$ which is a l.d.e. in v.

$\therefore$ I.F. $= e^{\int y\,dy} = e^{\frac{1}{2}y^2}$

$\therefore v.e^{\frac{1}{2}y^2} = c + \int y^3.e^{\frac{1}{2}y^2}dy$ [put $\frac{1}{2}y^2 = t$]

$= 2e^t(t-1) + c$.

Soln: $\frac{1}{x} = (2 - y^2) - ce^{-\frac{1}{2}y^2}$.

32(b) $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$. It is a Bernoulli's Eqn.

Dividing by $y(\log y)^2$, we get.

$\frac{1}{y(\log y)^2}\cdot\frac{dy}{dx}+\frac{1}{\log y}\cdot\frac{1}{x}=\frac{1}{x^2}$. Let us put $\frac{1}{\log y}=z$.

or, $-\frac{1}{(\log y)^2}\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\frac{dz}{dx}$

or, $-\frac{dz}{dx}+\frac{1}{x}\cdot z=\frac{1}{x^2}$

or, $\frac{dz}{dx}-\frac{1}{x}\cdot z=-\frac{1}{x^2}$. L.D.E. in z

I.F. = $e^{-\int\frac{dx}{x}}=e^{-\log x}=\frac{1}{x}$.

G.S. ⇒ $z\cdot\frac{1}{x}=c-\int\frac{1}{x^3}dx=c+\frac{1}{2x^2}$.

or, $\frac{1}{x\log y}=c+\frac{1}{2x^2}$.

or, $2x=\log y(1+2cx^2)$.

30(a) $xy-\frac{dy}{dx}=e^{-x^2}y^3$ ⇒ $\frac{dy}{dx}-xy=-e^{-x^2}y^3$

Dividing by y^3 ⇒ $y^{-3}\frac{dy}{dx}-xy^{-2}=-e^{-x^2}$.

Let us put, $-y^{-2}=z$ ⇒ $2y^{-3}\frac{dy}{dx}=\frac{dz}{dx}$

Then we have, $\frac{1}{2}\frac{dz}{dx}+x\cdot z=-e^{-x^2}$ L.D.E. in z.

or, $\frac{dz}{dx}+2x\cdot z=-2e^{-x^2}$.

I.F. = $e^{\int 2x\,dx}=e^{x^2}$

G.S. ⇒ $z\cdot e^{x^2}=c-2\int e^{x^2}\cdot e^{-x^2}dx=c-2x$.

or, $-y^{-2}e^{x^2}=c-2x$

or, $\frac{1}{y^2}=(2x-c)e^{-x^2}$.

31(a) $y(2xy+e^x)dx-e^x dy=0$.

$e^x\frac{dy}{dx}-2xy^2-e^xy=0$.

or, $\frac{dy}{dx}-2xy^2e^{-x}-y=0$ ⇒ $\frac{dy}{dx}-y=2xe^{-x}\cdot y^2$.

Dividing by y^2; $y^{-2}\frac{dy}{dx}-y^{-1}=2xe^{-x}$. put: $-y^{-1}=z$

⇒ $\frac{dz}{dx}+z=2xe^{-x}$. (l.d.e. in z). I.F. = e^x.

G.S. → $z\cdot e^x=c+\int 2xe^{-x}\cdot e^x dx=c+x^2$

⇒ $-\frac{1}{y}e^x=c+x^2$ ⇒ $x^2y+e^x+cy=0$

① Solve: $y^2 + (x - 1/y)\frac{dy}{dx} = 0$. It is nonlinear in y.

Let us try to see whether it is linear in x or not.

$$\frac{dx}{dy} = -\left(\frac{xy-1}{y^3}\right) \Rightarrow \frac{dx}{dy} + \frac{1}{y^2}x = 1/y^3 \quad \text{(Linear D.E. in } x)$$

I.F. $= e^{\int \frac{1}{y^2}dy} = e^{-1/y}$. Multiplying by the I.F.

$$\frac{d}{dy}(x.e^{-1/y}) = e^{-1/y}.\frac{1}{y^3}; \text{ Integrating,}$$

$$xe^{-1/y} = e^{-1/y}\left(1+\frac{1}{y}\right) + c, \text{ or, } x = 1 + \frac{1}{y} + ce^{1/y}.$$

② Reduce the D.E. $\sin y \frac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$ to a linear D.E. and hence solve it.

