Chapter 3

Non-linear Partial Differential Equations of Order One

Relevant Information on

1. Meanings of Various Classes of Integrals as applied to solutions of first order P.D.E.
2. General Method of equations of order one but of any degree: **Charpit's Method.**

3.1 Definitions of Various Classes of Integrals

1. A **solution** or an **integral** of a differential equation is a relation between the variables involved, by means of which and the derivatives obtained therefrom, the given differential equation is satisfied.

Below are given definitions of various classes of integrals of a partial differential equation of order one.

2. Complete Integral: Singular Integral

Let the non-linear partial differential equation of order one

$$f(x, y, z, p, q) = 0 \qquad (3.1.1)$$

be derived from

$$\phi(x, y, z, a, b) = 0 \tag{3.1.2}$$

by eliminating the arbitrary constants a and b. Then (3.1.2) is called the **complete integral** or **complete solution** of (3.1.1).

This complete solution represents a two-parameter family of surfaces which may or may not have an envelope. To find the envelope (when exists) we eliminate a and b from

$$\left.\begin{aligned} \phi(x, y, z, a, b) &= 0 \\ \frac{\partial \phi}{\partial a} = 0; \quad \frac{\partial \phi}{\partial b} &= 0 \end{aligned}\right\}$$

If the eliminant

$$\lambda(x, y, z) = 0 \tag{3.1.3}$$

satisfies (3.1.1), it is called the **singular solution** of (3.1.1). If $\lambda(x, y, z) = \xi(x, y, z) + \eta(x, y, z)$ where $\xi(x, y, z) = 0$ satisfies (3.1.1) while $\eta(x, y, z) = 0$ does not, then $\eta(x, y, z) = 0$ is the singular solution of (3.1.1). As in case of ordinary differential equations, the singular solution may be obtained from the P.D.E. by eliminating p and q from

$$\left.\begin{aligned} f(x, y, z, p, q) &= 0 \\ \frac{\partial f}{\partial p} = 0, \quad \frac{\partial f}{\partial q} &= 0 \end{aligned}\right\}$$

► **Example 3.1.1** $\phi = z - ax - by + (a^2 + b^2) = 0$ *is a* ***complete solution*** *of* $f = z - px - qy + (p^2 + q^2) = 0$ *(can be easily checked)*.

Eliminating a and b from

$$\left.\begin{aligned} \phi &= z - ax - by + a^2 + b^2 = 0 \\ \frac{\partial \phi}{\partial a} &= -x + 2a = 0; \quad \frac{\partial \phi}{\partial b} = -y + 2b = 0 \end{aligned}\right\}$$

we get

$$z = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{4}(x^2 + y^2) = \frac{1}{4}(x^2 + y^2).$$

This satisfies the differential equation and is the **singular solution**.

See that the **complete solution** represents a two-parameter family of planes which envelopes the paraboloid $x^2 + y^2 = 4z$.

3. **General solution:** If, in the complete solution (3.1.2) $\phi(x, y, z, a, b) = 0$ one of the constants, say b, is replaced by a known function of the other, say $b = \theta(a)$, then

$$\phi(x, y, z, a, \theta(a)) = 0$$

is a one-parameter family of the surfaces of (3.1.1).

If this family has an envelope, its equation may be found as usual by eliminating a from

$$\phi(x, y, z, a, \theta(a)) = 0 \text{ and } \frac{\partial}{\partial a}\phi(x, y, z, a, \phi(a)) = 0$$

and determining that part of the result which satisfies (3.1.1).

▶ **Example 3.1.2** *Let* $b = \theta(a) = a$ *in the complete solution of* Example (3.1.1), *given above.*

3.2 General Method of Solving the P.D.E. of First Order in Two Independent Variables x and y: Charpit's Method

Let

$$F(x, y, z, p, q) = 0 \tag{3.2.1}$$

be a given P.D.E. of first order in two independent variables x and y; z is a function of x and y; $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$. Since z depends on x and y, therefore

$$dz = pdx + qdy. \tag{3.2.2}$$

Now, if another relation can be found between x, y, z, p, q such as

$$f(x, y, z, p, q) = 0 \tag{3.2.3}$$

then p and q can be eliminated:

The values of p and q deduced from (3.2.1) and (3.2.3) can be substituted in (3.2.2) and the elimination of p and q is then possible.

The integral of the O.D.E. thus formed involving x, y, z will satisfy the given equation (3.2.1).

The problem thus reduces to find a relation of the form (3.2.3) which together with (3.2.1) will determine p and q that will render (3.2.2) integrable.

