47. CAUCHY INTEGRAL FORMULA

Another fundamental result will now be established.

Theorem. Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}. \tag{1}$$

Formula (1) is called the *Cauchy integral formula*. It tells us that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f on C.

When the Cauchy integral formula is written

$$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0), \tag{2}$$

it can be used to evaluate certain integrals along simple closed contours.

EXAMPLE. Let C be the positively oriented circle $|z| = 2$. Since the function

$$f(z) = \frac{z}{9 - z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, formula (2) tells us that

$$\int_C \frac{z\,dz}{(9 - z^2)(z + i)} = \int_C \frac{z/(9 - z^2)}{z - (-i)}\,dz = 2\pi i\left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

We begin the proof of the theorem by letting C_ρ denote a positively oriented circle $|z - z_0| = \rho$, where ρ is small enough that C_ρ is interior to C (see Fig. 64). Since the function $f(z)/(z - z_0)$ is analytic between and on the contours C and C_ρ, it follows

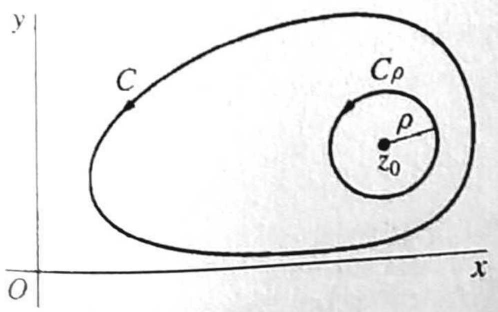

FIGURE 64

from the principle of deformation of paths (Corollary 2, Sec. 46) that

$$\int_C \frac{f(z)\,dz}{z-z_0} = \int_{C_\rho} \frac{f(z)\,dz}{z-z_0}.$$

This enables us to write

$$(3) \qquad \int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz.$$

But [see Exercise 10(*a*), Sec. 40]

$$\int_{C_\rho} \frac{dz}{z-z_0} = 2\pi i;$$

and so equation (3) becomes

$$(4) \qquad \int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz.$$

Now the fact that f is analytic, and therefore continuous, at z_0 ensures that, corresponding to each positive number ε, however small, there is a positive number δ such that

$$(5) \qquad |f(z)-f(z_0)| < \varepsilon \quad \text{whenever} \quad |z-z_0| < \delta.$$

Let the radius ρ of the circle C_ρ be smaller than the number δ in the second of these inequalities. Since $|z-z_0| = \rho$ when z is on C_ρ, it follows that the *first* of inequalities (5) holds when z is such a point; and inequality (1), Sec. 41, giving upper bounds for the moduli of contour integrals, tells us that

$$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{\varepsilon}{\rho}\,2\pi\rho = 2\pi\varepsilon.$$

In view of equation (4), then,

$$\left|\int_C \frac{f(z)\,dz}{z-z_0} - 2\pi i f(z_0)\right| < 2\pi\varepsilon.$$

Since the left-hand side of this inequality is a nonnegative constant that is less than an arbitrarily small positive number, it must equal to zero. Hence equation (2) is valid, and the theorem is proved.

48. DERIVATIVES OF ANALYTIC FUNCTIONS

It follows from the Cauchy integral formula (Sec. 47) that if a function is analytic at a point, then its derivatives of all orders exist at that point and are themselves analytic there. To prove this, we start with a lemma that extends the Cauchy integral formula so as to apply to derivatives of the first and second order.

Lemma. *Suppose that a function f is analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z is any point interior to C, then*

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} \quad \text{and} \quad f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3}. \tag{1}$$

Note that expressions (1) can be obtained *formally*, or without rigorous verification, by differentiating with respect to z under the integral sign in the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z}, \tag{2}$$

where z is interior to C and s denotes points *on* C.

