5th SEM. Lecture # 6. 15-09-2020

Characteristic Subgroup :

1. Let G be a group and H be its subgroup $[H \leq G]$. If $\phi \in Aut(G)$ and $\phi(H) = \{\phi(h) : h \in H\}$, then show that $\phi(H) \leq G$.

$\phi(H)$ is a non-empty subset of G, since $e \in H$, $\phi(e) = e (\in G) \in \phi(H)$.

Let $a, b \in \phi(H)$. So that $a = \phi(h_1)$, $b = \phi(h_2)$; for some $h_1, h_2 \in H$.

Now $ab^{-1} = \phi(h_1)[\phi(h_2)]^{-1} = \phi(h_1)\phi(h_2^{-1})$, [$\because \phi$ is a homomorph.]

$= \phi(h_1 h_2^{-1})$, since ϕ is a homomorph.

$\in \phi(H)$, since $H \leq G$, $h_1 h_2 \in H \Rightarrow h_1 h_2^{-1} \in H$.

$\therefore$ $\phi(H) \leq G$.

2. Let G be a group and N be a normal subgroup of G $[N \triangleright G]$. Show that $\phi(N) \triangleright G$, ϕ being an automorphism of G.

By 1, we have $\phi(N) \leq G$.

To prove: $\phi(N)$ is normal in G.

Let us take $g \in G$ and $a \in \phi(N)$ s.t. $a = \phi(n)$, $n \in N$, and $g = \phi(g_1)$ for $g_1 \in G$. [$\because \phi : G \to G$]

$\therefore$ $gag^{-1} = \phi(g_1)\phi(n)[\phi(g_1)]^{-1} = \phi(g_1)\phi(n)\phi(g_1^{-1})$

$= \phi(g_1 n g_1^{-1}) \in \phi(N)$, since $N \triangleright G$, $g_1 n g_1^{-1} \in N$.

$\Rightarrow$ $\phi(N) \triangleright G$.

Definition $\longrightarrow$ A subgroup C of G is said to be a characteristic subgroup of G, if $\phi(C) \subset C$, $\forall \phi \in Aut(G)$.

Note: char. subgroup is mapped into itself by every automorphism of the group, as $\phi(C) \subset C$.

Prove: A characteristic subgroup of G must be a normal subgroup of G.

BUT, the converse is NOT true.

For some $g \in G$, let us define an inner automorphism $\phi_g : G \to G$ by $\phi_g(x) = gxg^{-1}$, $x \in G$.

By the definition of the char. subgroup C of G, we have $\phi(C) \subset C$, $\forall \phi \in Aut(G)$.

$\therefore \phi_g(C) \subset C$, $\forall g \in G$, as $\phi_g \in Aut(G)$.

$\Rightarrow gag^{-1} \in C$, $\forall g \in G$, $\forall a \in C$.

$\Rightarrow \underline{C \rhd G}$.

Note: Every char. subgroup of G is normal, because every conjugation map is an inner automorphism.

Converse Part:

Example: For Klein's 4-group, $G = \{e, a, b, ab\}$ with $a^2 = e$, $b^2 = e$ and $ab = ba$. $o(a) = o(b) = o(ab) = 2$.

Let us take a subgroup $C = \{e, a\}$ of G, and $\phi = \begin{pmatrix} e & a & b & ab \\ e & b & a & ab \end{pmatrix}$ where ϕ is an automorphism of G.

$C = \{e, a\}$ is normal in G, because $\forall g \in G$, $gCg^{-1} \subset C$. (verify it). Since G is abelian, every subgroup of G is normal.

But $\phi(C) = \{\phi(e), \phi(a)\} = \{e, b\} \not\subset C$

$\Rightarrow \phi(C)$ is not a char. subgroup of G, though $\underline{C \rhd G}$.