On differentiating (3.2.1) and (3.2.3) with respect to x and y we shall obtain the following equations:

$$\left.\begin{aligned}&\text{(i)}\quad \frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0\\&\text{(ii)}\quad \frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0\end{aligned}\right\}\ F \textbf{ with respect to } x$$

$$\left.\begin{aligned}&\text{(iii)}\quad \frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0\\&\text{(iv)}\quad \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0\end{aligned}\right\}\ f \textbf{ with respect to } y$$

The elimination of $\frac{\partial p}{\partial x}$ from (i) and (iii) gives

$$\left(\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial x}\right)+p\left(\frac{\partial F}{\partial y}\frac{\partial f}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial z}\right)+\frac{\partial q}{\partial x}\left(\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)=0;$$

and the elimination of $\frac{\partial q}{\partial y}$ from (ii) and (iv) gives

$$\left(\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial y}\right)+q\left(\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial z}\right)+\frac{\partial p}{\partial y}\left(\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)=0.$$

On adding the L.H.S. of these two equations we see that the last bracketed terms cancel each other, since

$$\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial p}{\partial y}.$$

Hence adding and re-arranging we get

$$\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial p}+\left(\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial q}+\left(-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial z}+\left(-\frac{\partial F}{\partial p}\right)\frac{\partial f}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial y}=0. \qquad (3.2.4)$$

This is a linear equation of first order, which the auxiliary function f of equation (3.2.3) must satisfy. Its integrals are integrals of the auxiliary equations:

$$\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}=\frac{df}{0}. \tag{3.2.5}$$

The equations (3.2.5) are known as **Charpit's auxiliary equations.** Any integrals of these equations involving p or q or both can be traken for the required second relation (3.2.3). Actually the simplest relation involving p or q or both is taken as relation (3.2.3). After obtaining the relation (3.2.3) p and q are obtained from (3.2.1) and (3.2.3) and these values are substituted in (3.2.2). On integrating it we get the required complete solution of the given equation.

▶ **Example 3.2.1** *Find a complete integral of $px+qy=pq$.*

Solution: Here the given equation is

$$F(x,y,z,p,q)=px+qy-pq=0. \tag{1}$$

∴ Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}$$

or,

$$\frac{dp}{p+p\cdot 0}=\frac{dq}{q+q\cdot 0}=\frac{dz}{-p(x-q)-q(y-p)}=\frac{dx}{-(x-q)}=\frac{dy}{-(y-p)}. \tag{2}$$

Taking the first two fractions

$$\frac{dp}{p}=\frac{dq}{q} \Rightarrow \log p=\log q+\log a$$

$$\Rightarrow p=aq \quad (a \text{ is an arb. constant.}) \tag{3}$$

Substituting this value of p in (1) we get

$$aqx+qy-aq^2=0$$

or,

$$aq=ax+y \quad (q\neq 0). \tag{4}$$

From (3) and (4),

$$q=\frac{ax+y}{a}, \quad p=ax+y. \tag{5}$$

Putting these values of p and q in

$$dz = pdx + qdy$$

we get

$$\begin{aligned} dz &= (ax+y)dx + \frac{ax+y}{a}dy \\ \text{or,}\quad adz &= a(ax+y)dx + (ax+y)dy \\ \text{or,}\quad adz &= (ax+y)d(ax+y). \end{aligned}$$

Integrating

$$az = \frac{(ax+y)^2}{2} + b \quad (b \text{ is an arb. constant})$$

which is a complete integral, a, b being arbitrary constants.

▶ **Example 3.2.2** *Find a complete integral of* $q = 3p^2$.

Solution: Here the given equation is

$$f(x,y,z,p,q) = 3p^2 - q = 0. \tag{1}$$

$\therefore$ Charpit's auxiliary equations are

$$\begin{aligned} \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} &= \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}} \\ \text{or,}\quad \frac{dp}{0+p\cdot 0} &= \frac{dq}{0+q\cdot 0} = \frac{dz}{-6p^2+q} = \frac{dx}{-6p} = \frac{dy}{1}. \end{aligned} \tag{2}$$

Taking the first fraction of (2), $dp = 0$ so that

$$p = a. \tag{3}$$

Substituting this value of p in (1) we get

$$q = 3a^2. \tag{4}$$

Putting these values of p and q in $dz = pdx + qdy$, we obtain

$$dz = adx + 3a^2dy$$

so that

$$z = ax + 3a^2y + b \quad (a, b \text{ are arbitrary constants}).$$

This is a complete integral.

▶ **Example 3.2.3** *Verify that a complete integral of $z = pq$ (by using Charpit's Method) is $2\sqrt{z} = x\sqrt{a} + \left(\frac{1}{\sqrt{a}}\right)y + b$, $(a, b$ are arbitrary constants).*

Try yourself.

