Real Number System

~~Natural Numbers~~ :- We go from rationals to reals by two ways which are due to two famous mathematicians — Dedkind & Cantor.

Real Numbers :- The set consisting all rational as well as irrational numbers is called the set of all real numbers. The set of all real numbers is denoted by $\mathbb{R}$.

Properties of Real number system —

~~The pr~~

1. Algebric properties (Field Axioms)
2. Order Properties
3. Order Completeness / Completeness property
4. Archimedan Property of $\mathbb{R}$.
5. Density property of $\mathbb{R}$.

These properties are ~~call~~ also called defining properties of the real number system.

Real number system is the only system which satisfies all these properties.

Algebric Properties (Field Axioms)

$+, \cdot \rightarrow$ binary operations.

These operations satisfy some properties those are called algebric properties.

1. a ~~Properties w.r.to '+'~~

1. $a+b \in \mathbb{R} \quad \forall a,b \in \mathbb{R}$ ⇒ closure

2. $(a+b)+c = a+(b+c) \quad \forall a,b,c \in \mathbb{R}$

   ⇒ associativity w.r.to '+'

3. $\exists\, 0 \in \mathbb{R}$ s.t.

   $a+0 = a = 0+a \quad \forall a \in \mathbb{R}$

   ⇒ existence of identity w.r.to '+'

4. $\forall a \in \mathbb{R} \; \exists\, b \in \mathbb{R}$ s.t.

   $a+b=0$ & $b+a=0$

   ⇒ existence of inverse w.r.to '+'.

   $b=-a$, additive inverse of a.

Any set with ~~the~~ binary operation & with these properties is called group.

So, $(\mathbb{R},+)$ is a group.

5. $a+b = b+a \quad \forall a,b \in \mathbb{R}$

   ⇒ commutativity

~~6. a. Properties w.r.to '.'~~

6. $a \cdot b \in \mathbb{R}, \forall a,b \in \mathbb{R}$

7. $a\cdot(b\cdot c) = (a\cdot b)\cdot c \quad \forall a,b,c \in \mathbb{R}$

8. $a\cdot(b+c) = a\cdot b + a\cdot c \quad \forall a,b,c \in \mathbb{R}$

   ⇒ distributivity

9. $a\cdot b = b\cdot a \quad \forall a,b \in \mathbb{R}$

   ⇒ ~~co~~

10. $\exists\, 1 \in \mathbb{R}$ s.t. $1\cdot a = a = a\cdot 1 \quad \forall a \in \mathbb{R}$

11. $\forall a \in \mathbb{R}$ with $a \neq 0$, $\exists\, b \in \mathbb{R}$ s.t. $ab = 1 = ba$.

Theorem 2.4.2. Let $a, b, c \in \mathbb{R}$. Then

(i) $a + b = a + c$ implies $b = c$ (cancellation law for addition);

(ii) $a \neq 0$ and $a.b = a.c$ implies $b = c$ (cancellation law for multiplication).

Proof. (i) $a + b = a + c$.

$-a \in \mathbb{R}$, since $a \in \mathbb{R}$. Therefore $-a + (a + b) = -a + (a + c)$
or, $(-a + a) + b = (-a + a) + c$, by A2
or, $0 + b = 0 + c$, by A4
or, $b = c$.

(ii) $a.b = a.c$.

$\frac{1}{a} \in \mathbb{R}$, since $a \neq 0$. Therefore $(\frac{1}{a}).(a.b) = (\frac{1}{a}).(a.c)$
or, $(\frac{1}{a}.a).b = (\frac{1}{a}.a).c$, by M2
or, $1.b = 1.c$, by M4
or, $b = c$.

Theorem 2.4.3. Let $a \in \mathbb{R}$. Then

(i) $a.0 = 0$,

(ii) $(-1).a = -a$,

(iii) $-(-a) = a$,

(iv) $1/(1/a) = a$, provided $a \neq 0$.

Proof. (i) We have $0 + 0 = 0$ in $\mathbb{R}$.

Therefore $a.(0 + 0) = a.0$
or, $a.0 + a.0 = a.0$, by D
$-(a.0) \in \mathbb{R}$. Therefore $-(a.0) + [a.0 + a.0] = (-a.0) + a.0$
or, $[-(a.0) + a.0] + a.0 = 0$, by A2 and A4
or, $0 + a.0 = 0$, by A4
or, $a.0 = 0$, by A3.

