CHAPTER 4

# Dynamic Characteristics of Instruments

As the name implies, these characteristics describe the behaviour of a system with time when some input is given to the system. Suppose a resistor $R$ and a capacitor $C$ are connected in series. If now a voltage source $E_i$ is included in the circuit, charge builds up on the capacitor and the voltage across the resistor slowly builds up. This behaviour of the circuit can aptly be described by setting up an appropriate differential equation and solving it for certain boundary conditions. Of course, while setting up the equation, certain simplifying assumptions need be made to make the problem tractable. Here in the $RC$ circuit, it is tacitly assumed that neither the capacitor is leaky nor the resistor dissipates any energy in the form of heat. An actual instrumentation system is rather complex and therefore, it is necessary to introduce certain ideal conditions to make mathematical studies tractable. This is called *mathematical modelling* of the problem. Once a model has been built, the response of the system, termed the *dynamic response*, is studied with respect to a few idealised inputs. Transfer functions of systems are very useful to study their responses.

## 4.1 Transfer Function

The generalised relation between a particular input $q_i$ [$\equiv q_i(t)$] and the corresponding output $q_o$ with proper simplifying assumptions, can be written in the form

$$a_n \frac{d^n q_o}{dt^n} + \cdots + a_1 \frac{dq_o}{dt} + a_0 q_o = b_m \frac{d^m q_i}{dt^m} + \cdots + b_1 \frac{dq_1}{dt} + b_0 q_i \tag{4.1}$$

where $a$'s and $b$'s are combination of system parameters assumed to be constant.[1]

Taking Laplace transform[2] and assuming all initial conditions equal to zero, we get from Eq. (4.1)

$$(a_n s^n + \cdots + a_1 s + a_0) Q_o(s) = (b_m s^m + \cdots + b_1 s + b_0) Q_i(s)$$

The transfer function $G(s)$ is defined as

$$G(s) \equiv \frac{Q_o(s)}{Q_i(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} \tag{4.2}$$

[1] Any system satisfying a relation of the type given by Eq. (4.1) is called a *linear time-invariant* system. See Appendix B at page 858 for the definition of such a system.

[2] See Appendix C at page 861 for a brief introduction to Laplace transform.

## Properties of Transfer Function

1. The transfer function is a general relation between the Laplace transforms of the output and input quantities $Q_o(s)$ and $Q_i(s)$. It is not the instantaneous ratio of the time-varying quantities $q_o(t)$ and $q_i(t)$. For example, the relation between the current $i$ and the emf $e$ in an $LCR$ circuit is given by

$$e(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt$$

Taking Laplace transform, we get

$$\mathscr{L}\{e(t)\} \equiv E(s) = sLI(s) + RI(s) + \frac{I(s)}{sC}$$

Therefore, the transfer function,

$$G(s) = \frac{I(s)}{E(s)} = \frac{1}{sL + R + (1/sC)}$$

2. The transfer function does not give any insight about the structure of the system.
3. It offers a symbolic picture about the dynamic characteristics of the system as shown in Fig. 4.1.
4. If the transfer functions of individual components of the system are known, the overall characteristics of the system can be determined just by taking their product (Fig. 4.1), provided the loading effect between connected devices can be neglected.