To verify the first of expressions (1), we let d denote the smallest distance from z to points on C and use formula (2) to write

$$\begin{aligned}\frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}\,ds\\ &= \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z-\Delta z)(s-z)},\end{aligned}$$

where $0 < |\Delta z| < d$ (see Fig. 65). Evidently, then,

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2} = \frac{1}{2\pi i}\int_C \frac{\Delta z f(s)\,ds}{(s-z-\Delta z)(s-z)^2}. \tag{3}$$

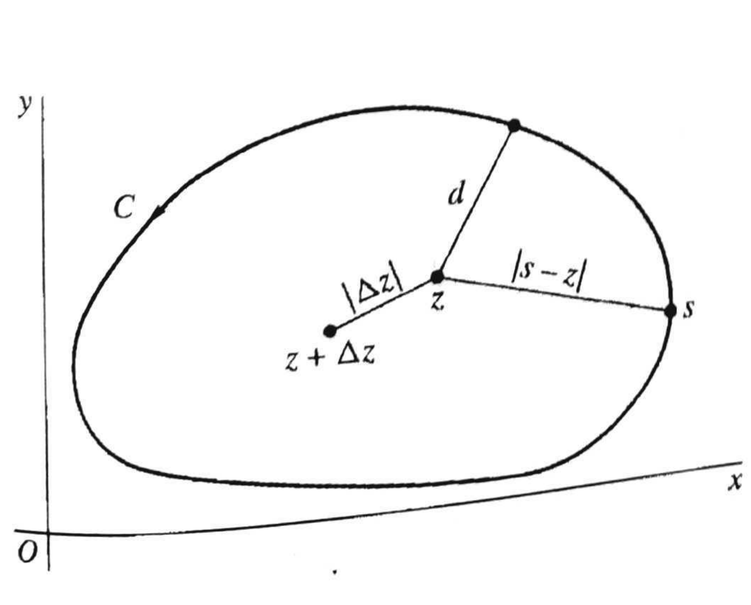

FIGURE 65

analytic function f' to conclude that its derivative f'' is analytic, etc. Theorem 1 is now established.

As a consequence, when a function

$$f(z) = u(x, y) + iv(x, y)$$

is analytic at a point $z = (x, y)$, the differentiability of f' ensures the continuity of f' there (Sec. 18). Then, since

$$f'(z) = u_x + iv_x = v_y - iu_y,$$

we may conclude that the first-order partial derivatives of u and v are continuous at that point. Furthermore, since f'' is analytic and continuous at z and since

$$f''(z) = u_{xx} + iv_{xx} = v_{yx} - iu_{yx},$$

etc., we arrive at a corollary that was anticipated in Sec. 25, where harmonic functions were introduced.

Corollary. *If a function $f(z) = u(x, y) + iv(x, y)$ is defined and analytic at a point $z = (x, y)$ then the component functions u and v have continuous partial derivatives of all orders at that point.*

One can use mathematical induction to generalize formulas (1) to

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s - z)^{n+1}} \qquad (n = 1, 2, \ldots). \tag{4}$$

The verification is considerably more involved than for just $n = 1$ and $n = 2$, and we refer the interested reader to other texts for it.* Note that, with the agreement that

$$f^{(0)}(z) = f(z) \quad \text{and} \quad 0! = 1,$$

expression (4) is also valid when $n = 0$, in which case it becomes the Cauchy integral formula (2).

When written in the form

$$\int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad (n = 0, 1, 2, \ldots), \tag{5}$$

expression (4) can be useful in evaluating certain integrals when f is analytic inside and on a simple closed contour C, taken in the positive sense, and z_0 is any point interior to C. It has already been illustrated in Sec. 47 when $n = 0$.

---

* See, for example, pp. 299–301 in Vol. I of the book by Markushevich, cited in Appendix 1.

EXAMPLE 1. If C is the positively oriented unit circle $|z| = 1$ and

$$f(z) = \exp(2z),$$

then

$$\int_C \frac{\exp(2z)\,dz}{z^4} = \int_C \frac{f(z)\,dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}.$$

EXAMPLE 2. Let z_0 be any point interior to a positively oriented simple closed contour C. When $f(z) = 1$, expression (5) shows that

$$\int_C \frac{dz}{z - z_0} = 2\pi i$$

and

$$\int_C \frac{dz}{(z - z_0)^{n+1}} = 0 \qquad (n = 1, 2, \ldots).$$

(Compare Exercise 10, Sec. 40.)