(ii) We have $1 + (-1) = 0$ in $\mathbb{R}$.
Then $[1 + (-1)].a = 0$

or, $a + (-1).a = 0$
$-a \in \mathbb{R}$. Therefore $-a + [a + (-1).a] = -a + 0$
or, $(-a + a) + (-1).a = -a$, by A2 and A3
or, $0 + (-1).a = -a$, by A4
or, $(-1).a = -a$, by A3.

(iii) We have $a + (-a) = 0$, by A4.
Since $-a \in \mathbb{R}, -a + \{-(-a)\} = 0$, by A4.
Therefore $-a + a = -a + \{-(-a)\}$.
or, $a = -(-a)$, by cancellation law for addition.

(iv) Since $a \neq 0, \frac{1}{a} \in \mathbb{R}$ and $a.(\frac{1}{a}) = 1$.
$a.\frac{1}{a} = 1 \Rightarrow \frac{1}{a} \neq 0$, because $\frac{1}{a} = 0 \Rightarrow 1 = 0$.
Since $\frac{1}{a} \neq 0, 1/(1/a) \in \mathbb{R}$ and $\frac{1}{a}.\{1/(1/a)\} = 1$.
Therefore $\frac{1}{a}.a = \frac{1}{a}.\{1/(1/a)\}$.
Since $\frac{1}{a} \neq 0$, $a = 1/(1/a)$, by cancellation law for multiplication.

Theorem 2.4.4. Let $a, b, c \in \mathbb{R}$. Then $a.b = 0$ implies $a = 0$, or $b = 0$.

Proof. Let $a \neq 0$. Then $\frac{1}{a} \in \mathbb{R}$ and $\frac{1}{a}.a = 1$.
$a.b = 0 \Rightarrow \frac{1}{a}.(ab) = \frac{1}{a}.0 \Rightarrow (\frac{1}{a}.a).b = 0 \Rightarrow b = 0$.
Therefore $a \neq 0 \Rightarrow b = 0$. Contrapositively, $b \neq 0 \Rightarrow a = 0$.

Therefore either $a = 0$ or $b = 0$.

Theorem 2.4.5. Let $a, b \in \mathbb{R}$. Then

(i) $a.(-b) = (-a).b = -(a.b)$,

(ii) $(-a).(-b) = a.b$.

Proof. We have $b + (-b) = 0$ in $\mathbb{R}$.
Therefore $a.[b + (-b)] = a.0$.
or, $a.b + a.(-b) = 0$, by D and theorem 2.4.3 (i)
$-(a.b) \in \mathbb{R}$. Therefore $-(a.b) + [a.b + a.(-b)] = -(a.b)$.
or, $[-(a.b) + a.b] + a.(-b) = -(a.b)$, by A2
or, $0 + a.(-b) = -(a.b)$, by A4
or, $a.(-b) = -(a.b)$, by A3.

Again $-a + a = 0$.
Therefore $[-a + a].b = 0.b$.
Proceeding similarly, we can prove $(-a).b = -(a.b)$.

Therefore $a.(-b) = (-a).b = -(a.b)$.

(ii) Let $p = -a$. Then $p \in \mathbb{R}$.
$(-a).(-b) = p.(-b) = -(p.b)$, by (i)
$= -[(-a).b] = -(-(a.b)) = a.b$, by theorem 2.4.3 (iii).

Order Properties

(i) Given $a, b \in \mathbb{R}$
$a \leq b$ or $b \leq a$

(ii) If $a, b \in \mathbb{R}$ and $a \leq b$ then $a+c \leq b+c$ for every $c \in \mathbb{R}$.

Q. $\alpha a \leq \alpha b$ ~~when $\alpha \geq 0$~~

(iii) If ~~and~~ $a, b \in \mathbb{R}$ & $0 \leq \alpha$, and $a \leq b$ then
$\alpha a \leq \alpha b$.

Any field in which such a order '$\leq$' is defined and which satisfies these three properties, is called an ordered field.

(iv) $a < b$ and $b < c \Rightarrow a < c$ $\forall a, b, c \in \mathbb{R}$

- $a \leq b \Rightarrow b \geq a$
  '$\geq$' is basically reverse of '$\leq$'
- $a < b \Rightarrow a \leq b$ but $a \neq b$
- $0 \leq x \Rightarrow x$ is non-negative
- $0 < x \Rightarrow x$ is called positive number
- $x < 0 \Rightarrow x$ is called negative number
- $x \leq 0 \Rightarrow x$ is called non-positive number
- $\mathbb{R}_+ = \{x \in \mathbb{R} / 0 \leq x\} \Rightarrow$ set of all non negative number
- $\mathbb{R}_- = \{x \in \mathbb{R} / x \leq 0\} \Rightarrow$ set of all non-positive number

of all positive real numbers

Theorem 2.4.7. Let $a \in \mathbb{R}$. Then

(i) $a > 0 \Rightarrow -a < 0$;

(ii) $a < 0 \Rightarrow -a > 0$.