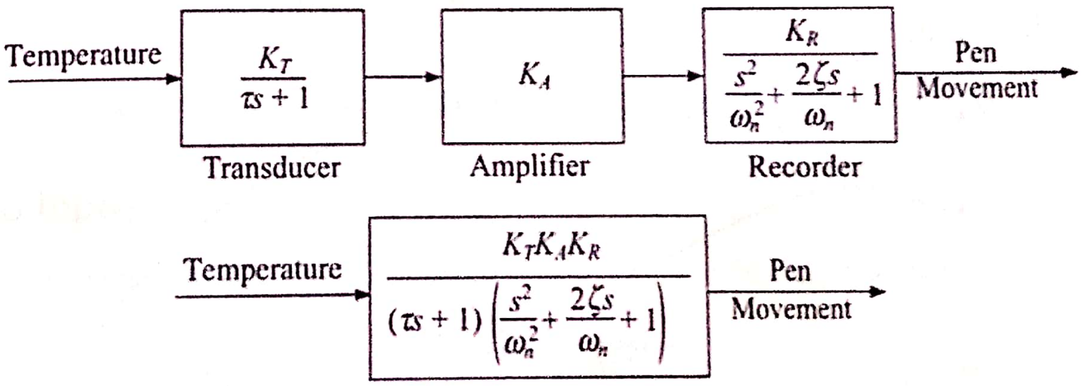


**Fig. 4.1** Individual transfer functions are multiplied to get the final transfer function.

So far while discussing dynamic characteristics, we have tacitly assumed that the inputs to the system are time-varying and we want to study the dynamic response of the system at different intervals of time. Such kind of a study is called a *time domain analysis*.

But time domain analysis is rather cumbersome and, of course, not necessary if the input varies periodically with time, such as $q_i = A_i \sin \omega t$. The output quantity $q_o$ in such cases will also be a sine wave, once the transients die out. The only changes that are expected are in the amplitude and the phase of the output. Since the input and output frequency are the same, the output is completely specified by giving the amplitude ratio $A_o/A_i$, and the phase shift angle $\phi$. Thus, the response of a system to a periodic input is completely studied if the amplitude ratio and phase shift are studied as a function of frequency (Fig. 4.2). This analysis is termed *frequency domain analysis*.

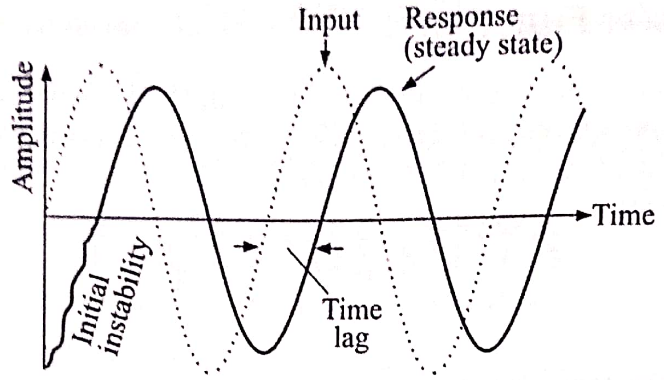


Fig. 4.2 A typical response of a system to a sinusoidal input

While a full-fledged time domain analysis with a sine wave input will also lead to the same results, much quicker and easier methods are offered by the concept of sinusoidal transfer function which is obtained simply by substituting $\jmath\omega$ for $s$ in the operational transfer function. Thus, the frequency domain transfer function for Eq. (4.2) is given by

$$G(\jmath\omega) = \frac{Q_o(\jmath\omega)}{Q_i(\jmath\omega)} = \frac{b_m(\jmath\omega)^m + \cdots + b_1\jmath\omega + b_0}{a_n(\jmath\omega)^n + \cdots + a_1\jmath\omega + a_0}$$

$G(\jmath\omega)$ for a given frequency $\omega$ is a complex quantity. Any complex quantity, $a + \jmath b$, can be expressed in the polar form $M\angle\phi$ where $M(= \sqrt{a^2 + b^2})$ is the magnitude and $\phi(= \tan^{-1}\frac{b}{a})$ is the angle. It can be proved that $M$ and $\phi$ corresponding to $G(\jmath\omega)$ equal the amplitude ratio and the leading phase angle, respectively. By leading phase angle we mean that if the output lags behind the input, $\phi$ is negative (Fig. 4.3).

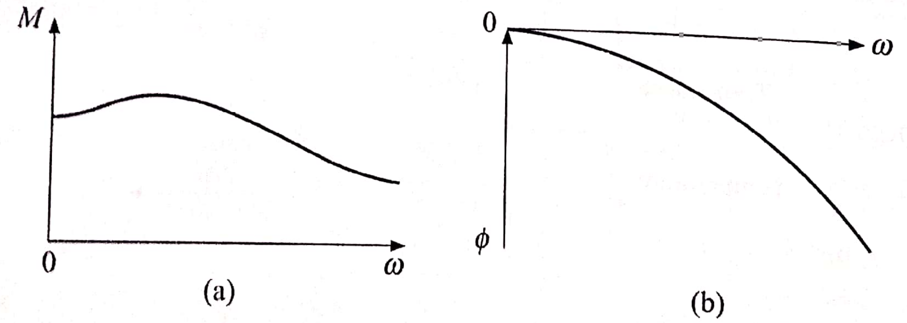


Fig. 4.3 Frequency response of system.