Remark: Determine $Aut(G)$, G being Klein's 4-group.

$$Aut(G) = \left\{ \begin{pmatrix} e & a & b & ab \\ e & a & b & ab \end{pmatrix}, \begin{pmatrix} e & a & b & ab \\ e & a & ab & b \end{pmatrix}, \begin{pmatrix} e & a & b & ab \\ e & b & a & ab \end{pmatrix}, \begin{pmatrix} e & a & b & ab \\ e & b & ab & a \end{pmatrix}, \begin{pmatrix} e & a & b & ab \\ e & ab & a & b \end{pmatrix}, \begin{pmatrix} e & a & b & ab \\ e & ab & b & a \end{pmatrix} \right\}. \quad |Aut(G)| = 6.$$

Since $o(a) = o(b) = o(ab) = 2$, $o(e) = 1$, $o(\phi(a)) = o(\phi(b)) = o(\phi(ab)) = 2$.

There are $3!$ possible permutations of $\{a, b, ab\}$.

Examples:

(1) $Z(G)$, the centre of a group G is a char. subgroup of G.

<u>To prove:</u> $\phi(Z) \subset Z$, where $Z = Z(G)$, $\phi \in Aut(G)$.

Let $x \in Z(G)$. $\Rightarrow xg = gx$, $\forall g \in G$.

Since ϕ is one-one $\Rightarrow \phi(xg) = \phi(gx)$

" ϕ is homomorphi $\Rightarrow \phi(x)\phi(g) = \phi(g)\phi(x)$

$\left.\begin{array}{l}\text{For } x \in Z, \exists z \in \phi(Z) \\ \text{s.t. } \phi(x) = z.\end{array}\right\} \Rightarrow z\phi(g) = \phi(g)z$, $\forall g \in G$.

Since ϕ is onto $\Rightarrow zg' = g'z$, $\forall g' \in G$. $[\underbrace{g' = \phi(g)}_{\in G}]$

$\Rightarrow z \in Z(G)$.

$\therefore z \in \phi(Z) \Rightarrow z \in Z$

$\therefore$ $\underline{\phi(Z) \subset Z}$.

(2) Every subgroup of a cyclic group is characteristic

Let $G = \langle a \rangle = \{a^n : n \in Z\}$ and $C \leq G$

s.t. $C = \langle a^m \rangle$.

Let $z \in \phi(C) = \{\phi(x) : x \in C\}$, so that $x = (a^m)^q$, $q \in Z$.

Then $z = \phi(x) = \phi\{(a^m)^q\} = \phi(a^{mq}) = [\phi(a)]^{mq}$

$= [\{\phi(a)\}^q]^m = (a^p)^m$, since $G = \langle\phi(a)\rangle$, then $\{\phi(a)\}^q = a^p$ for some $p \in Z$.

$= (a^m)^p \in C$.

$\therefore z \in \phi(C) \Rightarrow z \in C$.

$\therefore$ $\underline{\phi(C) \subset C} \Rightarrow C$ is a char. subgroup of G.

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Commutator subgroup :-

Commutator : An element of the group G which is of the form $ghg^{-1}h^{-1}$, for some $g, h \in G$, is called a commutator, $[g,h]$.

The identity element e is always a commutator and it is the only commutator if and only if G is abelian.

For, $e = [e,e] = eee^{-1}e^{-1} \Rightarrow e$ is a commutator.

Now if $ghg^{-1}h^{-1} = e$, then $\Leftrightarrow gh(hg)^{-1} = e$

$\Leftrightarrow gh = e(hg) = hg$, for $g, h \in G$

$\Leftrightarrow$ G is abelian.

Definition :- The subgroup of a group G that is generated by all the commutators of the group, is called the commutator subgroup, denoted by $[G,G]$, defined by

$G' = [G,G] = \langle ghg^{-1}h^{-1} : g, h \in G \rangle$.

REMARK: This subgroup is important because it is the smallest normal subgroup of G s.t. the quotient group G/G' is abelian.

⊛ In some sense it provides a measure of how far the group is from being abelian; the larger the normal subgroup is, the "less abelian" the group is.

⊛ If G is abelian, then $ghg^{-1}h^{-1} = e$ for $\forall g, h \in G$.

$\Rightarrow G' = [G,G] = \{e\}$.

$\therefore$ Quotient group $G/G' (= G)$, which is abelian.