# Fourier Series

## Periodic Functions:

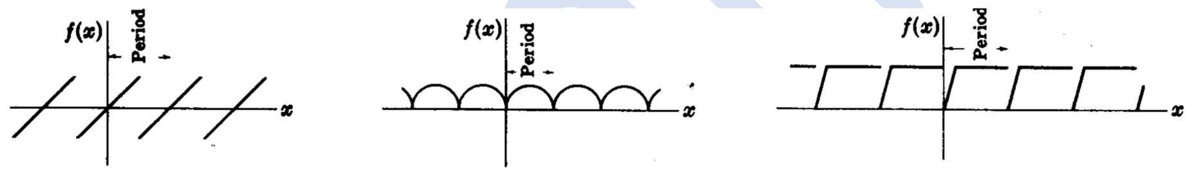
A function f(x) is said to have a period T or to be periodic with period T if for all x, f(x + T)= f(x), where T is a positive constant. The least value of T> 0 is called the least period or simply the period of f(x).

Ex 1.
The function sin x has periods 2π, 4π, 6π, ..., since sin (x2π), sin (x4π), sin (x6π), ... all equal sin x. However, 2π is the least period or the period of sin x.

Ex 2.
The period of sin nx or cos nx, where n is a positive integer, is 2π /n.

Ex 3.
The period of tan x is π.

Ex 4.
A constant has any positive number as a period.




(a) (b) (c)

If the value of each ordinate f(x) repeats itself at equal intervals in the abscissa, then f(x) is said to be a periodic function.

## Orthogonality of sine and cosine functions:

A set of functions $\{\varphi_1(x), \varphi_2(x), \ldots.\}$ is an orthogonal set of functions in the interval [a,b] if any two functions in the set are orthogonal to each other.
i.e. $(\varphi_m(x), \varphi_n(x))=\int_a^b \varphi_m(x)\,\varphi_n(x)dx = 0 \; for\; m \neq n$

**# Show that {1, cosx, sinx, cos2x, sin2x, ….} form an orthogonal set in (-π, π).**

The following integrals are useful in Fourier Series:

1. $\int_0^{2\pi} \sin mx \sin nx\, dx = \begin{cases} 0, \; if\; m \neq n \\ \pi, \; if\; m = n \\ 0, \; if\; m = n = 0 \end{cases}$
2. $\int_0^{2\pi} \cos mx \cos nx\, dx = \begin{cases} 0, \; if\; m \neq n \\ \pi, \; if\; m = n \\ 2\pi, \; if\; m = n = 0 \end{cases}$
3. $\int_0^{2\pi} \sin mx \cos nx\, dx = 0, for\; any\; integer\; values\; of\; m\; and\; n$

**Dirichlet Conditions (Statement only):**

If a function is periodic or non-periodic and satisfies some specific conditions then it can be expanded by an infinite series of sine and cosine terms, as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{------------------- (1)}$$

This infinite series is called Fourier Series. Where $a_0$, $a_n$, $b_n$ are called Fourier Coefficients or Fourier Constants or Euler's formulae.

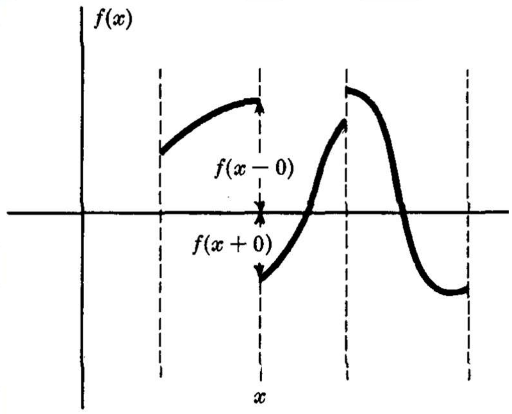
The specific conditions, mentioned above are known as Dirichlet's conditions. These conditions are as follows:

1. The function f(x) is periodic, single valued and bounded
2. The function f(x) should has finite number of discontinuities in one period (or f(x) should be piecewise continuous).
3. The function f(x) should has finite number of extrema (i.e. maxima and minima) in a period.

These conditions are sufficient conditions for expansion of a function in Fourier Series.

**#Note:**

A function f(x) is said to be piecewise continuous in an interval if
(i) the interval can be divided into a finite number of subintervals in each of which f(x) is continuous and
(ii) the limits of f(x) as x approaches the endpoints of each subinterval are finite.
Another way of stating this is to say that a piecewise continuous function is one that has at most a finite number of finite discontinuities. An example of a piecewise continuous function is shown in the figure below.




