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Problems on Bernoulli's equation [continued...]

① Solve: $\frac{dy}{dx} + \frac{1}{x}y = x^2y^6$. [Bernoulli's eqn.].

Dividing bothsides by y^6, we obtain

$y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x^2$. Let us substitute: $y^{-5} = z$, then $-5y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$.

$\Rightarrow -\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2$

$\Rightarrow \frac{dz}{dx} - \frac{5}{x}.z = -5x^2$. [L.D.E. in z] $\longrightarrow$ ①

I.F. $= e^{-\int\frac{5}{x}dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5}$.

Multiplying ① by the I.F. x^{-5}, we get

$x^{-5}.\frac{dz}{dx} - \frac{5}{x}.z.x^{-5} = -5x^2.x^{-5}$

a, $x^{-5}.\frac{dz}{dx} - 5x^{-6}.z = -5x^{-3}$.

a, $\frac{d}{dx}(x^{-5}.z) = -5x^{-3}$; Integration gives

$\int d(x^{-5}.z) = c + \int -5x^{-3}dx$

$\Rightarrow z.x^{-5} = c + \frac{5}{2}x^{-2} \Rightarrow y^{-5}.x^{-5} = c + \frac{5}{2}.x^{-2}$.

$\therefore$ The complete primitive/G.S. is given by

$y^{-5} = cx^5 + \frac{5}{2}x^3$. (Ans.)

② Solve: $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$. [Bernoulli's Eqn.]

Dividing bothsides by $\sqrt{y}$, we obtain

$y^{-1/2}\frac{dy}{dx} + \frac{x}{1-x^2}.\sqrt{y} = x$; Let us put $\sqrt{y} = z$

$\Rightarrow \frac{1}{2}y^{-1/2}\frac{dy}{dx} = \frac{dz}{dx}$.

$\therefore 2\frac{dz}{dx} + \frac{x}{1-x^2}.z = x$

$\Rightarrow \frac{dz}{dx} + \frac{x}{2(1-x^2)}.z = \frac{x}{2}$ [l.d.e. in z] $\longrightarrow$ ①; I.F. $= e^{\int\frac{x}{2(1-x^2)}dx}$

To evaluate $\int\frac{x}{2(1-x^2)}dx$, let us put $1-x^2 = t$

a, $-2x\,dx = dt$

$= -\int\frac{1}{4}\frac{dt}{t} = -\frac{1}{4}\log t = -\frac{1}{4}\log(1-x^2)$.

$\therefore$ I.F. $= e^{\int \frac{x}{2(1-x^2)}dx} = e^{-\frac{1}{4}\log(1-x^2)} = \frac{1}{(1-x^2)^{1/4}}$.

Multiplying ① by the I.F. $\left(= \frac{1}{(1-x^2)^{1/4}}\right)$ and then integrating bothsides, we obtain

$$z\cdot(1-x^2)^{-1/4} = c + \int \frac{x}{2(1-x^2)^{1/4}}dx$$

$$= c + \int \frac{-du}{4\cdot u^{1/4}} = c - \frac{1}{4}\cdot\frac{u^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} \quad \left[\begin{array}{l}\text{Put } 1-x^2 = u\\ \therefore x\,dx = -\frac{du}{2}\end{array}\right]$$

or, $\sqrt{y}\cdot(1-x^2)^{-1/4} = c - \frac{1}{3}u^{3/4} = c - \frac{1}{3}(1-x^2)^{3/4}$.

or, $\sqrt{y} = c(1-x^2)^{1/4} - \frac{1}{3}(1-x^2)$. This is the g.s.

③ Solve: $x^2y - x^3\frac{dy}{dx} = y^4\cos x$. [Bernoulli's Eqn.].

Transposing, $\frac{dy}{dx} - \frac{1}{x}y = -\frac{1}{x^3}\cos x\cdot y^4$

Dividing by y^4, we get: $y^{-4}\frac{dy}{dx} - \frac{1}{x}y^{-3} = -\frac{1}{x^3}\cos x$.

Let us put $y^{-3} = z \Rightarrow -3y^{-4}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore -\frac{1}{3}\frac{dz}{dx} - \frac{1}{x}\cdot z = -\frac{1}{x^3}\cos x$

or, $\frac{dz}{dx} + \frac{3}{x}\cdot z = \frac{3}{x^3}\cos x$ [l.d.e. in z] $\longrightarrow$ ①

I.F. $= e^{\int\frac{3}{x}dx} = e^{3\log x} = x^3$.