▶ **Example 3.2.4** *Given $F(x, y, z, p, q) = zpq - p - q = 0$, verify that Charpit's auxiliary equations are*

$$\frac{dp}{p^2q} = \frac{dq}{pq^2} = \frac{dz}{-p(qz-1) - q(pz-1)} = \frac{dx}{-(qz-1)} = \frac{dy}{-(pz-1)}.$$

Now take first two fractions, obtain $p = \frac{1+a}{z}$ and $q = \frac{1+a}{az}$ and obtain a complete integral

$$z^2 = 2(1+a)\left[x + \frac{1}{a}y\right] + b$$

▶ **Example 3.2.5** *Verify that a complete integral of $p^2 - y^2q = y^2 - x^2$ is*

$$z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{a^2}{y} - y + b.$$

[Take the fractions, $\frac{dp}{2x} = \frac{dx}{-2p}$ and proceed as above.]

▶ **Example 3.2.6** *Find a complete integral of $z^2(p^2z^2 + q^2) = 1$.*

[I.A.S. 1997]

Here $F(x, y, z, p, q) = p^2z^4 + q^2z^2 - 1 = 0$.

Charpit's auxiliary equations are

$$\frac{dp}{p(4p^2z^3 + 2zq^2)} = \frac{dq}{q(4p^2z^3 + 2zq^2)} = \frac{dz}{-2p^2z^4 - 2q^2z^2}$$
$$= \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2}.$$

Taking the first two fractions, $\frac{dp}{p} = \frac{dq}{q}$ so that $p = aq$.
Solving for p and q,

$$p = \frac{a}{z\sqrt{a^2z^2+1}}, \quad q = \frac{1}{z\sqrt{a^2z^2+1}}$$

so that $dz = pdx + qdy \quad \Rightarrow \quad adx + dy = z\sqrt{a^2z^2+1}dz$.

Integrating,

$$ax + y = \int (a^2z^2+1)^{1/2} \cdot z\,dz$$

$$ax + y + b = \frac{1}{3a^2}(a^2z^2+1)^{3/2}, \quad (\text{putting } a^2z^2+1 = t^2)$$

or, $$9a^4(ax+y+b)^2 = (a^2z^2+1)^3$$

which is a complete integral, a and b being arbitrary constants.

▶ **Example 3.2.7** *Find a complete integral of*

$$\text{(i) } q = (z+px)^2; \qquad \text{(ii) } p = (z+qy)^2.$$

(i) Here the given equation is $F(x,y,z,p,q) = (z+px)^2 - q = 0$
Charpit's auxiliary equations are

$$\frac{dp}{2p(z+px)+2p(z+px)} = \frac{dq}{2q(z+px)} = \frac{dz}{-2px(z+px)+q}$$
$$= \frac{dx}{-2x(z+px)} = \frac{dy}{0}.$$

Taking the second and fourth fractions, $\frac{dq}{q} = -\frac{dx}{x}$.

Integrating $\log q = \log a - \log x$ so that $q = a/x$.

Hence from the given equation $p = \frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}$.

$$\therefore \quad dz = pdx + qdy = \left(\frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}\right)dx + \frac{a}{x}dy$$

or, $$xdz + zdx = \sqrt{a}\frac{dx}{\sqrt{x}} + ady$$

or, $$d(xz) = \sqrt{a}x^{-\frac{1}{2}}dx + ady.$$

Integrating $xz = 2\sqrt{a}\sqrt{x} + ay + b$ (a, b being arbitrary constants)

(ii) Similar method: A complete integral: $yz = ax + \sqrt{ay} + b$.

▶ **Example 3.2.8** *Solve for a complete integral of* $yzp^2 - q = 0$.

Try yourself.

Solution: $z^2(a^2 - y^2) = (x+b)^2$ **or** $z^2 = 2ax + a^2y^2 + b$.

▶ **Example 3.2.9** *Find a complete integral, a singular solution and general solution of* $(p^2 + q^2)y = qz$. **[Delhi B.Sc. Hons 1989]**

Solution: Here the given equation is

$$F(x, y, z, p, q) = (p^2 + q^2)y - qz = 0. \quad (1)$$

$\therefore$ Charpit's auxiliary equations are

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y + qz - 2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2q + z}. \quad (2)$$