Thus, the sinusoidal transfer function for an $LCR$ circuit is

$$G(\jmath\omega) = \frac{X(\jmath\omega)}{F(\jmath\omega)} = \frac{1}{\jmath\omega L + R + (1/\jmath\omega C)}$$

## 4.2 Standard Inputs to Study Time Domain Response

Standard inputs generally used to study the dynamic behaviour of measurement systems and their Laplace transforms are as follows:

## Step Input

The functional form [Fig. 4.4(a)] is given by

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ A & \text{if } t > 0 \end{cases}$$

The Laplace transform of the step input is

$$\mathscr{L}\{f(t)\} = \frac{A}{s}$$

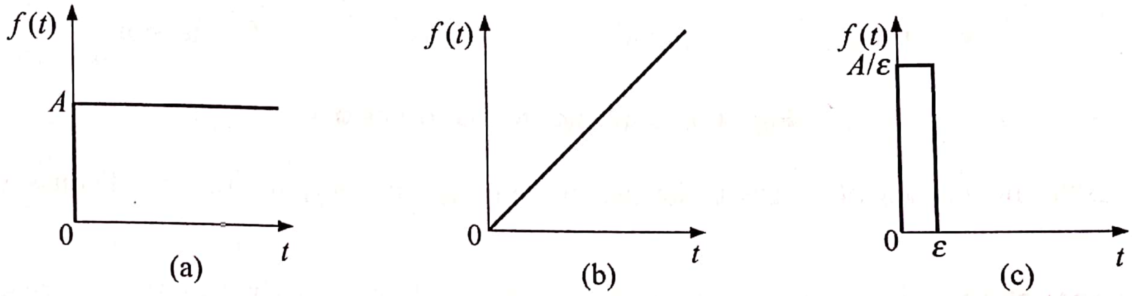


**Fig. 4.4** Standard inputs: (a) step function, (b) ramp function, and (c) impulse function.

## Ramp Input

The functional form [Fig. 4.4(b)] and the Laplace transform are

$$f(t) = At \qquad \text{and} \qquad F(s) = \frac{A}{s^2}$$

## Impulse Input

The impulse function, [Fig. 4.4(c)] related to the Dirac $\delta$-function is defined as

$$f(t) = A\delta(t)$$

where,

$$\delta(t) = \begin{cases} \lim\limits_{\varepsilon \to 0} \frac{1}{\varepsilon} & \text{for } 0 \le t \le \varepsilon \\ 0 & \text{for } t > \epsilon \end{cases}$$

By definition, $\int_0^\infty \delta(t)dt = 1$. Therefore, the corresponding Laplace transform[3] is $\mathscr{L}\{\delta(t)\} = 1$ Thus,

$$F(s) = A$$

Armed with this background knowledge we will now study dynamic responses of differen orders of instruments. But before that we will see what characteristics we want to watch. I other words, what are the dynamic characteristics of instruments.

[3]See Appendix C.1 at page 863 for a derivation.

## 4.3 Dynamic Characteristics

Like the static characteristics, dynamic characteristics can also be divided into two basic categories:

1. Desirable
2. Undesirable

The tree looks like Fig. 4.5.

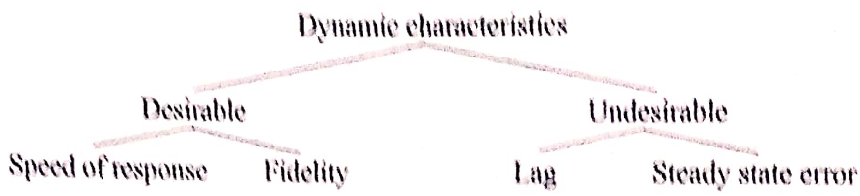


**Fig. 4.5** Dynamic characteristics tree.

Although the names of characteristics are self-explanatory, we briefly discuss what they mean.

**Speed or response.** The speed of response indicates how quickly the system reacts to the input signal.

**Fidelity.** The fidelity of a dynamic system denotes how faithfully the system outputs the input signal and what is the distortion, if any.