Multiplying ① by x^3 (= I.F.) and then integrating,

$$z\cdot x^3 = c + \int \frac{3}{x^3}\cos x\cdot x^3\,dx = c + 3\sin x$$

or, $y^{-3}\cdot x^3 = c + 3\sin x \Rightarrow x^3 = y^3(c + 3\sin x)$. g.S.

where c is an arbitrary constant.

Nonlinear D.E. of First order and higher degree:

General form: $A_0 p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots + A_{n-1} p + A_n = 0$;
where $A_0, A_1, \cdots, A_n$ are functions of x & y; $p \equiv \frac{dy}{dx}$.

There is no general method for solving this type of DE as such. However there exist certain techniques for solving this type of DE:

1. Equation solvable for x
2. " " " y
3. " " " p

① Eqn. solvable for x:

Let the DE be $f(x,y,p)=0$ and this can be put in the form: $x = \phi(y,p) \longrightarrow ①$.

Example:

(i) Solve: $x = py - p^2 \longrightarrow ①$

Differentiating ① w.r.t. y, we obtain

$$\frac{dx}{dy} = p + y\frac{dp}{dy} - 2p\frac{dp}{dy} \Rightarrow \frac{1}{p} = p + (y-2p)\frac{dp}{dy}.$$

$$\Rightarrow \left(\frac{1}{p} - p\right)\frac{dy}{dp} = y - 2p \Rightarrow \frac{dy}{dp} = \frac{y \cdot p}{1-p^2} - \frac{2p^2}{1-p^2}.$$

$$\Rightarrow \frac{dy}{dp} + \frac{p}{p^2-1}\cdot y = \frac{2p^2}{p^2-1} \cdot [\text{L.D.E. in } y] \longrightarrow ②$$

$$\text{I.F.} = e^{\int \frac{p}{p^2-1}dp} = e^{\frac{1}{2}\log|p^2-1|} = \sqrt{p^2-1}.$$

Multiplying ② by the I.F. and then integrating,

$$y \cdot \sqrt{p^2-1} = c + \int \frac{2p^2}{p^2-1}\cdot\sqrt{p^2-1}\,dp = c + \int \frac{2p^2}{\sqrt{p^2-1}}dp$$

$$= c + 2\int \frac{p^2-1+1}{\sqrt{p^2-1}}dp = c + 2\int\sqrt{p^2-1}\,dp + 2\int\frac{dp}{\sqrt{p^2-1}}.$$

$$= c + 2\cdot\left[\frac{p}{2}\sqrt{p^2-1} - \frac{1}{2}\cosh^{-1}p\right] + 2\cosh^{-1}p.$$

or, $y = p + (c + \cosh^{-1}p)(p^2-1)^{-1/2} \longrightarrow ③$

From ①, we get, $x = \frac{p}{\sqrt{p^2-1}}(c + \cosh^{-1}p) \longrightarrow ④$ [using ③]

③ & ④ together represent the g.s. in parametric form of the given DE ①.

(ii) Solve: $x + \dfrac{p}{\sqrt{1+p^2}} = a \longrightarrow$ ① (Eqn. not containing y).

Differentiating ① w.r.t. y, we get

$$\frac{dx}{dy} + \left[\frac{1}{\sqrt{1+p^2}} - \frac{p\cdot 2p}{2\sqrt[3]{1+p^2}}\right]\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{1}{p} + \left[\frac{(1+p^2) - p^2}{\sqrt[3]{1+p^2}}\right]\frac{dp}{dy} = 0 \Rightarrow \frac{dp}{dy} = -\frac{\sqrt[3]{1+p^2}}{p}.$$

$$\Rightarrow dy = -\frac{p\,dp}{\sqrt[3]{1+p^2}}$$; Integrating bothsides,

$$\int dy = c - \int \frac{p\,dp}{(1+p^2)\sqrt{1+p^2}}.$$ Let us put $1+p^2 = t^2$ or, $p\,dp = t\,dt$

$$\Rightarrow y = c - \int \frac{t\,dt}{(t^2)^{3/2}} = c - \int t^{-2}dt = c + \frac{1}{t} = c + \frac{1}{\sqrt{1+p^2}}$$

$$\therefore\ y = c + \frac{1}{\sqrt{1+p^2}} \longrightarrow ②.$$

From ①, we get $\dfrac{p^2}{1+p^2} = (x-a)^2. \Rightarrow 1 - \dfrac{p^2}{1+p^2} = 1-(x-a)^2$

$$\Rightarrow \frac{1}{1+p^2} = 1-(x-a)^2 \longrightarrow ③.$$

$\therefore$ Eliminating p between ① & ②, we get

$$y = c + \sqrt{1-(x-a)^2}\ ;\ \text{using ③}$$

or, $(y-c)^2 = 1-(x-a)^2 \Rightarrow \underline{(x-a)^2 + (y-c)^2 = 1}.$

This is the g.s. of ①, c being an arbitrary constant.

