Lecture # 4 . 12 8/9/2020 .

Applications of Factor group to Automorphism groups

Theorem :- $G/Z(G) \simeq Inn(G)$.

Proof: Let us consider the mapping

$T: G/Z(G) \longrightarrow Inn(G)$ by $T(gZ) = I_g$, $gZ \in G/Z(G)$

T is well defined and one-to-one. Because, for $gZ, hZ \in G/Z(G)$ and $gZ = hZ$, we have

$\Leftrightarrow \quad g^{-1}h \in Z \; ; \; g, h \in G$.

$\Leftrightarrow \quad g^{-1}hx = xg^{-1}h \, ; \, x \in G$

$\Leftrightarrow \quad g(g^{-1}hx) = g(xg^{-1}h)$

$\Leftrightarrow \quad hx = (gxg^{-1})h$

$\Leftrightarrow \quad hxh^{-1} = gxg^{-1}$.

$\Leftrightarrow \quad I_h = I_g$, or, $I_g = I_h$.

$\Leftrightarrow \quad T(gZ) = T(hZ)$.

$\boxed{\begin{array}{l} \Rightarrow \text{well defined} \\ \Leftarrow \text{one-one} \end{array}}$

T is onto : Because, let $I_g \in Inn(G)$.
Then $g \in G. \Rightarrow gZ \in G/Z(G)$,
such that $T(gZ) = I_g$.

T is homomorphism: Because,

$T(gZ * hZ) = T(ghZ)$, $[\because Z(G) = Z \trianglelefteq G]$.
$= I_{gh}$

~~Now~~ Now $I_{gh}(x) = (gh)x(gh)^{-1} = g(hxh^{-1})g^{-1}$

$= I_g(hxh^{-1}) = I_g \circ I_h(x) = (I_g \circ I_h)(x)$.

$\Rightarrow I_{gh} = I_g \circ I_h$.

$\therefore \; T(gZ * hZ) = T(ghZ) = I_{gh} = I_g \circ I_h$
$= T(gZ) \circ T(hZ)$.

$\therefore$ T is an isomorphism.

Hence $G/Z(G) \simeq Inn(G)$. (proved)

Remark: One application of the above theorem is noted down in the Example ⑤ page no. 11.

4. If G be an infinite cyclic group, prove that $O(Aut(G)) = 2$.

Since G is an infinite cyclic group, $G \simeq (Z, +)$.

To determine all automorph.. of $(Z,+)$.

Let T be an automorph.. of $(Z,+)$.

Since $Z = \langle 1 \rangle$, image group $Z = \langle T(1) \rangle$.

As the only generators of $(Z,+)$ are 1 & -1,

$\therefore$ $T(1) = 1$, or -1.

If $T(1) = 1$, then $T(n) = T(\underbrace{1+1+\cdots+1}_{(n\text{-terms})}) = nT(1)$
$= n$, $n \in Z$.

$\Rightarrow$ T is an identity automorph..

If $T(1) = -1$, then $T(n) = nT(1) = -n$, $n \in Z$.

$\therefore$ $O(Aut(G)) = 2$. (proved)

5. Find the number of $Inn(S_3)$.

Since S_3 is an non-abelian group, its centre $Z(S_3)$ is a proper subgroup of S_3, and $S_3/Z(S_3)$ is a non-cyclic group.

Since $O(S_3) = 6$, $O(Z(S_3)) | 6$, but $O(Z(S_3)) \neq 6$.

If $O(Z(S_3)) = 2$, $O(S_3/Z(S_3)) = \dfrac{O(S_3)}{O(Z(S_3))} = \dfrac{6}{2} = 3$, a prime.

$\Rightarrow S_3/Z(S_3)$ is cyclic, a contradiction.

If $O(Z(S_3)) = 3$, $O(S_3/Z(S_3)) = \dfrac{6}{3} = 2$, a prime.

$\Rightarrow S_3/Z(S_3)$ is cyclic, a contradiction.

$\therefore$ $O(Z(S_3)) = 1$. Then $O(S_3/Z(S_3)) = O(S_3) = 6$.

~~$\Rightarrow O(S_3/Z(S_3)) \simeq S_3$.~~

Since $Inn(S_3) \simeq O(S_3/Z(S_3))$, then $O(Inn(S_3)) = 6$.

Automorphism Group [Aut(G)] of infinite cyclic group

Prob: If G is an infinite cyclic group find $Aut(G)$.

Solution: Let $G = \langle a \rangle = \{a^i \mid i \in \mathbb{Z}\}$ be an infinite cyclic group.

Let $T \in Aut(G)$.

$\therefore$ $T : G \to G$ is 1-1, onto, homomorphism

Let $a \in G \Rightarrow T(a) \in G = \langle a \rangle$

$\Rightarrow T(a) = a^m$ for some $m \in \mathbb{Z}$.

Since T is onto, so for $a \in G$, $\exists\, x \in G$

s.t. $T(x) = a$

Where $x \in G \Rightarrow x = a^n$, for some $n \in \mathbb{Z}$

Now $a = T(x) = T(a^n)$

$= \{T(a)\}^n$, since T is homomorph-

$= (a^m)^n = a^{mn}$

$\Rightarrow a^{mn-1} = e \Rightarrow mn - 1 = 0$

$\Rightarrow mn = 1$

$\Rightarrow m = +1$ or -1

$\therefore$ $T(a) = a$ or $T(a) = a^{-1}$

$Aut(G) = \{I_G, T(a) = a^{-1}\}$

$o(Aut(G)) = 2$

$\Rightarrow$ $Aut(G)$ is cyclic & $\simeq \mathbb{Z}_2$.

Eg: $G = (\mathbb{Z}, +)$ is an infinite cyclic group.

$Aut(\mathbb{Z}) \simeq \mathbb{Z}_2$.