What is Bose Einstein Condensate?

It is a state of matter which is formed by cooling a gas of bosons (to temperatures very close to absolute zero). When boson atoms are cooled down, they slow down and their energies decreases, because of their quantum nature, the atoms behave as waves that increase in size as temperature decreases. At very low temperature, the size of the waves becomes larger than the average distance between two atoms. Then, at this very low temperature, all of the bosons are able to be at the very same energy in the same quantum state. They all form a single collective quantum wave called a Bose- Einstein condensate.

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BE Condensation vs. Gas-to-Liquid Condensation

Classical physics analogy: let's fill a container with a non-ideal gas and start lowering the temperature. The gas density remains constant until the condensation of gas (vapor) occurs. Since the density of liquid is much higher than that of gas, the gas density decreases with T.

Despite this similarity, the Bose-Einstein condensation is an ***entirely different phenomenon***.

One essential difference is that in the gas-to-liquid transition, which is due to interparticle attraction, the liquid and gas phases occupy different regions of space. The BE condensation is driven by exchange interactions. Each particle in the BE condensate has a wave function that fills the ***entire volume*** of the container.

The BE condensation is ***the condensation in the momentum space.***
The phase separation occurs in the k-space, not in the coordinate space. The condensed bosons have essentially zero momentum (i.e., their wave vector k is as small as the size of the container permits).

A common misconception about BE condensation is that it requires "brute force" cooling, when $k_B T$ becomes much smaller than the differences of energies of the quantum states of the system - that would be a trivial effect, though the temperatures would have been ***inaccessibly low*** for any macroscopic system. The point is that condensation can happen at ***much higher temperatures***, when $k_B T$ is still large compared to the inter-level $\Delta\varepsilon$.

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BOSE - EINSTEIN CONDENSATION

We know that the average number of bosons in the energy state ϵ_s is given by

$$\langle n_s \rangle = \frac{1}{e^{\alpha+\beta\epsilon_s}-1}$$

since $\sum \langle n_s \rangle = N$

$$\therefore N = \sum_s \frac{1}{e^{\alpha+\beta\epsilon_s}-1}$$

$$= \sum_s \frac{1}{e^{\beta(\epsilon_s-\mu)}-1} \quad \text{—— (1)} \qquad \text{for B.E, } \alpha = -\mu/kT = -\mu\beta$$

or replacing the summation by integration

$$N = \frac{4\pi V}{h^3}\int_0^{\infty} \frac{1}{e^{\beta(\epsilon_s-\mu)}-1}\, p^2 dp \quad \text{—— (2)}$$

We know, BE distribution goes over to MB distribution when $e^{-\beta\mu} \gg 1$

Therefore

$$\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\left(\frac{V}{N}\right) \gg 1$$

This condition is satisfied when T is high. Therefore, at high tempr μ is large and negative. with the decrease in tempr, the magnitude of μ also decreases. At what tempr will μ be equal to zero? To answer this question we put $\mu=0$ in eqn (2) and denote the corresponding tempr by T_b.

Therefore
$$N = \frac{4\pi V}{h^3}\int_0^{\infty} \frac{1}{e^{p^2/2mkT_b}-1}\, p^2 dp$$

Let $x = p^2/2mkT_b$ and we get

$$N = (2\pi V)\left(\frac{2mkT_b}{h^2}\right)^{3/2}\int_0^{\infty}\frac{x^{1/2}\,dx}{e^x-1}$$

$$= (2\pi V)\left(\frac{2mkT_b}{h^2}\right)^{3/2}\int_0^{\infty}\frac{x^{3/2-1}}{e^x-1}\,dx$$

$$= (2\pi V)\left(\frac{2mkT_b}{h^2}\right)^{3/2}\cdot \Gamma(3/2)\cdot \zeta(3/2)$$

$$= 2\pi V\cdot\left(\frac{2mkT_b}{h^2}\right)^{3/2}\cdot \frac{\sqrt{\pi}}{2}\times 2.612$$

or, $$N = V\left(\frac{2\pi mkT_b}{h^2}\right)^{3/2}\cdot 2.612 \quad \text{—— (3)}$$

or, $$\boxed{T_b = \frac{h^2}{2\pi mk}\left(\frac{N}{2.612\,V}\right)^{2/3}} \quad \text{—— (4)}$$

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T_b, therefore, is a characteristic tempr that depends on particle mass m and particle density N/V and is known as Bose tempr.

