CHAPTER 3

# Estimation of Static Errors and Reliability

While reporting a measured value of a quantity, it is necessary to indicate the possible error in the measurement. A question may be asked, how do we estimate a measurement error? Before going into that, let us define a few necessary parameters.

## 3.1 Definition of Parameters

The parameters we need to define are

1. Error
2. Scale range
3. Scale span
4. Limiting error
5. Probable error

### Error

If $X_m$ is the measured value of the quantity, and $X_t$ is the true value of the quantity then the absolute static error is defined as

$$\varepsilon_0 = |X_m - X_t|$$

Often a relative static error is reported. Its definition is

$$\varepsilon_r = \frac{|X_m - X_t|}{X_t} = \frac{\varepsilon_0}{X_t} \tag{3.1}$$

Mostly, an error is much less than the true value making $\varepsilon_r < 1$. Now from Eq. (3.1), we get

$$\varepsilon_r = \frac{X_m}{X_t} - 1$$

$$\Rightarrow \quad X_t = \frac{X_m}{1+\varepsilon_r} \cong X_m(1-\varepsilon_r)$$

While defining the absolute and relative static errors, we have used the term *true value*. The questions are, what is the true value of a physical quantity and how do we know it? It is said that if we make an infinite number of measurements with the help of a *calibrated* measuring instrument and observe that the individual measurements agree between themselves within a specified degree of accuracy, we may assume that the measured value is the true value of the quantity.

## Scale Range

The terms *scale range* is defined as follows. If $X_{min}$ and $X_{max}$ are the minimum and maximum values that an instrument can measure, then

$$\text{Scale range} = \text{Between } X_{min} \text{ and } X_{max}$$

Sometimes, the *dynamic range* of an instrument is specified.

### Dynamic range

The dynamic range is defined as

$$\text{Dynamic range } N = \frac{\text{Range of operation}}{\text{Resolution}}$$

It is a common practice to specify dynamic range in dB as

$$\text{Dynamic range} = 20 \log_{10} N$$

Thus, an instrument having a 40 dB dynamic range means that it can handle input sizes of 100 to 1.

**Example 3.1**

A voltmeter has a range of [4 V, 20 V] and a resolution of 1 mV. The dynamic range of the instrument is

(a) 21 dB (b) 60 dB (c) 72 dB (d) 84 dB

**Solution**

The range of operation of the instrument is $(20 - 4) = 16$ V and the resolution is $1 \times 10^{-3}$ V. So,

$$\text{Dynamic range} = 20 \log \frac{16}{1 \times 10^{-3}} = 84\,\text{dB}$$

Therefore, the answer is (d).

## Scale Span

If $X_{min}$ and $X_{max}$ are the minimum and maximum values that an instrument can measure, then the scale span is defined as

$$\text{Scale span} = X_{max} - X_{min}$$

**Example 3.2**

A voltmeter is calibrated between 10 V and 250 V. The scale span and scale range are respectively

(a) 250, 250 (b) 240, 250 (c) 250, 240 (d) 240, 240

**Solution**

The nearest answer is (d). But it is better said that the scale range is 10 to 250 V.

## 3.2 Limiting Error

Suppose the length of a rod is being measured with the help of a vernier scale which has a vernier constant of 0.1 mm. One may measure the length only once by the vernier scale and report value as $L \pm 0.1$ mm, if the measured value is $L$ mm. This reported error is called the *limiting error* (or *guarantee error*), because this is the maximum error which might have occurred during the measurement, assuming that the vernier scale has no calibration error. Many components (e.g. resistor, capacitor) or instruments are sold by manufacturers with some limits in their values or readings and indicated by gold or silver bands. These are limiting errors of the components.

### Probable Error

Alternatively, in the foregoing example, one may measure the same length a number of times, take the arithmetic mean of the values obtained, calculate the error by one of the statistical methods (discussed later) and report the value as $L \pm \Delta L$ mm. This error may be termed *probable error.*

Estimation of limiting error for a single measurand is pretty simple, though it may be a bit involved for a measurement involving many measurands each having its own limiting error, while for probable errors even a single measurand demands attention.

We will take up both these methods of estimation one after another. It will be seen that the statistical treatment offers a more optimistic picture in the final analysis.

### Combination of Limiting Errors

Suppose $u$ and $v$ are measurands and $X$ is the final result. For simplicity we consider two measurands only though it is easy to generalise the results for many measurands. A few fundamental mathematical operations such as addition, subtraction, multiplication, division, raising to powers which connect the measurands to the final result are individually considered below.

