Fourier Series

Expansion of functions with arbitrary period:

So far in our discussion, we have assumed that the period of the function is 2π. Now we are going to discuss Fourier Series expansion for function $f(x)$ with any arbitrary period $2l$ (say).

Let us assume that, the function $f(x)$ is defined in the interval $(-l, l)$. Now we want to change the function to the period of 2π, so that we can use the formulae of a_n and b_n as discussed earlier.

As $2l$ is the period for variable x.

Therefore, 2π is the period for variable $= \frac{x}{2l} \times 2\pi = \frac{\pi x}{l}$

Now, putting $y = \frac{\pi x}{l}$ or $x = \frac{yl}{\pi}$, the function $f(x)$ with periodicity $2l$ is transformed to the function $f(\frac{yl}{\pi})$ with periodicity 2π.

Now, $f(\frac{yl}{\pi})$ can be expanded in Fourier series as

$$f(\frac{yl}{\pi}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos ny + \sum_{n=1}^{\infty} b_n \sin ny$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{yl}{\pi})dy = \frac{1}{\pi}\int_{-l}^{l} f(x)d(\frac{\pi x}{l})$, $putting\ y = \frac{\pi x}{l}$

Therefore, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

Again $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{yl}{\pi}) \cos ny\, dy = \frac{1}{\pi}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} d\left(\frac{\pi x}{l}\right)$, $putting\ y = \frac{\pi x}{l}$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx$$

Similarly, $b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l} dx$.

Ex. 1

Expand the following function in Fourier series

$$f(x) = \begin{cases} 0, for -5 \le x \le 0 \\ 3, for\ 0 \le x \le 5 \end{cases}$$

Ans:

Given interval for the function is (-5, 5) with periodicity 10.

Therefore, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{1}{5}\int_{-5}^{5} f(x)dx = \frac{1}{5}\left\{\int_{-5}^{0} f(x)dx + \int_{0}^{5} f(x)dx\right\}$

$$a_0 = \frac{1}{5}\int_{0}^{5} 3dx = 3$$

And $a_n = \frac{1}{5}\int_{-5}^{5} f(x) \cos\frac{n\pi x}{5} dx = \frac{1}{5}\int_{0}^{5} 3\cos\frac{n\pi x}{5} dx = \frac{3}{5} \times \frac{5}{n\pi}\left[\sin\left(\frac{n\pi x}{5}\right)\right]_0^5 = 0$

Similarly,

$b_n = \frac{1}{5}\int_{-5}^{5} f(x) \sin\frac{n\pi x}{5} dx = \frac{1}{5}\int_{0}^{5} 3\sin\frac{n\pi x}{5} dx = \frac{3}{5} \times \frac{5}{n\pi}\left[-\cos\left(\frac{n\pi x}{5}\right)\right]_0^5$

$$= \frac{3}{n\pi}\{1 - (-1)^n\}$$

Therefore, $b_n = \begin{cases} 0\ if\ n\ is\ even \\ \frac{6}{n\pi}\ if\ n\ is\ odd \end{cases}$

Therefore, Fourier series expansion of given function becomes

$$f(x) = \frac{3}{2} + \sum_{n=1,3,5\ldots}^{\infty} \frac{6}{n\pi} \sin\frac{n\pi x}{5}$$

Half Range Series:

If the given function is defined in the interval (0, π), then it is immaterial whatever the function may be outside the interval (0, π). To get a cosine series expansion in the interval (-π, π), we have to extend the function $f(x)$ in the interval (-π, π) as an even function.

Then Euler's formulae can be written as:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx\,,\; a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\,, and\;\; b_n = 0$$

Similarly, to expand the function as sine series, we have to extend the function $f(x)$ in the interval (-π, π) as an odd function. And Euler's formulae can be written as:

$$a_n = 0,\;\; and\; b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\;\,dx.$$

Let us consider an example:

Ex. 1.

(a) Expand $f(x) = x, 0 \le x \le \pi$ in Fourier cosine series.

Ans:

Here the given function $f(x) = x, 0 \le x \le \pi$ has to be expanded as cosine series, so we can extend the function $f(x)$ in the interval (-π, π) as an even function as,

$$f(x) = \begin{cases} x, for\; 0 \le x \le \pi\; (Given) \\ -x, for -\pi \le x \le 0\; (Extended) \end{cases}$$

As the function $f(x)$ is now even functionin the interval (-π, π), therefore, $b_n = 0$

And $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi} \times \left[\frac{x^2}{2}\right]_0^{\pi} = \pi$

Similarly, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\;\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right\}$

$$= \frac{2}{\pi n^2}\{(-1)^n - 1\}$$

Therefore, $a_n = \begin{cases} 0,\; if\; n\; is\; even \\ -\frac{4}{\pi n^2}\,, if\; n\; is\; odd \end{cases}$

Therefore, Fourier series expansion of given function becomes

$$f(x) = \frac{\pi}{2} - \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{\pi n^2}\cos nx$$

(b) From the above Fourier series expansion show that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Ans:

We have $f(x) = \frac{\pi}{2} - \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{\pi n^2}\cos nx$

Putting x=0 in the above expression, we can write

$$0 = \frac{\pi}{2} - \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{\pi n^2}$$

$$\frac{\pi}{2} = \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1,3,5...}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{8}$$

$$i.e \sum_{n=1,3,5...}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Q.1 Obtain the Fourier Series for the function $f(x) = \begin{cases} 1 - \frac{2x}{\pi}, for\ 0 \leq x \leq \pi \\ 1 + \frac{2x}{\pi}, for - \pi \leq x \leq 0 \end{cases}$

Hence deduce that

$$\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{8}$$

Q.2 Express $f(x) = x$ as Half range cosine series in $0 \leq x \leq 2$.

Q.3 Obtain the Fourier Series for the function $f(x) = \begin{cases} x, for\ 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, for \frac{\pi}{2} \leq x \leq \pi \end{cases}$