What will happen if the tempr is further reduced? μ cannot be positive nor can it become negative again. The only possibility is for μ to remain equal to zero after it has attained zero value.

A difficulty creeps in at this stage. The first term i.e. the ground state term with $\varepsilon_1 = 0$ – in the summation of eqn ① tends to infinity if $\mu = 0$. The reason for this contradiction is the following: for a boson gas, there is no limitation on the number of particles that can belong to a single state, and, yet, the ground state was given zero weight as is evident from the fact that while replacing summation by integration, the density of states which it contained is proportional to $V d^3p$ i.e. to $V\sqrt{\varepsilon}\, d\varepsilon$ and which vanishes at $\varepsilon = 0$

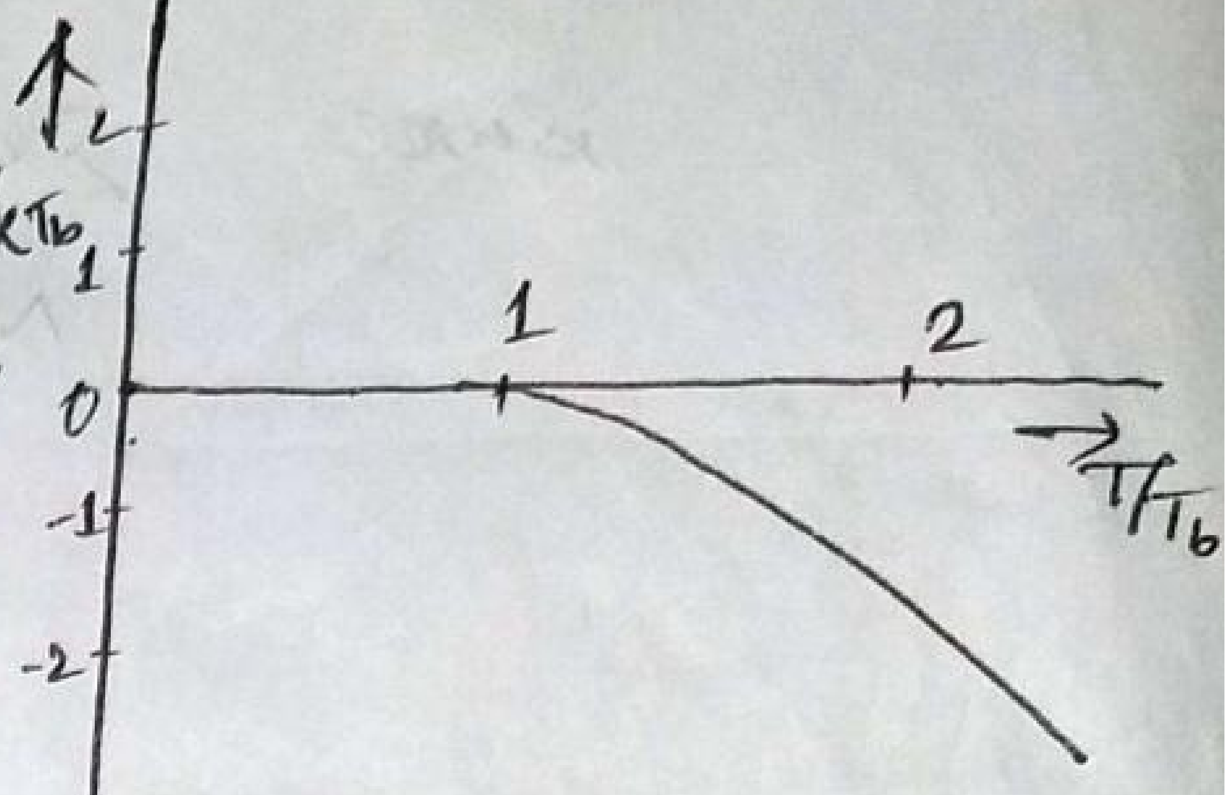

Fig-1
(the variation of μ of the ideal Boson gas.)

To remove this contradiction, we isolate the population of the ground state N in a separate term and write

$$N = N_0 + \frac{4\pi V}{h^3}\int_0^\infty \frac{p^2\,dp}{e^{p^2/2mkT}-1} \qquad (\mu = 0)$$

The integral is for excited states $\varepsilon = 0$. The lower limit can still be taken as zero, since is does not contribute towards the integral.

Integrating as before

$$N = N_0 + V\cdot\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\cdot 2.612$$

substituting for v from eqn ③

$$N = N_0 + \left(\frac{2\pi mkT}{h^2}\right)^{3/2}\cdot 2.612 \times \frac{N}{\left(\frac{2\pi mkT_0}{h^2}\right)^{3/2}\cdot 2.612}$$

$$= N_0 + N\left(T/T_0\right)^{3/2}$$

$$\text{or,}\quad N_0 = N\left\{1 - \left(T/T_b\right)^{3/2}\right\} \qquad \text{⑤}$$