**Addition and subtraction.** Here, $X = u \pm v$. Then

$$\frac{dX}{X} = \frac{du}{X} \pm \frac{dv}{X} = \frac{u}{X}\frac{du}{u} \pm \frac{v}{X}\frac{dv}{v} .$$

But because errors are $\pm\delta u$ and $\pm\delta v$, we have for both addition and subtraction,

$$\frac{\delta X}{X} = \pm\left(\frac{u}{X}\frac{\delta u}{u} + \frac{v}{X}\frac{\delta v}{v}\right)$$

**Multiplication and division.** Here, either $X = uv$ or $X = u/v$. Taking logarithm of both sides, the two cases can be written as

$$\ln X = \ln u \pm \ln v$$

Taking differentials, we get

$$\frac{dX}{X} = \frac{du}{u} \pm \frac{dv}{v}$$

But, because of the nature of the errors as stated before,

$$\frac{\delta X}{X} = \pm\left(\frac{\delta u}{u} + \frac{\delta v}{v}\right)$$

**Indices.** Say, $X = u^m v^n$. On logarithmic differentiation,

$$\frac{dX}{X} = m\frac{du}{u} + n\frac{dv}{v}$$

Hence,

$$\frac{\delta X}{X} = \pm\left(m\frac{\delta u}{u} + n\frac{\delta v}{v}\right)$$

**Example 3.3**

Three resistors have the following values: $R_1 = 200\ \Omega \pm 10\%$, $R_2 = 100\ \Omega \pm 5\%$ and $R_3 = 50\ \Omega \pm 5\%$. Determine the magnitude of the resultant resistances and the limiting errors if they are connected in (a) series, and (b) parallel.

**Solution**

(a) $R = R_1 + R_2 + R_3 = 350\ \Omega$. $\dfrac{\delta R_1}{R_1} = 0.1$ and $\dfrac{\delta R_2}{R_2} = \dfrac{\delta R_3}{R_3} = 0.05$.

Therefore,

$$\frac{\delta R}{R} = \frac{R_1}{R}\frac{\delta R_1}{R_1} + \frac{R_2}{R}\frac{\delta R_2}{R_2} + \frac{R_3}{R}\frac{\delta R_3}{R_3}$$

$$= \frac{200}{350}(0.1) + \frac{100}{350}(0.05) + \frac{50}{350}(0.05) \cong 0.079$$

Thus, $R = 350\ \Omega \pm 7.9\%$.

(b) Here $\dfrac{1}{R} = \dfrac{1}{200} + \dfrac{1}{100} + \dfrac{1}{50} = \dfrac{7}{200}$. Hence, $R \cong 28.6\ \Omega$.

Again,

$$d\left(\frac{1}{R}\right) = d\left(\frac{1}{R_1}\right) + d\left(\frac{1}{R_2}\right) + d\left(\frac{1}{R_3}\right)$$

which gives

$$\frac{dR}{R^2} = \frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2} + \frac{dR_3}{R_3^2}$$

Therefore,

$$\frac{\delta R}{R} = \frac{R}{R_1}\frac{\delta R_1}{R_1} + \frac{R}{R_2}\frac{\delta R_2}{R_2} + \frac{R}{R_3}\frac{\delta R_3}{R_3}$$

$$= \frac{28.6}{200}(0.1) + \frac{28.6}{100}(0.05) + \frac{28.6}{50}(0.05) \cong 0.057$$

Thus,

$$R = 28.6\ \Omega \pm 5.7\%$$

**Example 3.4**

The following are the data for a Hay's ac bridge: $R_1 = 1000\ \Omega \pm 1$ part in 10,000, $R_2 = 16,800\ \Omega \pm 1$ part in 10,000, $R_3 = 833 \pm 0.25\ \Omega$, $C = 1.43 \pm 0.001\ \mu F$. If frequency of the supply voltage is $50 \pm 0.1$ Hz and the formulae for $L$ and $R$ in the balanced condition of the bridge are

$$L = \frac{CR_1R_2}{1+\omega^2C^2R_3^2} \quad \text{and} \quad R = \frac{R_1R_2R_3C^2\omega^2}{1+\omega^2C^2R_3^2}$$

determine the values of $L$ and $R$ of the coil and their limits of error.

