KHARAGPUR COLLEGE

DEPARTMENT OF MATHEMATICS

STUDY MATERIALS

SUBJECT: MATHEMATICS HONOURS

CLASS: B. Sc. Hons.

SEMESTER: 5 TH

PAPER: C12T [GROUP THEORY-II]

UNIT: I, II, III & IV

Dr. Sangita Chakraborty

Associate Professor
Department of Mathematics
Kharagpur College

Email:
sangita@kharagpurcollege.ac.in

19/08/2020. RECAP LECTURE NOTE – 1 Schanrabty
5th sem. GROUP THEORY-2.

Homomorphism → A map $\phi : (G, \circ) \to (G', *)$ s.t. $\phi(a \circ b) = \phi(a) * \phi(b)$, $\forall a, b \in G$.

① It preserves the algebraic structure of the system.

② A one-to-one homomorph. is called monomorphism
An onto " " " epi "
A one-to-one & onto homomorph. " " iso ".

③ A homomorph. $\phi(a) = e_{G'}$, $\forall a \in G$, is called Trivial homom.
⇒ there always exists a homomorph. from a group to a group.

④ Let $\phi : (G, \circ) \to (G', *)$ be a homomorphism. Then
(i) $\phi(e_G) = e_{G'}$ (ii) $\phi(a^{-1}) = \{\phi(a)\}^{-1}$, $\forall a \in G$.
(iii) $\phi(a^n) = \{\phi(a)\}^n$, $n \in \mathbb{Z}$; $a \in G$
(iv) $o(\phi(a)) \mid o(a)$, if $o(a)$ is finite; $a \in G$.

⑤ Homomorphic image of $\phi = \mathrm{Im}\,\phi = \phi(G) = \{\phi(a) : a \in G\}$.
$\phi(G) \leq G'$.

⑥ If ϕ is an isomorphism, then
(i) G ableian $\Leftrightarrow$ G' also abelian / $\phi(G)$ abelian.
$\not\Leftarrow$ if ϕ is epimorph.
(ii) G cyclic $\Leftrightarrow$ G' " cyclic / $\phi(G)$ cyclic.
$\not\Leftarrow$ if ϕ is epimorph.
if $G = \langle a \rangle \Rightarrow G'$ or $\phi(G) = \langle \phi(a) \rangle$.

⑦ $\mathrm{Ker}\,\phi = \{a \in G : \phi(a) = e_{G'}\} \subseteq G$.
(i) $\mathrm{Ker}\,\phi \trianglelefteq G$. [i.e, $\mathrm{Ker}\,\phi$ is a normal subgroup of G]
(ii) ϕ is monomorph. $\Leftrightarrow \mathrm{Ker}\,\phi = \{e_G\}$.
(iii) if ϕ is epimorph, then ϕ is isomorph. $\Leftrightarrow \mathrm{Ker}\,\phi = \{e_G\}$.

⑧ If ϕ is an isomorphism, then
(i) $o(a) = o(\phi(a))$, $\forall a \in G$. [isomorph. preserves the order]
(ii) G & G' have the same cardinality.

9. ϕ isomorph $\Rightarrow \phi^{-1}$ also isomorph.
10. $G \simeq G' \Rightarrow G' \simeq G$.
11. ϕ & ψ isomorph $\Rightarrow \psi \circ \phi$ also isomorph...
12. Two finite cyclic ~~and~~ groups of the same order are isomorphic.
13. Two infinite cyclic groups are isomorphic.
14. A finite cyclic group of order $n \simeq (Z_n, +_n)$.
15. Isomorphism Theorem:-
    $\phi : G \to G'$ be an onto homomorphism and $H = \ker\phi$. Then $G/H \simeq G'$.

NOTE. An isomorphism preserves
(i) the commutative property of groups,
(ii) the cyclic " " ",
(iii) the order of the elements of groups.

Automorphism →

- An isomorphism from a group G onto itself is called an automorphism.
- The set of all automorphisms of G is denoted by $Aut(G)$.
- Theorem → $Aut(G)$ forms a group under the mapping composition.

PROOF → Let $\alpha, \beta, \gamma \in Aut(G)$. Then the maps $\alpha : G \to G$, $\beta : G \to G$, $\gamma : G \to G$ are all bijections.

$\alpha o \beta : G \to G$ is then a bijection.