At $T \to 0$, $N_0 \to N$, all bosons at $T = 0$ are in the ground state. This macroscopic occupation of the zero momentum ground state is called Bose-Einstein condensation of bosons. It may be noted that we are ~~referring~~ referring here to the 'condensation in the momentum space, and not to the actual condensation in the gas. As T increases the number of particles in the ground state decreases, the remaining particles being distributed into other states.

for $T < T_0$ the system may be regarded as a mixture of two phases: ① a gaseous phase with $N' = N(T/T_0)^{3/2}$ particles distributed over the excited states, i.e. state for which $\varepsilon \neq 0$ and ⓘⓘ a condensed phase with $N_0 = N - N'$ particles in the ground state with $\varepsilon = 0$. for $T > T_0$ the system will be in the gaseous phase alone

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for $T<T_b$. since the particles in the ground state ($\varepsilon=0$) do not contribute to the energy, the total energy is determined by the particles in excited state $\varepsilon>0$

Therefore
$$E = N'\varepsilon = \frac{4\pi V}{h^3}\int_0^\infty \frac{p^2}{2m}\cdot\frac{p^2 dp}{e^{p^2/2mkT}-1}$$
$$= \frac{4\pi V}{2mh^3}\int_0^\infty \frac{p^4 dp}{e^{p^2/2mkT}-1}$$

$$\text{or, } E = \frac{\pi V}{mh^3}\int_0^\infty (2mkT)^{5/2}\frac{x^{3/2}}{e^x-1}dx \quad \left\langle \because x=\frac{p^2}{2mkT}\right)$$

$$\text{or, } E = \frac{\pi V}{mh^3}(2mkT)^{5/2}\int_0^\infty \frac{x^{3/2}dx}{e^x-1}$$
$$= \frac{\pi V}{mh^3}(2mkT)^{5/2}\cdot\Gamma(5/2)\,\zeta(5/2)$$
$$= \frac{\pi V}{mh^3}(2mkT)^{5/2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\times 1.341$$
$$= \frac{3V}{4m}\cdot(\pi/h^2)^{3/2}\cdot(2mkT)^{5/2}\times 1.341 \quad \text{—— (6)}$$