**Solution**

Given:
$$\frac{\delta R_1}{R_1} = \frac{\delta R_2}{R_2} = 1 \text{ part in } 10,000 = 0.01\%$$

$$\frac{\delta R_3}{R_3} = \frac{0.25}{8.33} \times 100\% = 0.03\%$$

$$\frac{\delta C}{C} = \frac{0.001}{1.43} \times 100\% \cong 0.07\%$$

$$\frac{\delta\omega}{\omega} = \frac{0.1}{50} \times 100\% = 0.2\%$$

Thus,
$$L = \frac{(1.43 \times 10^{-6})(1000)(16800)}{1+(100\pi)^2(1.43 \times 10^{-6})^2(833)^2} \cong 21.1 \text{ H}$$

$$\frac{\delta L}{L} = 3\frac{\delta C}{C} + \frac{\delta R_1}{R_1} + \frac{\delta R_2}{R_2} + 2\frac{\delta R_3}{R_3} + 2\frac{\delta\omega}{\omega}$$

$$= [(3 \times 0.07) + 0.01 + 0.01 + (2 \times 0.03) + (2 \times 0.2)]\% = 0.69\%$$

Therefore,
$$L = 21 \text{ H} \pm 0.69\%$$

Similarly,
$$R = \frac{(1000)(16800)(833)(1.43 \times 10^{-6})^2(100\pi)^2}{1+(100\pi)^2(1.43 \times 10^{-6})^2(833)^2} = 2477\ \Omega$$

$$\frac{\delta R}{R} = \frac{\delta R_1}{R_1} + \frac{\delta R_2}{R_2} + 3\frac{\delta R_3}{R_3} + 4\frac{\delta C}{C} + 4\frac{\delta\omega}{\omega}$$

$$= [0.01 + 0.01 + (3 \times 0.03) + (4 \times 0.07) + (4 \times 0.2)]\% = 1.19\%$$

Therefore,
$$R = 2477\ \Omega \pm 1.19\%$$

**Example 3.5**

A 0–10 ampere ammeter has a guaranteed accuracy of 1% of the full-scale deflection. The limiting error while reading 2.5 A is

(a) 1% (b) 2%
(c) 4% (d) none of the above

Suppose, we have measured the heights of 200 students of a college with the height of each student recorded in inches. A student's height may be any value such as 62.35 inches. It does not make sense to figure out how many students have heights of 62.35 inches. It may so happen that we may not find another student exactly 62.35 inches tall. It is better if we divide heights within a few 'class'es or 'cell's such as 56–58 inches, 58–61 inches and so on. The cell demarcation is arbitrary, but we care to see that each cell possesses a midpoint which is a convenient whole number. Let the grouped data so obtained be graphed as given in Fig. 3.1.

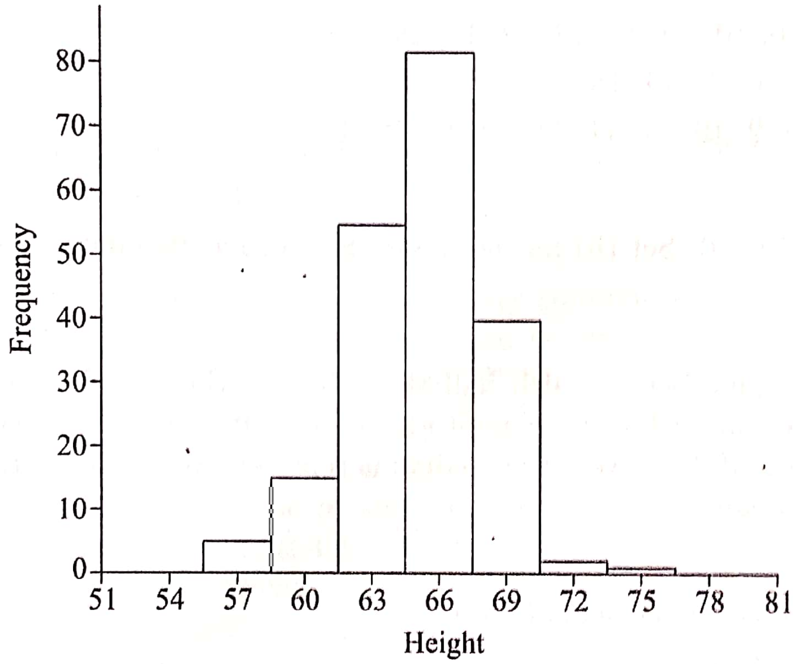


**Fig. 3.1** Frequency distribution of the height of 200 students.

This graph is called the *frequency distribution* or *histogram*. Next, the question arises as to how we can characterise the frequency distribution of a sample with a single descriptive measure, or simple *statistic*. In fact, there are two highly useful descriptions: one is the *central point* of the distribution and the other is the *spread*.