**Lag.** As the name implies, the lag indicates what time the system takes to output the input signal.

**Steady state error.** The steady state error is defined as

$$e_{ss} = \lim_{t \to \infty} e_m$$

where,

$$e_m = q_i - \frac{q_o}{K}$$

Here, $q_i$ is the input signal

$q_i/K$ is the normalised output

$K$ is the amplification factor

It will be seen that in the ultimate analysis, the desirable and undesirable characteristics all depend on the speed of response, which, in turn, is related to the time constant $\tau$ of the system.

We will study dynamic characteristics of instruments according to their order. The question is, what do we mean by the order of an instrumentation system?

### *Order of a system*

The order of a dynamic system is indicated by the highest power of the Laplace variable $s$ in the rationalised denominator of the corresponding transfer function.

Suppose, the transfer function of an instrumentation system is given by

$$G(s) = \frac{s+9}{s^2+2s+9}$$

Here, the denominator is given by

$$s^2+2s+9$$

which consists of a power of 2 for the Laplace variable $s$. Therefore, it represents a second order system. Now, the same transfer function can be written as

$$G(s) = \frac{1+(9/s)}{s+2+(9/s)}$$

Does it indicate that it is a first order system? The answer is obviously 'no' because the denominator contains a fraction.

*Note:* The power of $s$ being related to the degree of the differential equation describing the instrumentation system, actually it is the degree of the differential equation which determines the order of a dynamic system. But then, more often than not the systems are described by their transfer functions.

Let us now embark upon the study of different dynamic systems.

## 4.4 Zero Order Instrument

Suppose all $a$'s and $b$'s except $a_0$ and $b_0$ in Eq. (4.1) are zero, degenerating it to an algebraic equation

$$a_0 q_o(t) = b_0 q_i(t) \tag{4.3}$$

A *zero order* instrument is defined as one which closely obeys this equation over its range of operation. On rearranging Eq. (4.3), we get

$$q_o(t) = \frac{b_0}{a_0} q_i(t) = K q_i(t) \tag{4.4}$$

where $K = b_0/a_0$ = static sensitivity.

Equation (4.4) being an algebraic relation, it is apparent that the output $q_o$ faithfully follows the input $q_i$ with no distortion or time lag of any sort. The zero order instrument, therefore, may be considered as ideal having a perfect dynamic response.

A potentiometer used for measuring displacements[4] may be shown to be a zero-order instrument. In such an arrangement, a wire-wound resistance, provided with a sliding contact, is excited with a voltage. Assuming that the resistance is distributed linearly along its length $L$, we have

$$E_o = \frac{x}{L} E_i \equiv Kx \tag{4.5}$$

where $E_o$ and $E_i$ are the output and input voltages, $x$ is the displacement and $K$ is a constant.

[4]See Section 6.2 at page 173.

*Note:* We have made the following tacit assumptions to write Eq. (4.5):

1. The increase in resistance is continuous. But in actuality, for a wire-wound type potentiometer the wound wire has a finite diameter and hence the resistance increases in steps as the sliding contact (called *wiper*) moves (see Fig. 6.6 at page 175).
2. The winding is purely resistive, which is not true because all such windings have inductive and capacitive effects.
3. Electric loading by the voltage measuring instrument is negligible. In case there is loading, the relation will not be linear.
4. There is no mechanical loading by the sliding contact (i.e. wiper) and, therefore, no heat generation during sliding motions.

## 4.5 First Order Instrument

If the dynamic relation between the input and output of an instrument assumes the form

$$a_1\frac{dq_o(t)}{dt} + a_0q_o(t) = b_0q_i(t)$$

it is called a *first order* instrument. However, this relation can be written with two rather than three coefficients as follows:

$$\tau\frac{dq_o(t)}{dt} + q_o(t) = Kq_i(t) \tag{4.6}$$

Here, $K = b_0/a_0$ and $\tau = a_1/a_0$. $\tau$ is called the time constant because it has the dimension of time in physical processes.