Let us put $\cos y = z$, in order to make it linear

$$\frac{dz}{dx} + (2\cos x)z = \sin^2 x \cos x. \quad \text{(L.D.E. in } z).$$

I.F. $= e^{\int 2\cos x\,dx} = e^{2\sin x}$.

Multiplying by the I.F. and then integrating

$$\frac{d}{dx}(z.e^{2\sin x}) = e^{2\sin x}.\sin^2 x\cos x$$

$$z.e^{2\sin x} = \int e^{2\sin x}.\sin^2 x \cos x\,dx + c. \left[\text{put } u = \sin x\right]$$

or,

G.S.: $\cos y = \frac{1}{2}\sin^2 x - \frac{1}{2}\sin x + \frac{1}{4} + ce^{-2\sin x}$.

③. Solve: $(x^2y^3 + 2xy)dy = dx$.

$$\frac{dx}{dy} - (2y)x = y^3x^2 \quad \text{(Bernoulli's Eqn.)}.$$

Dividing by x^2, $x^{-2}\frac{dx}{dy} - 2y.x^{-1} = y^3$.

Let us put $x^{-1} = z$, $-\frac{dz}{dy} - 2yz = y^3$

or, $\frac{dz}{dy} + 2yz = -y^3$ [Linear D.E. in z]

I.F. $= e^{\int 2y\,dy} = e^{y^2}$.

Multiplying by the I.F., $\frac{d}{dy}(e^{y^2}z) = -y^3.e^{y^2}$.

Integrating $\Rightarrow$ $ze^{y^2} = -\int y^3e^{y^2} + c = -\frac{1}{2}e^{y^2}(y^2-1) + c$ [put $y^2 = u$]

or, $\frac{1}{x} = \frac{1}{2}(1-y^2) + ce^{-y^2}$.

Ex. A curve passes through the point (0, 1) and the gradient at (x, y) on it is $y(xy-1)$. Find the eqn of the curve.

We have $\frac{dy}{dx} = y(xy-1) = xy^2 - y$

a, $\frac{dy}{dx} + y = xy^2$; It is a Bernoulli's eqn.

Dividing both sides by y^2,

$y^{-2}\frac{dy}{dx} + \frac{1}{y} = x$; put $\frac{1}{y} = v \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow -\frac{dv}{dx} + v = x \Rightarrow \frac{dv}{dx} - v = -x$ which is l.d.e. in v.

$\therefore$ I.F. $= e^{-\int dx} = e^{-x}$.

$v.e^{-x} = c + \int(-x)e^{-x}dx = xe^{-x} - e^{-x} + c$

$\Rightarrow v - (x-1) = c.e^{x}$

$\Rightarrow \frac{1}{y} - (x-1) = c.e^{x}$. put, $x = 0, y = 1$.

$\Rightarrow 1 + 1 = c \Rightarrow c = 2$

$\therefore \frac{1}{y} - (x-1) = 2e^{x}$.

Equation $f'(y)\frac{dy}{dx} + P.f(y) = Q$, where P and Q are functions of x or constants.

Put $f(y) = v$.

$\therefore \frac{dv}{dx} + P.v = Q$.

eg. Solve! $\sec^2 y\frac{dy}{dx} + (\tan y).2x = x^3$

Put $\tan y = v \Rightarrow \sec^2 y\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + v.2x = x^3$; which is l.d.e. in v.

I.F. $= e^{\int 2x dx} = e^{x^2}$

$\therefore v.e^{x^2} = c + \int x^3 e^{x^2}dx$; put $x^2 = t$.