## Measures of Central Tendency

Statistically speaking, a central point or average is a value which is typical or representative of a set of data. Since the average tends to lie centrally within a set of data arranged according to magnitude, it is also called the central point or a *measure of central tendency*.

Generally six types of averages are defined. They are:

1. Mode
2. Median
3. Arithmetic mean or simply, mean
4. Geometric mean
5. Harmonic mean, and
6. Root mean square

### Mode

Mode is defined as the most frequent value. In the frequency distribution of heights of students (Fig. 3.1), the mode value is 66 inches.

The mode may not exist, or even it does it may not be unique.

**Example 3.7**
What are the mode values for the following three sets?

(a) 2, 3, 5, 6, 9, 10, 10, 10, 11, 12, 12, 14, 18
(b) 3, 5, 6, 9, 10, 11, 12, 14, 18
(c) 2, 3, 5, 5, 5, 6, 9, 10, 11, 11, 11, 12, 12, 14, 18

**Solution**
The mode of set (a) is 10. Set (b) has no mode. Set (c) has two modes, 5 and 11.

### Median

The median is the value below which half the values in the sample fall. So, if the number of data $N$, arranged according to magnitude, is odd, it is the value corresponding to the $(N \div 2 + 0.5)$th data. If $N$ is even, the median is represented by the average of the $(N \div 2)$th and $(N \div 2 + 1)$th points.

**Example 3.8**
Find the medians of the given sets of data:

(a) 2, 3, 4, 4, 6, 7, 7, 7, 9
(b) 3, 3, 7, 8, 12, 13, 16, 19

**Solution**

(a) The number of data points is 9 which is odd. So, the $(9 \div 2) + 0.5 = 5$th point, i.e. 6, is the median.
(b) Here the number of data points, 8, is even. So, the median is the average of the $8 \div 2 = 4$th and 5th points, i.e. $(8 + 12) \div 2 = 10$.

### Arithmetic mean

Of all these types of averages, the arithmetic mean $\mu$ is considered the most probable value of the measurand and is defined as

$$\mu = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where $x_i$'s are individual data and $n$ is the number of measurements.

If the data $x_1, x_2, \ldots, x_k$ occur $f_1, f_2, \ldots, f_k$ times respectively, (i.e. occur with frequencies $f_1, f_2, \ldots, f_k$), the arithmetic mean is

$$\mu = \frac{f_1x_1 + f_2x_2 + \cdots + f_kx_k}{f_1 + f_2 + \cdots + f_k} = \frac{1}{n}\sum_{i=1}^{n} f_ix_i$$

**Example 3.9**

If 10, 16, 12 and 4 occur with frequencies 5, 3, 4 and 2 respectively then what is the arithmetic mean?

Solution

The arithmetic mean is

$$\mu = \frac{(5)(10) + (3)(16) + (4)(12) + (2)(4)}{5 + 3 + 4 + 2} = 11$$

### *Geometric mean*

The geometric mean $g_m$ of a set of $n$ numbers $x_1, x_2, x_3, \ldots, x_n$ is the $n$th root of the product of number. Written mathematically

$$g_m = \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$$

**Example 3.10**

The mass of a substance is being measured in a faulty common balance having unequal arm lengths. Show that the true mass of the substance is the geometric mean of the masses determined by placing the substance once on the left pan and next time on the right pan of the balance.

Solution

Let the true mass of the substance be $m$ and the lengths of the left and right arms of the balance be $x_1$ and $x_2$ respectively. Initially, the substance is placed on the left pan and a mass $m_1$ on the right pan balances it. Then

$$m x_1 = m_1 x_2$$

or

$$m = m_1 \frac{x_2}{x_1}$$

Next, the mass is placed on the right pan and a mass $m_2$ on the left pan balances it. Then

$$m x_2 = m_2 x_1$$

or

$$x_2 = \frac{m_2}{m} x_1$$

Substituting this value of $x_2$ in the first equation, we get

$$m = m_1 \frac{x_2}{x_1} = \frac{m_1}{x_1} \cdot \frac{m_2}{m} x_1$$

or

$$m^2 = m_1 m_2$$

or

$$m = \sqrt{m_1 m_2}$$

### *Harmonic mean*

The harmonic mean $h_m$ of a set of $n$ numbers $x_1, x_2, x_3, \ldots, x_n$ is the reciprocal of the arithmetic mean of the reciprocals of number. Written mathematically,

$$h_m = \frac{1}{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

**Example 3.11**

A person travels from X to Y at an average speed of 60 km/h and returns by the same route at an average speed of 50 km/h. Find the average speed for the round trip.