**Expansion of periodic functions in a series of sine and cosine functions:**

Now we know that, if f(x) satisfies Dirichlet's conditions and exist in the interval (-π, π), then function f(x) can be expanded as infinite Fourier Series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_n \cos nx + \cdots)$$
$$+(b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots + b_n \sin nx + \cdots)$$

Where Euler's formulae can be expressed as

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \,,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \text{ and}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

# **Note** that

$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$ signifies the average value of function f(x) in the interval (-π, π).

**Determination of Fourier coefficients:**

We have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ --------------------(1)

1) To determine the value of Fourier Coefficient $a_0$, let multiply both sides of equation (1) by $dx$ and then taking average over 2π as

   $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_n \cos nx\, dx + \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} b_n \sin nx\, dx$

   Second and third term of RHS in the above equation become zero (due to orthogonality relations as derived earlier).

   Therefore, $\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx = \frac{a_0}{2} \times 2\pi = \pi a_0$

   And $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$----------------------(2)

2) To determine the value of Fourier Coefficient $a_n$, let multiply equation (1) by $cos(nx)dx$ and then taking average over 2π as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a_0}{2}\cos nx\, dx + \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_n \cos nx \cos nx\, dx$$
$$+ \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} b_n \sin nx \cos nx\, dx$$

   First and third term of RHS in the above equation become zero (due to orthogonality relations as derived earlier).

   Therefore, $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_n \cos nx \cos nx\, dx$

   $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} a_n \cos nx \cos nx\, dx$

   $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{2\pi} a_n \times \pi = \frac{a_n}{2}$

   Therefore, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$------------------------(3)

3) To determine the value of Fourier Coefficient $b_n$, let multiply equation (1) by $sin(nx)dx$ and then taking average over 2π as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{a_0}{2}\sin nx\, dx + \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_n \cos nx \sin nx\, dx$$
$$+ \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} b_n \sin nx \sin nx\, dx$$

   First and second term of RHS in the above equation become zero (due to orthogonality relations as derived earlier).

   Therefore, $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} b_n \sin nx \sin nx\, dx$

   $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} b_n \sin nx \sin nx\, dx$

   $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{2\pi} b_n \times \pi = \frac{b_n}{2}$

   Therefore, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$----------------------(4)

Q.1. Expand the function $f(x) = \begin{cases} 0 \; for -\pi < x < 0 \\ 1 \quad for\ 0 < x < \pi \end{cases}$ in Fourier series.

**Even and odd functions and their Fourier expansions:**

1. A function is said to be even function (symmetric with respect to y-axis or function axis) if, $f(-x) = f(x)$. Here function-axis (y axis) is mirror for the reflection of the curve.

The area under such a curve from -π to π is double the area from 0 to π.

i.e. $\int_{-\pi}^{\pi} f(x)dx = 2\int_0^{\pi} f(x)dx$

Therefore,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{\pi}^{0} f(-x)\sin(-nx)\,d(-x) + \int_0^{\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{\pi}^{0} f(x)\sin(-nx)\,d(-x) + \int_0^{\pi} f(x)\sin nx\,dx\right\}\ as\ f(-x) = f(x)$$

$$= \frac{1}{\pi}\left\{\int_{\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\int_0^{\pi} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right\} = 0$$

So for an even function we have to calculate only $a_0\ and\ a_n$.

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)dx + \int_0^{\pi} f(x)dx\right\}$

$$= \frac{1}{\pi}\left\{\int_{\pi}^{0} f(-x)d(-x) + \int_0^{\pi} f(x)dx\right\} = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)dx + \int_0^{\pi} f(x)dx\right\} = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

Similarly, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx$ , since *f*(*x*) and cos(*nx*) are both even functions, therefore product *f*(*x*)cos(*nx*) is also an even function.

2. A function is said to be odd function (symmetric with respect to origin or skew-symmetric) if, $f(-x) = -f(x)$.

   Here area under such a curve from -π to π is zero , i.e. $\int_{-\pi}^{\pi} f(x)dx = 0$

   Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$ , since f(x) is odd function

   $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$ , as product of odd and even functions is odd

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$ , as product of two odd functions is even function.

Q.2. Expand the following functions in Fourier series:

i) $f(x) = sinx,\ \ for -\pi < x < \pi$

ii) $f(x) = cos2x,\ \ for -\pi < x < \pi$

iii) $f(x) = x^2,\ \ for -\pi < x < \pi$

iv) $f(x) = |x|,\ \ for -\pi < x < \pi$