Substituting for v from eqn (3)
$$E = \frac{3}{4m}(\pi/h^2)^{3/2}\cdot(2mkT)^{5/2}\cdot N\left(\frac{h^2}{2\pi mkT_b}\right)^{3/2}\times\frac{1.341}{2.612}$$
$$\text{or, } E = 0.77\,NK\frac{T^{5/2}}{T_b^{3/2}} \quad \text{—— (7)}$$

The heat capacity of the ideal boson gas can be determined from the equation
$$C_v = \left(\frac{\partial E}{\partial T}\right)_v = 1.93\,NK\frac{T^{3/2}}{T_b^{3/2}} \quad \text{—— (8)}$$

We see from eqn (8) that at the condensation tempr $T=T_b$ the heat capacity exceeds the classical value $\frac{3}{2}NK$.

$$\text{Now, } PV = \frac{2}{3}E = \frac{2}{3}\times\frac{3V}{4m}(\pi/h^2)^{3/2}\cdot(2mkT)^{5/2}\times 1.341 \quad \langle \text{using eqn (6)} \rangle$$

$$\text{or, } P(T) = \frac{1}{2m}(\pi/h^2)^{3/2}\cdot(2mkT)^{5/2}\times 1.341$$

Substituting for m from eqn (3)

$$P(T) = (\pi/h^2)^{3/2}\cdot(kT)^{5/2}\cdot\left(\frac{N}{V}\right)\cdot\left(\frac{h^2}{\pi kT_b}\right)^{3/2}\cdot\frac{1.341}{2.612}$$
$$P(T) = \frac{1.341}{2.612}\,\frac{kT^{5/2}}{T_b^{3/2}}\cdot(N/V) \quad \text{—— (9)}$$

Therefore pressure at the transition point i.e. at $T=T_b$
$$P(T_b) = 0.5134\left(\frac{N}{V}kT_b\right) \quad \text{—— (10)}$$

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Thus the pressure of an ideal Bose gas at the transition tempr T_b is about one-half of that of an equivalent Boltzmann gas. From eqn (9) and (10)

$$\frac{P(T)}{P(T_b)} = \left(T/T_b\right)^{5/2}$$

Therefore $P(T) = P(T_b)\left(T/T_b\right)^{5/2}$

$$= 0.5134\left(\frac{NKT_b}{V}\right)\left(T/T_b\right)^{5/2}$$

$$= 0.5134\left\{N\left(T/T_b\right)^{3/2}\right\}\frac{KT}{V}$$

or, $P(T) = 0.5134\,\frac{N'KT}{V} \quad (T \leq T_b)$ —— (11)