**Solution**

Let the distance between the two places be $x$ km. Then, if $t_1$ and $t_2$ be the time (in h) taken for the onward and return trips, we have

$$t_1 = \frac{x}{60}$$

$$t_2 = \frac{x}{50}$$

Therefore, the average speed $v_{\text{av}}$ for the round trip is

$$v_{\text{av}} = \frac{\text{total distance}}{\text{total time}} = \frac{2x}{t_1 + t_2} = \frac{2x}{\frac{x}{60} + \frac{x}{50}} = \frac{2}{\frac{1}{60} + \frac{1}{50}}$$

We observe that $v_{\text{av}}$ is nothing but the harmonic mean of the two speeds.

### *Root mean square*

The root mean square (rms) or quadratic mean of a set of $n$ numbers $x_1, x_2, x_3, \ldots, x_n$ is defined as

$$\text{rms} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}$$

This averaging is quite common in engineering and physical applications.

## Measures of Spread

The average height, in our earlier example, may be the most important characteristic (statistic) of the students. However, it is also equally important to know how the observations spread out or disperse around the average value. Like the measures of central tendency, there are several measures of spread or dispersion:

1. Deviation
2. Mean absolute deviation
3. Variance, and
4. Standard deviation

### *Deviation*

Usually written as $d_i$, deviation is the scatter of an individual datum from the mean. Symbolically,

$$d_i = x_i - \mu$$

Obviously,

$$\Sigma d_i = 0$$

### Mean absolute deviation

The mean absolute deviation[1] $D$ of a set of data is defined as the average of absolute values of deviations, i.e.

$$D = \frac{1}{n}\sum_{i=1}^{n}|d_i| = \frac{1}{n}\sum_{i=1}^{n}|x_i - \mu|$$

If $x_1, x_2, \ldots, x_n$ occur with frequencies $f_1, f_2, \ldots, f_n$ respectively, then

$$D = \frac{1}{n}\sum_{i=1}^{n} f_i|x_i - \mu|$$

**Example 3.12**
Find the mean absolute deviation of heights of 100 male students of a class as given in table below.

| *Height* (in) | 60 – 62 | 63 – 65 | 66 – 68 | 69 – 71 | 72 – 74 |
|---|---|---|---|---|---|
| *No. of students* | 5 | 18 | 42 | 27 | 8 |

**Solution**
Here, the arithmetic mean

$$\mu = \frac{(61 \times 5) + (64 \times 18) + (67 \times 42) + (70 \times 27) + (73 \times 8)}{5 + 18 + 42 + 27 + 8}$$

$$= 67.45 \text{ in}$$

The rest of the calculation is presented in the following table:

| *Height* (in) | $\|x_i - \mu\| = \|x_i - 67.45\|$ | $f_i$ | $f_i\|x_i - \mu\|$ |
|---|---|---|---|
| 60–62 | 6.45 | 5 | 32.25 |
| 63–65 | 3.45 | 18 | 62.10 |
| 66–68 | 0.45 | 42 | 18.90 |
| 69–71 | 2.55 | 27 | 68.85 |
| 72–74 | 5.55 | 8 | 44.40 |
| | $\Sigma =$ | **100** | **226.50** |

Therefore, the mean absolute deviation $D = \dfrac{226.50}{100} = 2.26$ in

### Variance

Intuitively, the mean absolute deviation is a good measure of spread; but it is mathematically intractable. One difficulty is the problem of differentiating an absolute value function. To

[1]Often referred to as *average deviation*.

obviate this difficulty, the *variance*, which is nothing but the mean squared deviation, was defined as

$$\text{Variance} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

The statistical theories distinguish between a *sample* and a *population* of data. The sample denotes that the number of data is less than or equal to 20 while the population indicates the size more than 20. According to the statistical theories, though the variance or MSD is a good measure of dispersion for a population, the divisor should be $(n-1)$ rather than $n$ to make it an *unbiased estimator* for a sample.