Taking Laplace transform of Eq. (4.6), we get

$$\tau sQ_o(s) + Q_o(s) = KQ_i(s)$$

or

$$(1+\tau s)Q_o(s) = KQ_i(s)$$

Therefore, the transfer function is given by

$$G(s) \equiv \frac{Q_o(s)}{Q_i(s)} = \frac{K}{1+\tau s}$$

The familiar *RC* circuit is an example of a first order arrangement. Here the relation between $e_i$ (= voltage, input) and $Q$ (= charge, output) is given by,

$$R\frac{dQ}{dt} + \frac{Q}{C} = e_i$$

or

$$\tau\frac{dQ}{dt} + Q = Ke_i$$

where, $\tau = RC$ and $K = C$.

## Examples of First Order Instruments

### *Mercury-in-glass thermometer*

The common mercury-in-glass thermometer, (Fig. 4.6) behaves as a first order instrument as can be seen from the analysis given below.

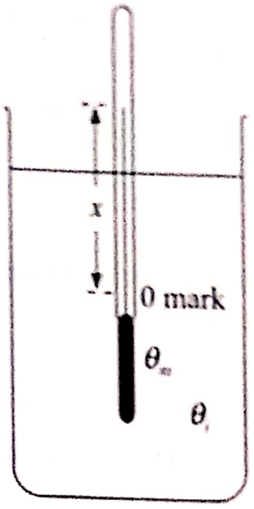


Fig. 4.6 Mercury-in-glass thermometer.

Let $V$ be the volume of the bulb

$A_b$ be the area of the bulb conducting heat

$\gamma_a$ be the coefficient of apparent expansion of mercury

$\theta_m$ be the temperature attained by mercury at any instant

$x$ be the corresponding height of the mercury column (output)

$A$ be the area of cross-section of the capillary tube

$K_g$ be the thermal conductivity of glass

$\rho$ be the density of mercury

$c$ be the specific heat of mercury

$\theta_i$ be the temperature of the liquid (input)

$\theta$ be the wall thickness of the glass bulb,

then $$x = \frac{\gamma_a V \theta_m}{A}$$

which gives $$\theta_m = \frac{Ax}{\gamma_a V}$$

Now, the heat conducted from the liquid to mercury through the bulb (i.e. heat lost by the liquid) during the interval $dt$ equals

$$\frac{K_g A_b}{\theta}(\theta_i - \theta_m)dt$$

The corresponding heat gained by mercury in the bulb is

$$(\rho c V)\, d\theta_m$$

Equating heat lost to heat gained and substituting the value of $\theta_m$, we get after some rearrangement

$$\tau \frac{dx}{dt} + x = K\theta_i$$

where,

$$\tau = \frac{\rho c V \theta}{K_g A_b}$$

$$K = \frac{\gamma_a V}{A}$$

### Thermocouple

The other common temperature measuring devices, such as thermocouple and thermistor, are also first order systems. To verify that, we consider a thermocouple which is dipped into a hot liquid. For simplicity, we assume that

(a) the heat transfer takes place only by conduction

(b) the emf vs temperature curve of the thermocouple is linear

(c) the other junction of the thermocouple is kept at room temperature

If $A$ is the heat transfer area of the thermocouple
$K_t$ is the thermal conductivity of the thermocouple material
$\theta$ is the temperature attained by the thermocouple at any instant
$\theta_i$ is the temperature of the hot liquid
$m$ is the mass of the thermocouple junction
$c$ is the specific heat of the junction material
$E$ is the developed emf in the thermocouple

we have

$$E = K\theta \tag{4.7}$$

where $K$ is a constant. Heat conducted from the liquid to the thermocouple junction during a small time interval $dt$ is

$$K_t A(\theta_i - \theta)dt$$

The corresponding heat gained by the thermocouple junction is

$$mc\, d\theta$$

Thus,

$$mc\, d\theta = K_t A(\theta_i - \theta)dt$$

which gives

$$\frac{mc}{K_t A}\frac{dE}{dt} + E = K\theta_i \quad \text{[applying Eq. (4.7)]} \tag{4.8}$$

or

$$\tau \frac{dE}{dt} + E = K\theta_i \tag{4.9}$$

where

$$\tau = \frac{mc}{K_t A}$$

Equation (4.9) shows that the thermocouple is a first order system.

