Ex1) Let $A = \{p \in \mathbb{Q}^+$ positive rational no. s.t. $p^2 < 2\}$

$B = \{p \in \mathbb{Q}^+ \mid p^2 > 2\}$

A is bounded above, all the elements of B will act as an upper bound for A.

B has a lower bound, and all the elements of A behave as a lower bound of this B.

But neither B nor A has a least upper bound and the greatest lower bound.

So,

2. $E_1 = \{n \in \mathbb{Q}$ s.t. $n < 0\}$

$E_2 \{n \in \mathbb{Q}$ s.t. $n \leq 0\}$

$\sup(E_1) = 0$ & $\sup(E_2) = 0$

$\sup(E_1) \notin E_1$ but $\sup(E_2) \in E_2$.

3. $\sup\{1,2,3\} = 3$, $\inf\{1,2,3\} = 1$

4. If $a < b$, then $b = \sup[a,b) = \sup(a,b)$ and $a = \inf[a,b) = \inf(a,b]$.

5. If $S = \{q \in \mathbb{Q} \mid e < q < \pi\}$, then $\inf(S) = e$, $\sup(S) = \pi$.

6. If $S = \{x \in \mathbb{R} \mid x^2 < \pi\}$, the $\inf S = -\sqrt{3}$ & $\sup S = \sqrt{3}$.

8. Suppose that $A = \{x \in \mathbb{R} / 1 < x < 2\}$.

2 is an upper bound of A.

let $M > 1$ be an upper bound of A
We will prove that $M \geq 2$.

Suppose the contrary that
$1 < M < 2$.

Let r be a rational no. such that
$M < r < 2$ then $r \in A$.

and $M < r$ which contradict the fact that M is an upper bound of A.

Hence we must have $2 \leq M$ so that
$\sup\{A\} = 2$.

Since 2 is not an element of A.
so, 2 is not max^m of A.

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Theorem:- Supremum of a set is unique.

Proof:- Let S be any subset of $\mathbb{R}$.

Suppose that u_1 and u_2 are both supremum of S.

If $u_1 < u_2$, then the hypothesis that u_2 is a supremum implies u_1 cannot be an upper bound of S.

Similarly, we can say ~~that~~ than $u_2 < u_1$ is not possible.

Therefore, we must have $u_1 = u_2$.

Theorem:- Infimum of a set is unique.

- In order ~~to~~ for a nonempty set S in $\mathbb{R}$ to have a supremum, it must have an upper bound.

- Supremum of a set may or may not be an element of the set.

$S_2 = \{x : 0 \le x \le 1\}$

$\cancel{0 = \sup S_2}$ $\quad 0 = \inf S_2$

$1 \quad\quad 1 = \sup S_2$

Let $A = \{x \in \mathbb{R} : 1 < x < 2\}$.

The Completeness Property of $\mathbb{R}$:-

Every nonempty subset of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

This is also called the Supremum Property of $\mathbb{R}$.

Suppose that S is a nonempty subset of $\mathbb{R}$ that is bounded below. Then the nonempty set $\bar{S} = \{-s : s \in S\}$ is bounded above, and the supremum property implies that $u = \sup \bar{S}$ exist in $\mathbb{R}$. Which implies S has an infimum in $\mathbb{R}$.

Example: Let S be a non-empty ~~set~~ subset of $\mathbb{R}$ that is bounded above, and let a be any number in $\mathbb{R}$. Define the set $a+S = \{a+s : s \in S\}$.
Then $\sup(a+S) = a + \sup(S)$.

$\Rightarrow$ Let $u = \sup S$, then $x \leq u \ \forall x \in S$.

$x \leq u$

$\Rightarrow a+x \leq a+u$.

$\Rightarrow a+u$ is an upper bound for the set $a+S$.

Therefore $\sup(a+S) \leq a+u$. $\rightarrow$ (i)

If v is any ~~element in~~ upper bound of the set $a+S$, then $a+x \leq v \ \forall x \in S$.

$\Rightarrow x \leq v-a \ \forall x \in S$

$\Rightarrow v-a$ is an upper bound S.

Therefore

$u = \sup S \le v - a$

$\Rightarrow u \le v - a$

$\Rightarrow u + a \le v$

Since v is any upper bound of $a+S$, we can replace v by $\sup(a+S)$ to get $a+u \le \sup(a+S)$. — (ii)

Combining (i) & (ii), we conclude that

$\sup(a+S) = a+u = a + \sup S$.

(b) Let A and B are two nonempty subsets of $\mathbb{R}$ s.t.

$a \le b \quad \forall a \in A \;\&\; \forall b \in B$.

Prove that $\sup(A) \le \inf(B)$.

$\Rightarrow$ For given $b \in B$, we have $a \le b \; \forall a \in A$

$\Rightarrow$ b is an upper bound of A

$\Rightarrow \sup A \le b$,

And for given $a \in A$, $a \le b \; \forall b \in B$

$\Rightarrow$ a is a lower bound of B

$\Rightarrow a \le \inf B$.

Now, ~~$a \le b$~~

$\sup A \le a \le b \le \inf B$

$\Rightarrow \sup A \le \inf B$.

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