$= c + \frac{1}{2}e^t(t-1)$

$\tan y.e^{x^2} = c + \frac{1}{2}e^{x^2}(x^2-1)$

35 (d) Find the particular solutions for given values of x and y:
$\sin x \frac{dy}{dx} + 4y\cos x = \cot x$; $x = \pi/2$, $y = 4/3$,

~~$\cot - \cos x = z$, $\Rightarrow + \sin x \frac{dx}{dy} = \frac{dz}{dy}$~~

~~$\therefore \frac{dz}{dx} - 4yz =$~~

~~$\sin x + 4y\cos x \cdot \frac{dx}{dy} = \cot x \cdot \frac{dx}{dy}$~~

$\frac{dy}{dx} + 4y\cot x = \frac{\cot x}{\sin x}$. It is a l.d.e. in y.

I.F. $= e^{\int 4\cot x\,dx} = e^{4\log\sin x} = \sin^4 x$.

g.s. $\Rightarrow$ $y.\sin^4 x = c + \int \frac{\cot x}{\sin x}.\sin^4 x\,dx = c + \int \cos x.\sin^2 x\,dx$.

a, $y\sin^4 x = c + \int \sin^2 x\, d(\sin x) = c + \frac{\sin^3 x}{3}$.

a, $3y\sin^4 x - \sin^3 x = 3c \cdot \Rightarrow \sin^3 x(3y\sin x - 1) = 3c$.

when $x = \pi/2$, $y = 4/3$ $\therefore$ $3c = \sin^3 \pi/2(3 \cdot 4/3 \sin \pi/2 - 1)$

a, $3c = 3 \Rightarrow c = 1$.

$\therefore$ Particular Soln is given by; $y\sin^4 x = \frac{1}{3}\sin^3 x + 1$.

30 (b) Reduce $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ into linear one and hence solve it,

Dividing bothsides by $\sec y$, we get.

$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$. [Let us put $\sin y = z$ a, $\cos y \frac{dy}{dx} = \frac{dz}{dx}$]

a, $\frac{dz}{dx} - \frac{z}{1+x} = (1+x)e^x$ (l.d.e. in z)

I.F. $= e^{-\int \frac{dx}{1+x}} = e^{-\log(1+x)} = \frac{1}{1+x}$

g.s. $\rightarrow$ $z \cdot \frac{1}{1+x} = c + \int \frac{(1+x)e^x}{1+x}dx = c + \int e^x dx = c + e^x$.

a, $\sin y = (1+x)(c + e^x)$.

34 If $\frac{dr}{d\theta} + 2r\tan\theta = \sin\theta$ and $r = 0$ when $\theta = \pi/3$, Show that the max. value of r is $1/8$.

It is a l.d.e. in r.

I.F. $= e^{\int 2\tan\theta\, d\theta} = e^{-2\log\cos\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$.

g.s. $\rightarrow$ $r.\sec^2\theta = c + \int \sin\theta.\sec^2\theta\, d\theta = c + \int \sec\theta\tan\theta\, d\theta$

a, $r\sec^2\theta = c + \sec\theta$. $r = 0$, $\theta = \pi/3$ gives $c = -\sec\pi/3$ $\therefore c = -2$.

$\therefore$ $r\sec^2\theta = \sec\theta - 2$;

or, $r = \cos\theta - 2\cos^2\theta$. $\frac{dr}{d\theta} = -\sin\theta + 4\sin\theta\cos\theta$

For max. value of r, $\frac{dr}{d\theta} = 0$

~~$\Rightarrow r = \frac{\sin\theta}{2\tan\theta} = \frac{1}{2}\cos\theta$~~

$\Rightarrow \sin\theta(4\cos\theta - 1) = 0$, either $\sin\theta = 0$, $\theta = 0$ or, $\cos\theta = 1/4$, $\theta = \cos^{-1}1/4$.

$\frac{d^2r}{d\theta^2} = -\cos\theta + 4\cos 2\theta = -1/4 + 4(2\cos^2\theta - 1)$
$= -1/4 + 4(2 \cdot \frac{1}{16} - 1) < 0$.
$\therefore$ r is max. when $\cos\theta = 1/4$.
$r_{max} = 1/4 - 2 \cdot \frac{1}{16} = 1/8$. (proved).