Do yourself:

(iii) Solve: $x = 4p + 4p^3$. g.s. $\rightarrow \begin{cases} x = 4p + 4p^3 \\ y = 2p^2 + 3p^4 + c. \end{cases}$

(iv) Solve: $y = 2px + y^2p^2$ $\begin{cases} \text{g.s.} \rightarrow x = \dfrac{y^2}{2c} - \dfrac{c^2}{2}. \\ \text{s.s.} \rightarrow 27y^4 + 32x^3 = 0. \end{cases}$

(v) Solve: $4yp^2 - 2px + y = 0$. g.s. $\rightarrow y^2 = 2cx - 4c^2$.

② Equations solvable for y:

The DE $f(x,y,p)=0$ can be put in the form:

$y=\phi(x,p)$.

Examples:

(i) Solve: $y=p^2x+p \longrightarrow ①$. $y=\phi(x,p)$.

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=(2px+1)\frac{dp}{dx}+p^2 \Rightarrow p-p^2=(2px+1)\frac{dp}{dx}$.

$\Rightarrow \frac{dx}{dp}+\frac{2p}{p(p-1)}\cdot x=\frac{1}{p(1-p)} \longrightarrow ②$ (It is a l.d.e. in x)

I.F. $= e^{\int \frac{2}{p-1}dp} = e^{2\log|p-1|} = (p-1)^2$.

Multiplying ② by I.F. $(p-1)^2$ and integrating, we get

$x\cdot(p-1)^2 = c+\int\frac{(p-1)^2}{p(1-p)}dp = c+\int\frac{1-p}{p}dp = c+\log p-p$

a, $x=(p-1)^{-2}[c+\log p-p] \longrightarrow ③$.

From ①, we get by using ③,

$y=p^2(p-1)^{-2}[c+\log p-p]+p \longrightarrow ④$

③ & ④ together constitute g.s. of ① in parametric form.

OR, Eliminating p between ① & ③, we get the g.s. of ①.

(ii) Solve: $y=2p+3p^2 \longrightarrow ①$. $y=\phi(p)$.

Differentiating bothsides of ① w.r.t. x, we get

$\frac{dy}{dx}=(2+6p)\frac{dp}{dx} \Rightarrow p=(2+6p)\frac{dp}{dx} \Rightarrow \frac{dx}{dp}=\frac{2+6p}{p}$

$\Rightarrow dx=\frac{2+6p}{p}dp$; Integrating bothsides, we get

$\int dx = c+\int\frac{2}{p}dp+\int 6\,dp \Rightarrow x=c+2\log p+6p \longrightarrow ②$

Eliminating p between ① & ② the g.s. of ① is obtained.

Differential Equation in CLAIRAUT'S FORM :

$y = px + f(p) \longrightarrow (1)$. Differentiation w.r.t. x gives—

$\cancel{p} = \frac{dy}{dx} = \cancel{p} + [x + f'(p)]\frac{dp}{dx} \Rightarrow [x + f'(p)]\frac{dp}{dx} = 0$

$\Rightarrow$ either $\frac{dp}{dx} = 0$; or, $x + f'(p) = 0$

$\Rightarrow p = c. \longrightarrow (2)$ $\qquad \Rightarrow x = -f'(p) \longrightarrow (3)$

From (1), $\underline{y = c.x + f(c)}$, [using (2)], which is the g.s.

From (1) & (3), $\underbrace{x = -f'(p)}_{\longrightarrow (4)}$; $y = -pf'(p) + f(p) \longrightarrow (5)$

Eliminating p between (4) & (5), Singular Solution ($\underline{\text{S.S.}}$) of (1) is obtained.

Geometrical interpretation;

(1) The g.s. $y = cx + f(c)$ represents the family of straight lines.

(2) The curve given by (4) & (5) represents the S.S. of the clairaut's eqn.

(3) The curve given by the S.S. touches every member of the family of straight lines given by the g.s., and also at each point of the curve (S.S.) it is touched by the some member of the family of lines (g.s.)

(4) The S.S. of the Clairaut's equation, is therefore, becomes the envelope of the family of straight lines given by the g.s.

Extended Form of Clairaut's Eqn.

$y = xf(p) + \phi(p) \longrightarrow (1)$ Lagrange's Equation.