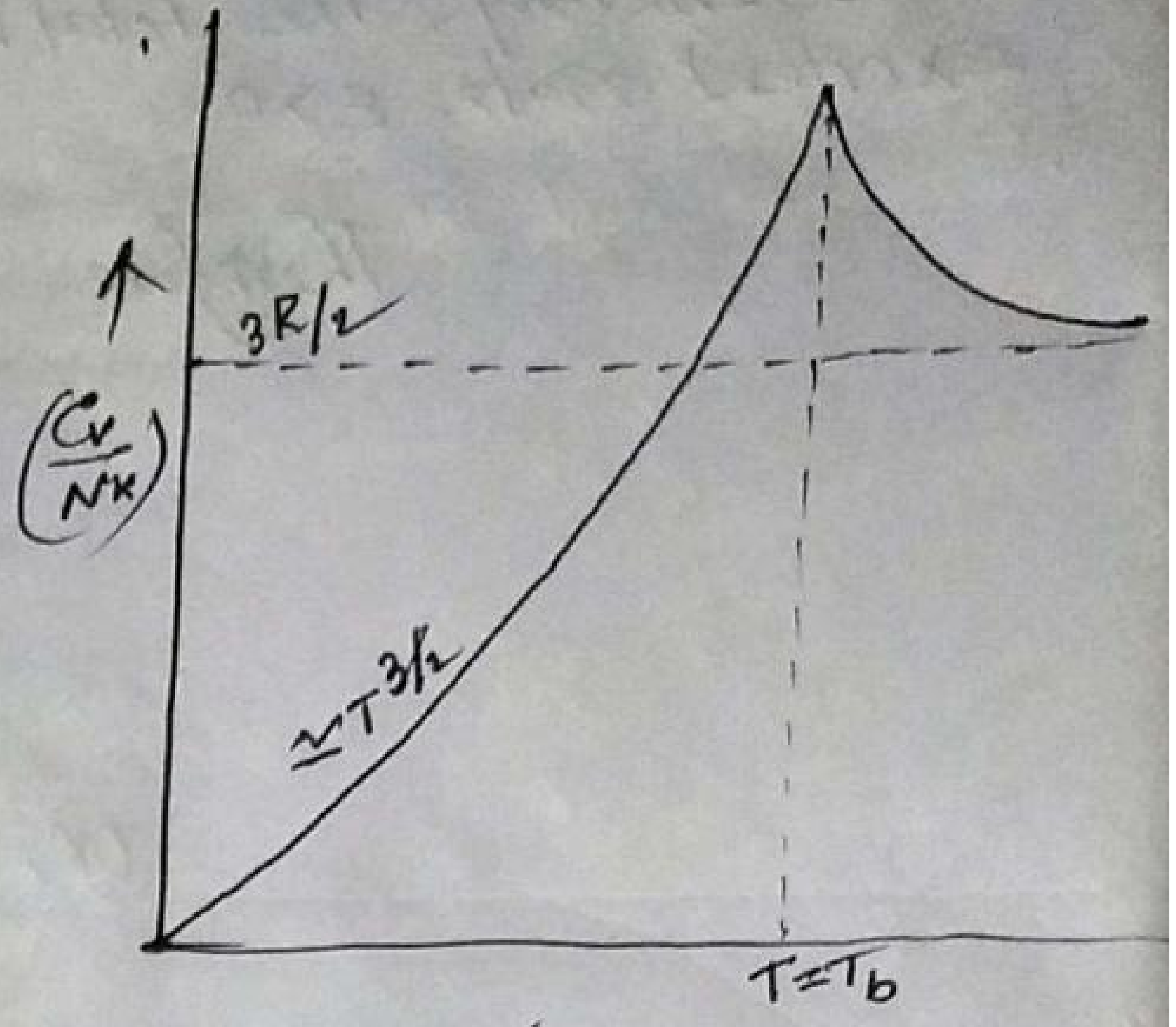

fig-2 $(T/T_b) \longrightarrow$

(specific heat of ideal Bose gas)

We conclude, therefore, that the particles in the condensed phase do not exert any pressure, but the pressure exerted by the particles in the excited state is about half of the pressure that would have been exerted if the gas were Boltzmannian.

Entropy of the boson gas is most expeditiously obtained from the relation

$$E = TS - pV + \mu N$$

$$= TS - pV \quad (\mu = 0)$$

or, $S = \frac{E + pV}{T} = \frac{5}{2}\frac{pV}{T} \quad \left(pV = \frac{2}{3}E\right)$

$$S = \frac{5}{2} \times 0.5134\, NK\left(T/T_b\right)^{3/2} \quad \langle \text{using eqn (11)} \rangle$$

or, $(S)_{T<T_b} = \frac{5}{2} \times .5134 \times \frac{C_V}{1.93}$ (using eqn (8))

$$\boxed{(S)_{T<T_b} = 0.665\,(C_V)_{T<T_b}}$$

from specific heat curve C_V shows a sudden drop for the tempr below T_b and consequently, entropy will also decrease suddenly. A decrease in entropy means decrease in disorder or increase in order. We have seen earlier that at $T<T_b$, a large number of particles condense into the ground state ($\varepsilon=0$) which attributed to zero entropy (entropy, $k\log_e 1 = 0$, since statistical weight of ground state is one). Since $\varepsilon = p^2/2m$, $\varepsilon = 0$, implies $p=0$ and therefore we can say equivalently, that condensed particles condense in the momentum space.