### Standard deviation

Denoted by $\sigma$ or $s$, standard deviation is defined in two ways—one for a sample $(n \leq 20)$ and the other for a population $(n > 20)$ as follows

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n}} \qquad n > 20$$

$$s = \sqrt{\frac{\Sigma d_i^2}{n-1}} \qquad n \leq 20$$

Variance which is just the squared standard deviation is thus written as $\sigma^2$ or $s^2$.

#### Example 3.13

A set of 10 independent measurements were made to determine the diameter of the bob of a simple pendulum. The measured values in cm were: 1.570, 1.597, 1.591, 1.562, 1.577, 1.580, 1.564, 1.586, 1.550 and 1.575. Determine (a) the arithmetic mean, (b) the average deviation, (c) the standard deviation, and (d) the variance.

Solution

The calculation is presented in a tabular form below with the last row in bold face letters indicating sums of corresponding columns:

| $x_i$ | $\lvert d \rvert$ | $d^2$ |
|---|---|---|
| 1.570 | 0.005 | 0.000025 |
| 1.597 | 0.022 | 0.000484 |
| 1.591 | 0.016 | 0.000256 |
| 1.562 | 0.013 | 0.000169 |
| 1.577 | 0.002 | 0.000004 |
| 1.580 | 0.005 | 0.000025 |
| 1.564 | 0.011 | 0.000121 |
| 1.586 | 0.011 | 0.000121 |
| 1.550 | 0.025 | 0.000625 |
| 1.575 | 0.000 | 0.000000 |
| **15.752** | **0.110** | **0.001866** |
| $\mu = 1.575$ | $D = 0.011$ | $s = 0.0143$<br>$s^2 = 0.000204$ |

The measurement can thus be reported as $1.575 \pm 0.014$ cm where the indicated error is the standard deviation.

## 3.4 Error Estimates from the Normal (or Gaussian) Distribution

If a rather large number of careful measurements are carried out of a measurand, and the frequency of occurrence of a particular value is plotted against the corresponding values, the resulting histogram usually assumes the form of a bell-shaped curve called the *normal* or *Gaussian*[2] *curve* as shown in Fig. 3.2. Here we have plotted the Gaussian distribution curve for a typical height measurement of students of a class as given in Example 3.12 at page 39.

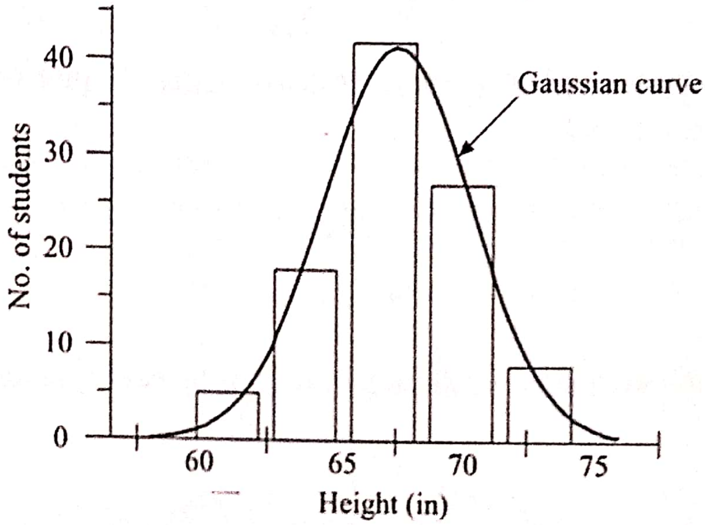


Fig. 3.2 Height vs. number of students of a class showing a Gaussian distribution.

Errors that are made in physical measurements do often have a normal distribution. This distribution is defined by the equation

$$y = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\} \tag{3.3}$$

where $\mu$ is the mean, $\sigma$ is the standard deviation, and $\pi \cong 3.14159$.

It is easily seen from the following calculation that the total area bounded by the normal distribution curve and the abscissa is 1.

$$\begin{aligned}\int_{-\infty}^{+\infty} y dx &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\} dx \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-u^2) du \qquad \text{where } u = \frac{x-\mu}{\sigma\sqrt{2}} \\ &= \frac{\Gamma(1/2)}{\sqrt{\pi}} = 1\end{aligned}$$

[2]Named after (Johann) Karl Friedrich Gauss (1777–1855), German mathematician. His contribution to electrostatics gave birth to the electromagnetic field theory apart from his seminal contributions to probability theory and other branches of mathematics.