$\Rightarrow \alpha o \beta$ is an automorphism, so $\alpha o \beta \in Aut(G)$.

$\therefore$ $Aut(G)$ is closed w.r.t. the composition of maps.

We have, $\alpha o (\beta o \gamma) = (\alpha o \beta) o \gamma$, as composition is associative.

$\therefore$ Associative property holds in $Aut(G)$.

The identity map $I_G : G \to G$ defined by

$I_G(a \otimes b) = a \otimes b$, $\forall a, b \in G$.

$= I_G(a) \otimes I_G(b)$. $\Rightarrow I_G$ is a homomorphism.

Also we have I_G is a bijection, and hence I_G is an automorphism $\in Aut(G)$.

For any $\alpha \in Aut(G)$, $\{I_G o \alpha\}(a) = I_G\{\alpha(a)\} = \alpha(a)$, $\forall a \in G$.

Also, $\{\alpha o I_G\}(a) = \alpha\{I_G(a)\} = \alpha(a)$, $\forall a \in G$.

$\Rightarrow I_G o \alpha = \alpha o I_G$, $\forall \alpha \in Aut(G)$.

$\therefore$ I_G is the identity element in $Aut(G)$.

For any $\alpha \in Aut(G)$, $\alpha : G \to G$ being a bijection, $\alpha^{-1} : G \to G$ exists and also a bijection.

Let $a, b, x, y \in G$ s.t. $\alpha(a) = x$, $\alpha(b) = y$; $\alpha^{-1}(x) = a$, $\alpha^{-1}(y) = b$.

NOTE: $\otimes$ is the b.c. of the group G

②

$$\alpha\circ\alpha^{-1}(x)=\alpha\{\alpha^{-1}(x)\}=\alpha(a)=x.\Rightarrow \alpha\circ\alpha^{-1}=I_G$$
$$\alpha^{-1}\circ\alpha(a)=\alpha^{-1}(x)=a.\Rightarrow \alpha^{-1}\circ\alpha=I_G$$

Now $\alpha^{-1}(x\circledast y)=\alpha^{-1}\{\alpha(a)\circledast\alpha(b)\}=\alpha^{-1}\circ\{\alpha(a\circledast b)\}$, [$\because \alpha$ is a homo...]
$=(\alpha^{-1}\circ\alpha)(a\circledast b)=I_G(a\circledast b) \quad =a\circledast b=\alpha^{-1}(x)\circledast\alpha^{-1}(y).$

$\Rightarrow \alpha^{-1}$ is a homomorphism, also it being a bijection, α^{-1} is an automorphism, hence $\alpha^{-1}\in Aut(G)$.

$\therefore$ <u>Inverse</u> of any $\alpha\in Aut(G)$ exists in $Aut(G)$.

$\therefore$ $Aut(G)$ forms a group under the composition of mapping.

<u>REMARK</u>: ① $Aut(G)$ is a non-abelian group, since $\alpha\circ\beta\neq\beta\circ\alpha$, in general.

② $Perm(G)\equiv$ the set of permutations on G.
$Aut(G)\leq Perm(G)\longrightarrow$ show it.
Let $\alpha,\beta\in Aut(G)$, then $\alpha\circ\beta^{-1}:G\to G$ exists and is a bijection.

Let $x,y\in G$. Then
$(\alpha\circ\beta^{-1})(x\circledast y)=\alpha\circ\{\beta^{-1}(x\circledast y)\}=\alpha\circ\{\beta^{-1}(x)\circledast\beta^{-1}(y)\}$, [$\because \beta^{-1}$ is a homomorphism]
$=\{(\alpha\circ\beta^{-1})(x)\}\circledast(\alpha\circ\beta^{-1})(y)$, [$\because \alpha$ is a homomorphism]
$\Rightarrow \alpha\circ\beta^{-1}$ is a homomorphism, also it being a bijection, it is an automorphism.
$\therefore$ $\alpha\circ\beta^{-1}\in Aut(G)$ for $\alpha,\beta\in Aut(G)$.

Also, $Aut(G)$ is a non-empty subset of $Perm(G)$, as $I_G\in Aut(G)$, I_G is also called <u>trivial</u> automorphism.

$\therefore$ $Aut(G)\leq Perm(G)$.

<u>Note</u>: In general, there are many bijections which donot preserve the structure of the group.