- Explain Bose-Einstein condensation. How does it differ from ordinary condensation ? Derive an expression for the critical temperature at which this phenomenon sets in.

- Show that the molar specific heat of a strongly degenerate Bose gas is given as

$$C_V = 1.92\,R\left(\frac{T}{T_C}\right)^{3/2}$$

Represent it graphically. .

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LIQUID HELIUM :→

As an application of Bose Einstein statistics, we may investigate the qualitative nature of the superfluid transition of liquid helium at 2.2K. Ordinary He consists almost entirely of neutral atoms of the isotope $_2He^4$. As the total angular momentum of these atoms is zero, their discussion must fall under the jurisdiction of Bose Einstein statistics.

Helium exhibits peculiar properties at low tempr. It is found that

(i) helium gas at atmospheric pressure condenses at 4.2K (its critical tempr being 5.2K) into a liquid of very low density, about 0.124 gm/cm^3.

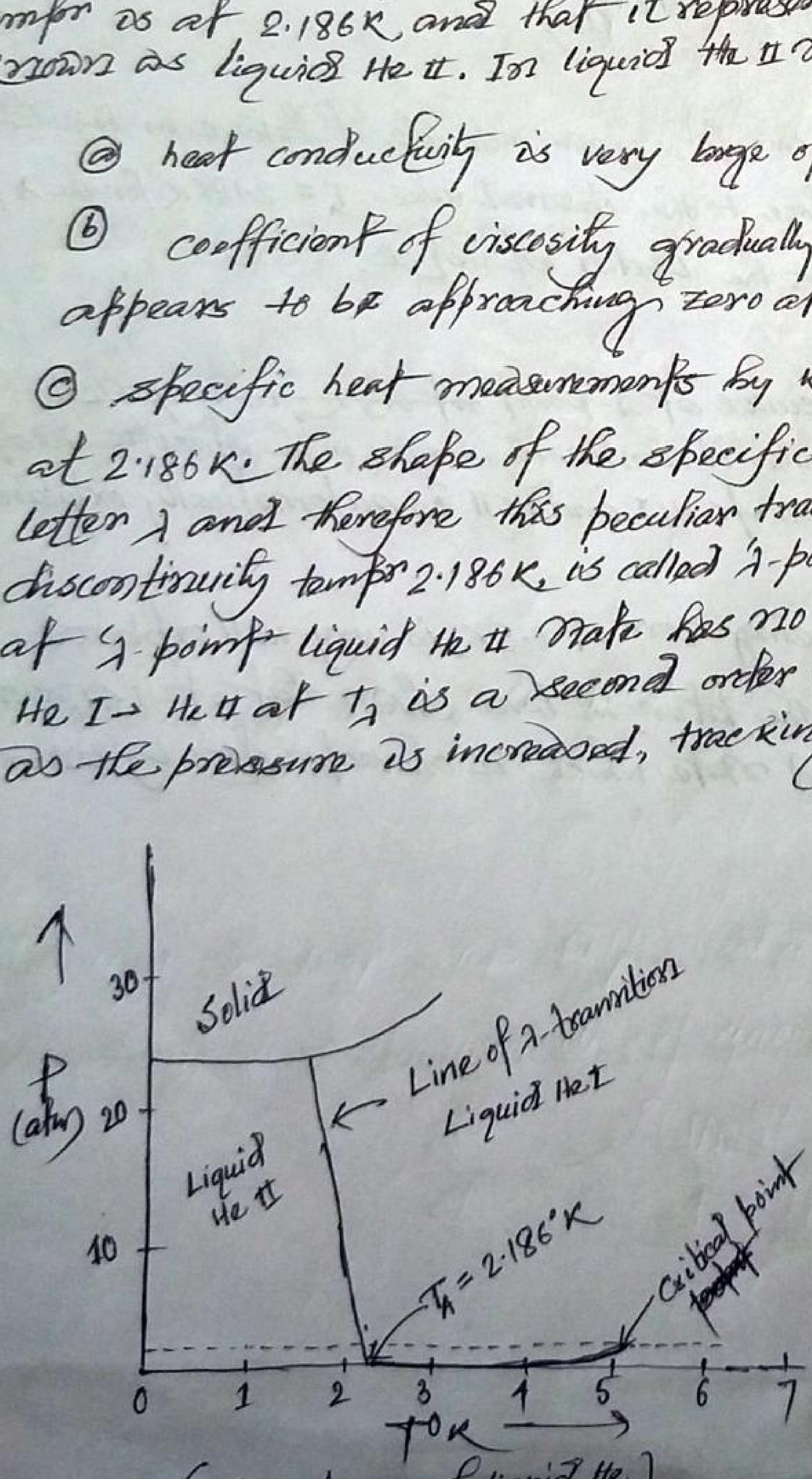
(ii) further cooling to about 0.82K does not freeze it and it is believed that it remains liquid all the way down to absolute zero. The solid state of helium does not form unless it is subjected to an external pressure of atleast 23 atmosphere.

