Convergence of Sequence Using Definition :—

(1) Show that $\lim_{n \to \infty} \frac{1}{n} = 0$.

Let $\varepsilon > 0$ is given.

Now, $|x_n - x| = |\frac{1}{n} - 0|$

$|x_n - x| < \varepsilon$

$\Leftrightarrow |\frac{1}{n}| < \varepsilon$

$\Leftrightarrow \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$

Let us choose $n_0 = [\frac{1}{\varepsilon}] + 1$.

Therefore for every $\varepsilon > 0$, $\exists$ $n_0 (= ([\frac{1}{\varepsilon}] + 1))$ in $\mathbb{N}$ s.t.

$|\frac{1}{n} - 0| < \varepsilon \quad \forall n \geq n_0$.

Therefore, $\lim_{n \to \infty} \frac{1}{n} = 0$.

(2) $\lim_{n \to \infty} \frac{n^2+1}{n^2} = 1$

Let $\varepsilon > 0$ is given

Now, $|x_n - x| = |\frac{n^2+1}{n^2} - 1|$

$= |\frac{n^2+1-n^2}{n^2}| = \frac{1}{n^2}$

$|x_n - x| < \varepsilon$

$\Leftrightarrow \frac{1}{n^2} < \varepsilon$

$\Leftrightarrow n > \frac{1}{\sqrt{\varepsilon}}$

Let us choose $n_0 = \left[\frac{1}{\sqrt{\varepsilon}}\right] + 1$.

Therefore, for every $\varepsilon > 0$, there exists $n_0 \left(= \left[\frac{1}{\varepsilon}\right] + 1\right) \in \mathbb{N}$ s.t.

$$\left|\frac{n^2+1}{n^2} - 1\right| < \varepsilon \quad \forall n \geq n_0.$$

So, $\lim_{n \to \infty} \frac{n^2+1}{n^2} = 1$.

(3) $\lim_{n \to \infty} x_n = 2$, $x_n = \frac{2n+1}{n+1}$

Let $\varepsilon > 0$ is given

$$|x_n - 2| = \left|\frac{2n+1}{n+1} - 2\right|$$

$$= \left|\frac{2n+1-2n-2}{n+1}\right|$$

$$= \frac{1}{n+1}$$

Now,

$|x_n - 2| < \varepsilon$

$\Leftrightarrow \frac{1}{n+1} < \varepsilon \Leftrightarrow \varepsilon(n+1) > 1$

$\Rightarrow n+1 > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{\varepsilon} - 1$

Let us choose $n_0 = \left[\frac{1}{\varepsilon} - 1\right] + 1$.

Therefore for every $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$

such that

$$\left|\frac{2n+1}{n+1}-2\right| < \varepsilon \quad \forall\, n \geq n_0 .$$

so, $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$.

(4) $x_n = 2$, $\lim_{n\to\infty} x_n = 2$.

let $\varepsilon > 0$.

$$|x_n - 2| = |2-2| = 0 < \varepsilon \quad \forall\, n \geq 1$$

Therefore for every $\varepsilon > 0$ $\exists$ $n_0 (=1) \in \mathbb{N}$ such that

$$|x_n - 2| < \varepsilon \quad \forall\, n \geq n_0 .$$

Hence, $\lim_{n\to\infty} x_n = 2$, $x_n = 2$ $\forall\, n$.

(5) $\lim_{n\to\infty} \frac{1}{n^{1/k}} = 0$ ~~[illegible]~~ $k \in \mathbb{N}$.

let $\varepsilon > 0$ is given

$$|x_n - 0| = \left|\frac{1}{n^{1/k}} - 0\right| = \frac{1}{n^{1/k}}$$

$|x_n - 0| < \varepsilon$

$\Leftrightarrow$ $\frac{1}{n^{1/k}} < \varepsilon$

$\Leftrightarrow$ $n^{1/k} > \frac{1}{\varepsilon}$ $\Leftrightarrow$ $n > \left(\frac{1}{\varepsilon}\right)^k$

Let $n_0 = \left[\left(\frac{1}{\varepsilon}\right)^k\right] + 1$.

So, for every $\varepsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$\left|\frac{1}{n^{1/k}} - 0\right| < \varepsilon \quad \forall\, n \geq n_0 . \Rightarrow$$

(6) $\lim_{n\to\infty} \frac{\cos n\pi}{\sqrt{n}} = 0$

$|x_n - 0|$

$= \left|\frac{\cos n\pi}{\sqrt{n}}\right| = \frac{|(-1)^n|}{\sqrt{n}} = \frac{1}{\sqrt{n}}$ [$\because$ $|\cancel{\cos x}| \cancel{\leq} 1$ $\cancel{\forall x \in \mathbb{R}}$]

[$\because$ $\cos n\pi = (-1)^n$]

$|x_n - 0| < \varepsilon$

$\Leftrightarrow \frac{1}{\sqrt{n}} < \varepsilon$

$\Leftrightarrow \sqrt{n} > \frac{1}{\varepsilon} \Leftrightarrow n > \left(\frac{1}{\varepsilon}\right)^2$

Let $n_0 = \left[\frac{1}{\varepsilon^2}\right] + 1$

Therefore for every $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$\left|\frac{\cos n\pi}{\sqrt{n}} - 0\right| < \varepsilon \quad \forall\, n \geq n_0.$

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