## Dynamic Response of First Order Instruments

### *Step response*

As shown before, here $Q_i(s) = A/s$. Therefore,

$$Q_o(s) = G(s)Q_i(s) = \frac{KA}{s(\tau s+1)} = KA\left(\frac{1}{s} - \frac{\tau}{\tau s+1}\right) = KA\left(\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right) \tag{4.10}$$

Taking inverse Laplace transform of Eq. (4.10), we get

$$q_o(t) = KA[1 - \exp(-t/\tau)] \tag{4.11}$$

On non-dimensionalising, Eq. (4.11) becomes

$$\frac{q_o}{KA} = 1 - \exp(-t/\tau) \tag{4.12}$$

**Measurement error.** The measurement error $e_m$ is

$$e_m = q_i - \frac{q_o}{K} = A - A[1 - \exp(-t/\tau)]$$

$$\Rightarrow \quad \frac{e_m}{A} = \exp(-t/\tau) \tag{4.13}$$

**Steady-state error.** From Eq. (4.13) we find that the steady-state error for the step input of a first order instrument is

$$e_{ss} = \lim_{t\to\infty} e_m = 0$$

An idea about the growth of the output with time can be obtained from Table 4.1 which has been shown in graphical form in Fig. 4.7.

**Table 4.1** Values of non-dimensionalised parameters for step response of the first order instrument

| $t/\tau$ | 0 | 1 | 2 | 3 | 4 | 5 | $\infty$ |
|---|---|---|---|---|---|---|---|
| $q_o/KA$ | 0.000 | 0.632 | 0.865 | 0.950 | 0.982 | 0.993 | 1.000 |

From these analyses we can infer that

1. The speed of response depends only on the value of $\tau$.
2. The response reaches within 5% of its final value at $3\tau$.
3. The steady-state value can be assumed to have reached around $5\tau$.

Going back to the case of mercury-in-glass thermometer, we find that

1. Reducing $\tau$ means reducing $\rho, c$ and $V$, and increasing $K_g$ and $A_b$; but reducing $V$ means reducing $A_b$ as well!
2. $\rho, c$ can be reduced by choosing an appropriate fluid for the thermometer;
3. By lowering $V$, the static sensitivity is lowered. This, in turn, means that a fast responding thermometer is less sensitive.

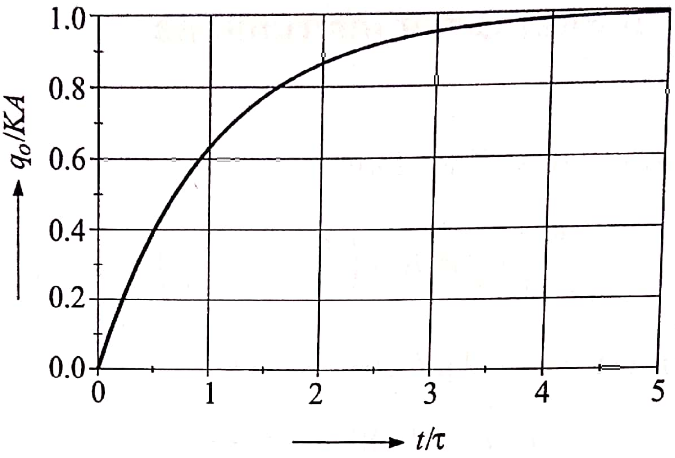


Fig. 4.7 Step response of the first order instrument.

**Example 4.1**

A thermometer, initially at 70°C, is suddenly dipped in a liquid at 300°C. After 3 s, the thermometer indicates 200°C. After what time is the thermometer expected to give a reliable reading, say well within 1% of the actual value?

Solution

This is obviously a case of a step input, the step being not from an initial zero value, but a finite value of $\theta_0 = 70$°C. So if we denote $\theta_3$ as the thermometer reading after 3 s, our equation is

$$\theta_3 = (300 - 70)\left[1 - \exp\left(-\frac{3}{\tau}\right)\right] + 70 = 200\,^\circ\text{C} \quad \text{(given)}$$

or
$$\exp\left(-\frac{3}{\tau}\right) = \frac{230 - 130}{230} = \frac{10}{23}$$

or
$$-\frac{3}{\tau} = \ln 10 - \ln 23 = -0.8329$$

or
$$\tau \cong 3.6 \text{ s}$$

Since a reliable reading can be obtained at $5\tau$, the required time is 18 s.