(iii) for He^4 in liquid phase, there is another phase transition, called λ-transition, which divides the liquid state into two phase He I and II. Onnes, while liquefying helium, noted that about 2.2K, density appeared to pass through an abrupt maximum and then decreasing slightly thereafter. Investigations also revealed that critical tempr is at 2.186K and that it represents a transition to a new state of matter known as liquid He II. In liquid He II state, it was found that,

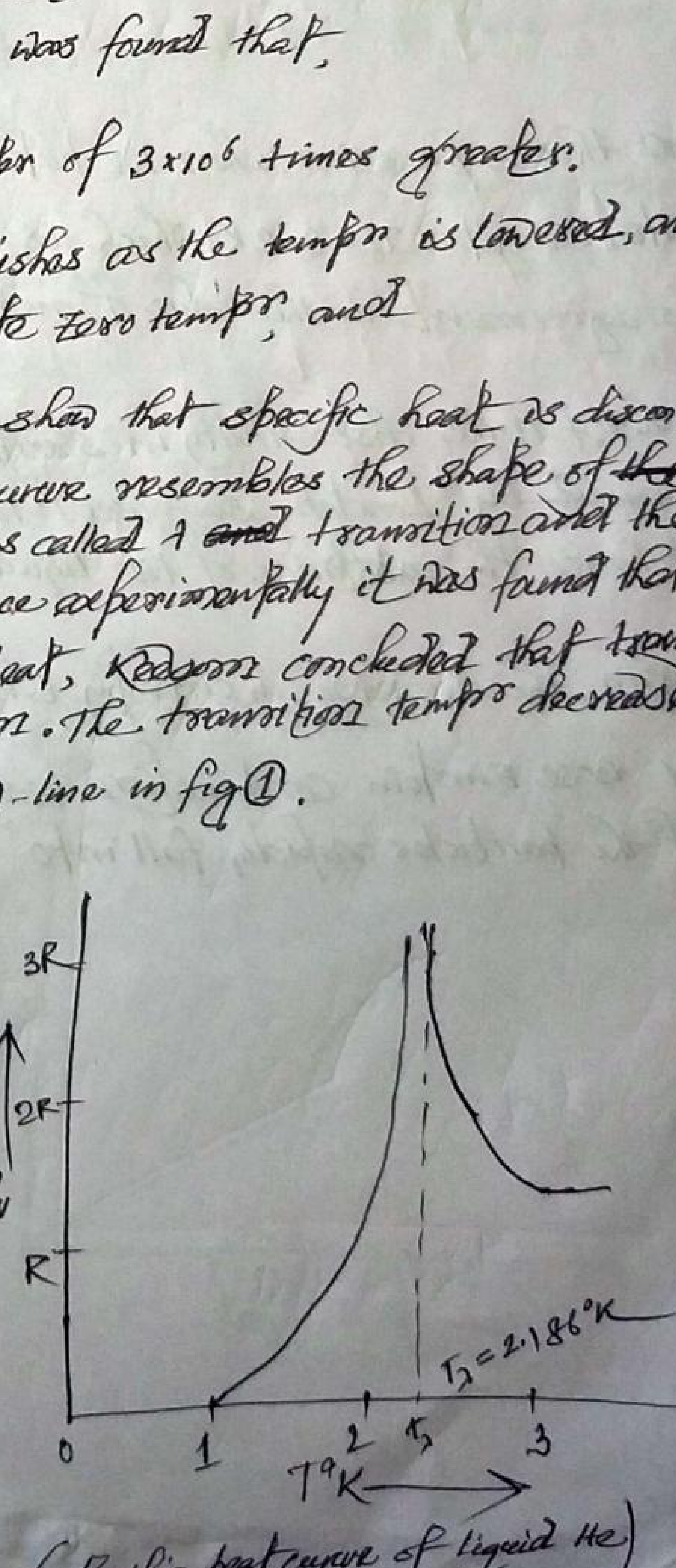
ⓐ heat conductivity is very large of the order of 3×10^6 times greater.