Proof. (i) $a \in \mathbb{R}$ and $a + (-a) = 0$, by A4.

By the law of trichotomy, $-a < 0$, or $-a = 0$, or $-a > 0$.

Let $-a > 0$.

$-a > 0$ and $a \in \mathbb{R} \Rightarrow -a + a > a$, by O3

$\Rightarrow 0 > a$, a contradiction.

Let $-a = 0$. Then $a + (-a) = a + 0 = a$,

and also $a + (-a) = 0$, by A4.

Therefore $a = 0$, a contradiction.

We conclude that $-a < 0$.

(ii) Similar proof.

Theorem 2.4.8. Let $a, b \in \mathbb{R}$. Then

(i) $a > 0, b > 0 \Rightarrow a + b > 0$,

(ii) $a < 0, b < 0 \Rightarrow a + b < 0$,

(iii) $a > 0, b > 0 \Rightarrow ab > 0$,

(iv) $a < 0, b < 0 \Rightarrow ab > 0$,

(v) $a > 0, b < 0 \Rightarrow ab < 0$.

Proof. (i) $a > 0$ and $b \in \mathbb{R} \Rightarrow a + b > b$, by O3

$a + b > b$ and $b > 0 \Rightarrow a + b > 0$, by O2.

(ii) Similar proof.

(iii) $a > 0, b > 0 \Rightarrow a.b > 0.b$, by O4

i.e., $ab > 0$.

(iv) $a < 0, b < 0 \Rightarrow a < 0, -b > 0$
$\Rightarrow a.(-b) < 0(-b)$, by O4
$\Rightarrow -ab < 0$
$\Rightarrow -(-ab) > 0$, by Theorem 2.4.7 (ii)
$\Rightarrow ab > 0$, by Theorem 2.4.3 (iii).

(v) Similar proof.

Theorem 2.4.9. Let $a, b, c, d \in \mathbb{R}$ and $a > b$, $c > d$. Then $a + c > b + d$.

Proof. $a > b$ and $c \in \mathbb{R} \Rightarrow a + c > b + c$, by O3.
$c > d$ and $b \in \mathbb{R} \Rightarrow b + c > b + d$, by O3.
$a + c > b + c$ and $b + c > b + d \Rightarrow a + c > b + d$, by O2.

Corollary. Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n \in \mathbb{R}$ and $a_i > b_i$ for $i = 1, 2, \ldots, n$.

Then $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n$.

Theorem 2.4.10. Let $a, b, c, d \in \mathbb{R}$ and $a > 0, b > 0, c > 0, d > 0$. Then $a > b, c > d \Rightarrow ac > bd$.

Proof. $a > b$ and $c > 0 \Rightarrow ac > bc$, by O4.
$c > d$ and $b > 0 \Rightarrow bc > bd$, by O4.
$ac > bc$ and $bc > bd \Rightarrow ac > bd$, by O2.

Corollary 1. Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n \in \mathbb{R}$ and $a_i > 0, b_i > 0$ for $i = 1, 2, \ldots, n$.

Then $a_i > b_i \Rightarrow a_1 a_2 \ldots a_n > b_1 b_2 \ldots b_n$.

Corollary 2. Let $a, b \in \mathbb{R}$ and $a > b > 0$. Then $a^n > b^n$ for all $n \in \mathbb{N}$.

Theorem 2.4.11. If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.

Proof. Since $a \neq 0$, either $a < 0$ or $a > 0$, by O1.

Case I. Let $a < 0$. Then $-a > 0$, by Theorem 2.4.7 (ii)
By O4, $a. - a < 0. - a$. Therefore $-a^2 < 0$.
This implies $a^2 > 0$, by Theorem 2.4.7 (ii)

Case II. Let $a > 0$.
By O4, $a.a > a.0$. Therefore $a^2 > 0$.

Combining the cases, we have $a^2 > 0$ for all $a \neq 0$.

Corollary. $1 > 0$, since $1 = 1.1 = 1^2$.

Theorem 2.4.12. Let $a \in \mathbb{R}$. Then

(i) $a > 0 \Rightarrow \frac{1}{a} > 0$, (ii) $a < 0 \Rightarrow \frac{1}{a} < 0$.

Proof left to the reader.