Taking the first two fractions, we get $pdp + qdq = 0$ and hence (on integrating)

$$p^2 + q^2 = a^2 \quad (3)$$

(3) and (1) give $q = \frac{a^2y}{z}$ and $p = \frac{a}{z}\sqrt{z^2 - a^2y^2}$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}dx + \frac{a^2y}{z}dz$$

or,

$$\frac{z\ dz - a^2y\ dy}{\sqrt{(z^2 - a^2y^2)}} = adx.$$

Integrating, $(z^2 - a^2y^2)^{1/2} = ax + b$

$$z^2 - a^2y^2 = (ax + b)^2 \quad (4)$$

which is a required **complete integral.**

To find **singular integral** we differentiate this complete integral partially w.r.t. a and b, and obtain

$$0 = 2ay^2 + 2(ax + b) \cdot x \quad (5)$$

$$0 = 2(ax + b). \quad (6)$$

Eliminating a and b between (4), (5) and (6) we get $z = 0$ which clearly satisfies (1) ($\because\ p = 0, q = 0$) and hence it is the **singular integral.**

General integral: Replacing b by some function of a, say $b = \phi(a)$ in (4) we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2. \tag{7}$$

Differentiating (7) partially w.r.t. a we get

$$-2ay^2 = 2[ax + \phi(a)][x + \phi'(a)]. \tag{8}$$

The general integral is obtained by eliminating from a (7) and (8).

▶ **Example 3.2.10** *Find a complete and singular integrals of*

$$2xz - px^2 - 2qxy + pq = 0.$$

[Delhi Hons. 1998, 2000]

Solution: Here the given equation is

$$F = 2xz - px^2 - 2qxy + pq = 0. \tag{1}$$

$\therefore$ Charpit's auxiliary equations are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

The second fraction gives $dq = 0$ **or** $q = a$.

Putting $q = a$ in (1) we get $p = \frac{2x(z-ay)}{x^2-a}$.

Hence from $dz = pdx + qdy$ we deduce

$$dz = \frac{2x(z - ay)}{x^2 - a}dx + ady \quad \textbf{or,} \quad \frac{dz - ady}{z - ay} = \frac{2x\ dx}{x^2 - a}.$$

Hence, on integration,

$$\log(z - ay) = \log(x^2 - a) + \log b$$

or, $z - ay = b(x^2 - a)$ or, $z = ay + b(x^2 - a)$ (2)

is the **required complete integral**, a, b being arbitrary constants.

Differentiating the complete integral w.r.t. a and b we get

$$0 = y - b \text{ and } 0 = x^2 - a, \text{ i.e., } a = x^2, b = y$$

substituting these values of a and b in (2) we get $z = x^2 y$ which is the **required singular integral.**

Examples III

(Exercises based on Charpit's Method)

Find a complete integral of the following partial differential equations: (Using Charpit's auxiliary equations)

1. $q = px + p^2$.
2. $pxy + pq + qy = yz$.
3. $p^2 + q^2 - 2px - 2qy + 1 = 0$. **[I.A.S. 1999]**
4. $z = px + qy + p^2 + q^2$. **[I.A.S. 1996]**
5. $p^2 + q^2 - 2px - 2qy + 2xy = 0$.
6. $p^2 x + q^2 y = z$.
7. $2x(q^2 z^2 + 1) = pz$. **[I.A.S. 1998]**
8. $2z + p^2 + qy + 2y^2 = 0$.
9. $(p + q)(px + qy) = 1$.
10. $2(z + px + qy) = yp^2$.
11. $p - 3x^2 = q^2 - y$.
12. $p(q^2 + 1) + (b - z)q = 0$.

Answers

1. $z = \frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+4a} + 2a\log\{x + \sqrt{x^2 - 4a}\}\right] + ay + b$. **2.** $(z - ax)(y + a)^a = bc^y$. **3.** $(a^2 + 1)z = \frac{1}{2}v^2 \pm \left\{\frac{1}{2}v\sqrt{v^2 - (a^2+1)}\right\} - \frac{1}{2}(a^2 + 1)\log\left\{v + \sqrt{v^2 - (a^2+1)}\right\} + b$ where $v = ax + y$. **4.** $z = ax + by + a^2 + b^2$. **5.** $2z = x^2 + y^2 + ax + ay \pm \frac{1}{\sqrt{2}}\left\{(x - y)\sqrt{(x-y)^2 - \frac{a^2}{2}}\right\} - \frac{a^2}{2}\log\left[(x - y) + \sqrt{(x+y)^2 - \frac{a^2}{2}}\right]$. **6.** $(1 + a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$. **7.** $z^2 + 2(a^2 + 1)x^2 + 2ay + b$. **8.** $2y^2 z + y^2(x - a)^2 + y^4 = b$. **9.** $z(a + 1)^{1/2} = 2(ax + b)^{1/2} + b$. **10.** $yz - a(x/y) + (a^2/4y^2) = b$. **11.** $z = x^2 + ax \pm \frac{2}{3}(y + a)^{3/2} + b$. **12.** $2\sqrt{a(z - b) - 1} = x + ay + b$.