Differentiating bothsides of (1) w.r.t. x, we get

$p = \frac{dy}{dx} = f(p) + [xf'(p) + \phi'(p)]\frac{dp}{dx} \Rightarrow \frac{dx}{dp} = \frac{xf'(p) + \phi'(p)}{p - f(p)}$

a, $\frac{dx}{dp} + x.\frac{f'(p)}{f(p) - p} = \frac{\phi'(p)}{p - f(p)}$. which is a l.d.e. in x.

Examples on Clairaut's Form

Obtain the complete primitive and singular solution;

(1) $y = px + p - p^2 \longrightarrow (1)$; Differentiating w.r.t. x,

$\cancel{p} = \frac{dy}{dx} = \cancel{p} + (x+1-2p)\frac{dp}{dx} \Rightarrow \frac{dp}{dx}(x+1-2p) = 0$

$\Rightarrow$ either $\frac{dp}{dx} = 0 \Rightarrow p = c \longrightarrow (2)$ [by integrating]

or $x+1-2p = 0 \Rightarrow x = 2p-1 \Rightarrow p = \frac{x+1}{2} \longrightarrow (3)$

Using (2), from (1) we get the g.s. of (1) as:

$y = cx + c - c^2$; c being an arbitrary constant

Using (3) in (1), we get the s.s. of (1) as follows;

$y = \frac{(x+1)x}{2} + \frac{x+1}{2} - \left(\frac{x+1}{2}\right)^2 = \frac{x+1}{4}\left[2x+2-x-1\right]$

or, $y = \frac{(x+1)^2}{4} \Rightarrow (x+1)^2 = 4y \leftarrow$ S.S.

(2) $y = px + \sqrt{1+p^2} \longrightarrow (1)$; Differentiating w.r.t. x,

$\cancel{p} = \frac{dy}{dx} = \cancel{p} + \left[x + \frac{1\times 2p}{2\sqrt{1+p^2}}\right]\frac{dp}{dx} \Rightarrow \left(x + \frac{p}{\sqrt{1+p^2}}\right)\frac{dp}{dx} = 0$

$\Rightarrow$ either $\frac{dp}{dx} = 0 \Rightarrow p = c \longrightarrow (2)$

or, $x + \frac{p}{\sqrt{1+p^2}} = 0 \Rightarrow x^2 = \frac{p^2}{1+p^2} \longrightarrow (3)$.

Using (2) in (1), we get the complete primitive:

$y = c.x + \sqrt{1+c^2}$; c being an arbitrary constant

Using (3) in (1), we get, $y = -\frac{p^2}{\sqrt{1+p^2}} + \sqrt{1+p^2} = \frac{-p^2+1+p^2}{\sqrt{1+p^2}}$

or, $y^2 = \frac{1}{1+p^2} \longrightarrow (4)$; Adding (3) & (4), we get

$x^2 + y^2 = \frac{p^2}{1+p^2} + \frac{1}{1+p^2} = 1$, i.e.,

$x^2 + y^2 = 1$ is the s.s. of (1).

Lagrange's Equation:

Example:

① Solve: $y = p^2 x + p^3 \longrightarrow (1)$ [Lagrange's eqn]

Differentiating bothsides of (1) w.r.t. x, we get:

$$p = \frac{dy}{dx} = p^2 + (2px + 3p^2)\frac{dp}{dx}$$

$$\Rightarrow \frac{dx}{dp} = \frac{2px + 3p^2}{p - p^2} = \frac{2x + 3p}{1 - p} \Rightarrow \underbrace{\frac{dx}{dp} + \frac{2x}{p-1} = \frac{3p}{1-p}}_{\longrightarrow (2)} \text{ (l.d.e. in } x\text{)}$$

$$I.F. = e^{\int \frac{2}{p-1} dp} = e^{2\log|p-1|} = (p-1)^2.$$

Multiplying the l.d.e. (2) by the I.F. $(p-1)^2$,

$$\frac{dx}{dp} \cdot (p-1)^2 + \frac{2x}{p-1} \cdot (p-1)^2 = \frac{3p \cdot (p-1)^2}{1-p}$$

or, $\frac{dx}{dp} \cdot (p-1)^2 + 2x(p-1) = 3p(1-p)$

or, $\frac{d}{dp}\left[x(p-1)^2\right] = 3p(1-p)$.

Integrating bothsides, we get

$$x(p-1)^2 = c + \int 3p(1-p)\, dp$$

$$= c + \frac{3p^2}{2} - p^3.$$

or, $x = \frac{2c + 3p^2 - 2p^3}{2(p-1)^2} \longrightarrow (3)$.

Eliminating p between (1) & (3), the g.s. of (1) will be obtained.