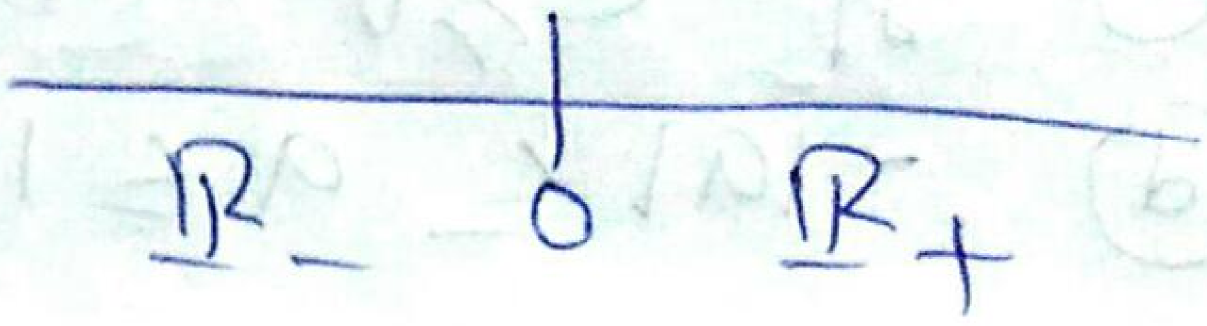
Real number system is also called real line. Reason being you can identify each real number with a pt. on a st. line




## Absolute Value funⁿ

$|\cdot| : \mathbb{R} \to \mathbb{R}_+$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

**Properties**

① $|x| \geq 0 \quad \forall x \in \mathbb{R}$

② $|x+y| \leq |x| + |y| \quad , \forall x, y \in \mathbb{R}$

$\Rightarrow$ triangle inequality

~~Because if x & y represents two sides of the triangle &~~

③ $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$

## Theorem:-

(a) $|ab| = |a||b| \quad \forall\, a, b \in \mathbb{R}$

(b) $|a|^2 = a^2 \quad \forall\, a \in \mathbb{R}$

(c) If $c \geq 0$, then $|a| \leq c$ iff $-c \leq a \leq c$

(d) $-|a| \leq a \leq |a| \quad \forall\, a \in \mathbb{R}$.

**Proof:-**

(a) If either $a$ or $b$ is $0$, then

$|ab| = 0 \cdot 0 \neq 6$

If $a > 0$, $b > 0$ then $ab > 0$

So that $|ab| = ab = |a|\,|b|$.

If $a > 0$, $b < 0$, then $ab < 0$

so that $|ab| = -ab = a \cdot (-b) = |a|\,|b|$.

(b) since $a^2 \geq 0$, we have $\cancel{a^2 =}$

$a^2 = |a^2| = |aa| = |a| \cdot |a| = |a|^2$.

(c) If $|a| \leq c$

then we have $a \leq c \quad -a \leq c$

$\Rightarrow -c \leq a \leq c$.

(d) We have $|a| \geq a \quad \forall\, a \in \mathbb{R}$

$\Rightarrow -|a| \leq a \leq a$

$\Rightarrow -|a| \leq a$

so, $-|a| \leq a \leq |a| \quad \forall\, a \in \mathbb{R}$.

## Triangle Inequality:

If $a, b \in \mathbb{R}$ then $|a+b| \le |a| + |b|$.

We have, $-|a| \le a \le |a|$ and
$-|b| \le b \le |b|$.

On adding, $-(|a|+|b|) \le a+b \le |a|+|b|$.

$\Rightarrow |a+b| \le |a|+|b|$.

Corollary :- If $a, b \in \mathbb{R}$, then

(a) $||a|-|b|| \le |a-b|$

(b) $|a-b| \le |a|+|b|$.

## The Completeness Property of $\mathbb{R}$

Ordered Set :- Let $S$ be a set. An order on $S$ is a relation, denoted by '<', with the following properties —

(i) If $x \in S$ and $y \in S$ then one ~~and~~ and only one of the statements
$x < y$, $x = y$, $y < x$ is true.

(ii) If $x, y, z \in S$, if $x < y$ & $y < z$ then $x < z$.

If a set $S$ together with the relation '<' satisfies (i) & (ii) is called an ordered set, or we say '<' is an order on $S$.

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Def$^n$:- An ordered set $S$ is a set in which an ordere is defined.

e.g., $\mathbb{Q}$ & $\mathbb{R}$ are ordered set.

1. $r, s \in \mathbb{R}$ define $r < s \Rightarrow s - r$ is a positive rational number.
2. $\mathbb{R}$ is an ordered set.
3. Set of complex numbers is not an ordered set.

suppose $i > 1$

$\Rightarrow i \cdot i > 1 \cdot 1$

$\Rightarrow -1 > 1 \Rightarrow$ which is absurd.

So, our assumption is wrong, our ordering is not correct.

We are unable to introduce the order in the set of all complex numbers. So, the set of complex numbers is not an ordered field.

Def$^n$:- Suppose $S$ is an ordered set and $E \subset S$. If there exists a $\beta \in S$, such that $x \le \beta \ \forall x \in E$, then we say $E$ is bounded above and $\beta$ is known as upper bound for $E$.

Def$^n$:- suppose $S$ is an ordered set and $E \ne \phi$, $E \subset S$. If there exists $\gamma \in S$ s.t. $\gamma \le x \ \forall x \in E$, then $E$ is bounded below and $\gamma$ is a lower bound of $E$.

## Def$^n$ (Least upper bound) :-

suppose $S$ is an ordered set, $E$ is bounded above. Suppose there exists an $\alpha \in S$ with the following properties,

(i) $\alpha$ is an upper bound of $E$, $x \le \alpha$.

(ii) (If we take number slightly lower that $\alpha$, then it should not be an upper bound)

If $\gamma < \alpha$ then $\gamma$ is not an upper bound of $E$.

$\alpha-\varepsilon \quad \alpha$

[or, $\forall \varepsilon > 0$, $\exists \beta \in E$ s.t. $\beta > \alpha - \varepsilon$].

Then $\alpha$ is called the least upper bound of $E$, (or we say supremum.

We denote it by

$$\alpha = \sup(E) = \text{lub}(E).$$

## Def$^n$ (Greatest lower bound) :-

suppose $S$ is an ordered set. $E \neq \phi$, $E \subseteq S$, and $E$ is bounded below. Suppose there exists a $\beta \in S$ with the following properties —

(i) $\beta$ is a lower bound of $E$, $\beta \le x$ $\forall x \in E$.

(ii) If $\delta > \beta$ then $\delta$ is not a lower bound of $E$

$\beta \quad \beta+\varepsilon$

[$\forall \varepsilon > 0$ $\exists \gamma \in S$ s.t. $\gamma < \beta + \varepsilon$]

Then $\beta$ is known as greatest lower bound of $E$ (or supremum of the set $E$) $\beta = \inf(E) = \text{glb}(E)$