ⓑ coefficient of viscosity gradually diminishes as the tempr is lowered, and appears to be approaching zero at absolute zero tempr, and

ⓒ specific heat measurements by Keesom show that specific heat is discontinuous at 2.186K. The shape of the specific heat curve resembles the shape of ~~the~~ letter λ and therefore this peculiar transition is called λ ~~and~~ transition and the discontinuity tempr 2.186K is called 'λ-point'. Since experimentally it was found that at λ-point liquid He II state has no latent heat, Keesom concluded that transition He I → He II at T_λ is a second order transition. The transition tempr decreases as the pressure is increased, tracking out λ-line in fig①.



(phase diagram of liquid He)



(specific heat curve of liquid He)

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Explanation based on Bose Einstein condensation model: London's Theory:

Explanation of these peculiarities of liquid He at low tempr. based on B-E statistics, was given by London who suggested that He II as a liquid analogous to B-E gas and that λ- transition in liquid helium is the counter part of Bose Einstein condensation in the ideal gas. In Bose Einstein gas, degeneracy is

$$\frac{1}{D} = \frac{n}{g_s V}\left(\frac{2\pi m K T}{h^2}\right)^{3/2}$$

London suggested that helium atoms are light enough and though the density (n/V) of the liquid is sufficiently high for the right hand side to be large and degeneracy to be well marked but is low enough for the liquid to behave as a gas. He concluded this λ- transition as a result of Bose Einstein condensation and gave an analogy betn λ point and Bose Einstein condensation tempr T_0 defined by equn

$$g_s\left(\frac{2\pi m K T_0}{h^2}\right)^{3/2} V = \frac{n}{f_{3/2}(0)} = \frac{n}{2.612}$$

$$\text{or,} \quad T_0 = \frac{h^2}{2\pi m K}\left(\frac{n}{2.612 V g_s}\right)^{2/3}$$

In this expression, when we put $v = 27.4\ cm^3$ for a gram molecule of helium in liquid state, we get $T_0 = 3.12K$ which is quite close to the observed value $T_\lambda = 2.186K$ for the λ point. This agreement in the value T_0 and T_λ favours the London explanation.

Further, for the discontinuity in specific heat curve at λ-point ~~at 0.5K, is very well explained by~~ London suggested that similar discontinuity occurs in the specific heat and hence the existance of two liquid components He I and He II is automatically explained.

Again, the decrease in entropy below T_λ, being zero at 0.5K is very well explained by Bose Einstein condensation because, in the latter, we have shown that at $T<T_0$ most of the particles rapidly fall into the ground state which is characteristics by zero energy.

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