### *Ramp response*

Here $Q_i(s) = A/s^2$. Therefore,

$$Q_o(s) \equiv G(s)Q_i(s) = \frac{KA}{s^2(1 + \tau s)} = KA\left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{1 + \tau s}\right)$$

On taking inverse Laplace transform, we get

$$q_o(t) = KA\left[t - \tau\left\{1 - \exp\left(-\frac{t}{\tau}\right)\right\}\right] \tag{4.14}$$

**Measurement error.** The measurement error is

$$
\begin{aligned}
e_m &= At - A\left[t - \tau\left\{1 - \exp\left(-\frac{t}{\tau}\right)\right\}\right] \\
&= A\tau\left\{1 - \exp\left(-\frac{t}{\tau}\right)\right\} \\
&= \underbrace{-A\tau \exp\left(-\frac{t}{\tau}\right)}_{\text{transient error}} + \underbrace{A\tau}_{\text{steady-state error}}
\end{aligned}
\tag{4.15}
$$

**Steady-state error.** The steady-state error is

$$e_{ss} = \lim_{t \to \infty} e_m = A\tau$$

The steady-state error obviously depends on $\tau$ which means that a small $\tau$ instrument is desirable.

**Lag.** An intriguing revelation is that the instrument reading always lags behind the actual value as if the instrument shows a value what the input was $\tau$ seconds ago (see Fig. 4.8). The situation will be clear from Example 4.2

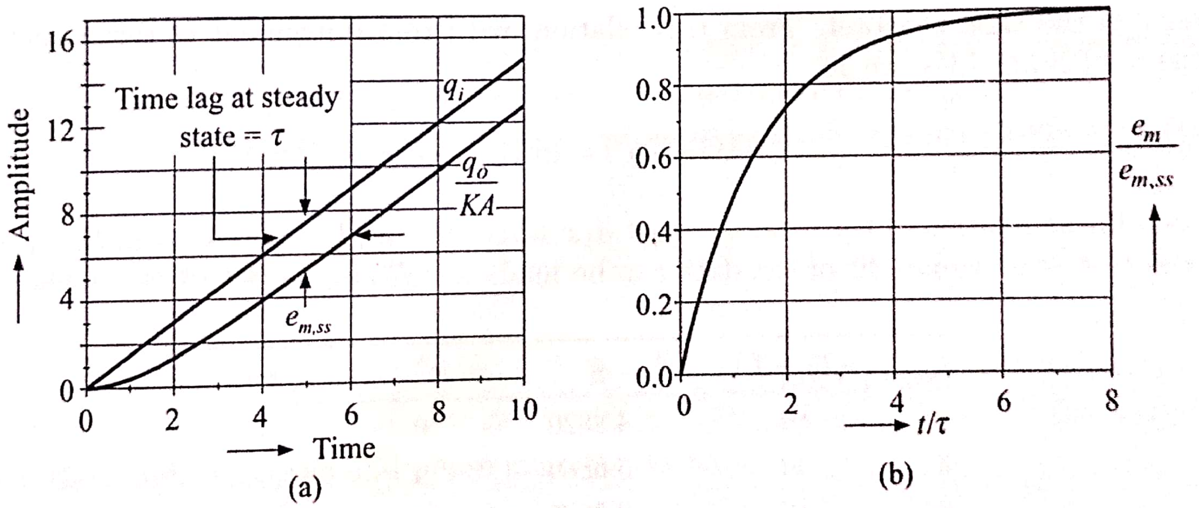


Fig. 4.8 Ramp response of the first order instrument: (a) actual response, and (b) error plot.

**Example 4.2**

A balloon carrying a first order thermometer ($\tau$ = 10 s) rises through the atmosphere at the rate of 10 m/s and radios the temperature and altitude readings back to the ground. At 3500 m the balloon says the temperature is 0°C. What is the true altitude at which 0°C occurs, provided the variation of temperature with altitude is 0.15°C per 30 m?

Solution

We assume that

(i) there is no lag in the altimeter reading,
(ii) the time lag for radio wave propagation is negligibly small, and
(iii) the only lag is in the thermometer reading.