We conclude this section with a theorem due to E. Morera (1856–1909). The proof here depends on the fact that the derivative of an analytic function is itself analytic, as stated in Theorem 1.

Theorem 2. *Let f be continuous on a domain D. If*

$$\int_C f(z)\,dz = 0 \tag{6}$$

for every closed contour C lying in D, then f is analytic throughout D.

In particular, when D is *simply connected*, we have for the class of continuous functions on D a converse of Theorem 1 in Sec. 46, which is the extension of the Cauchy–Goursat theorem involving such domains.

To prove the theorem here, we observe that when its hypothesis is satisfied, the theorem in Sec. 42 ensures that f has an antiderivative in D; that is, there exists an analytic function F such that $F'(z) = f(z)$ at each point in D. Since f is the derivative of F, it then follows from Theorem 1 above that f is analytic in D.

EXERCISES

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

   (*a*) $\displaystyle\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; (*b*) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$; (c) $\displaystyle\int_C \frac{z\,dz}{2z+1}$;

(*d*) $\int_C \frac{\cosh z}{z^4}\,dz$; (*e*) $\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz \quad (-2 < x_0 < 2)$.

Ans. (*a*) 2π; (*b*) $\pi i/4$; (c) $-\pi i/2$; (*d*) 0; (*e*) $i\pi \sec^2(x_0/2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(*a*) $g(z) = \dfrac{1}{z^2+4}$; (*b*) $g(z) = \dfrac{1}{(z^2+4)^2}$.

Ans. (*a*) $\pi/2$; (*b*) $\pi/16$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\,dz \qquad (|w| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(w) = \int_C \frac{z^3 + 2z}{(z-w)^3}\,dz.$$

Show that $g(w) = 6\pi i w$ when w is inside C and that $g(w) = 0$ when w is outside C.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}.$$

6. Let f denote a function that is *continuous* on a simple closed contour C. Following a procedure used in Sec. 48, prove that the function

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s - z}$$

is *analytic* at each point z interior to C and that

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

at such a point.

7. Let C be the unit circle $z = e^{i\theta}\ (-\pi \le \theta \le \pi)$. First show that, for any real constant a,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

49. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

This section is devoted to two important theorems that follow from the extension of the Cauchy integral formula in Sec. 48.

Lemma. *Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R (Fig. 67). If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M_R}{R^n} \qquad (n = 1, 2, \ldots). \tag{1}$$

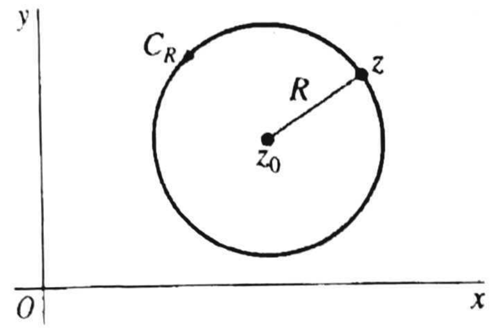

FIGURE 67

Inequality (1) is called *Cauchy's inequality* and is an immediate consequence of the expression

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z - z_0)^{n+1}} \qquad (n = 1, 2, \ldots),$$

which is a slightly different form of equation (5), Sec. 48. We need only apply inequality (1), Sec. 41, which gives upper bounds for the moduli of the values of contour integrals, to see that

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\cdot\frac{M_R}{R^{n+1}}2\pi R \qquad (n = 1, 2, \ldots),$$

where M_R is as in the statement of the lemma. This inequality is, of course, the same as inequality (1) in the lemma.

The lemma can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here, which is known as *Liouville's theorem*, states this result in a somewhat different way.

Theorem 1. *If f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.*

To start the proof, we assume that f is as stated in the theorem and note that, since f is entire, Cauchy's inequality (1) with $n = 1$ holds for any choices of z_0 and R:

(2) $$|f'(z_0)| \le \frac{M_R}{R}.$$

Moreover, the boundedness condition in the statement of the theorem tells us that a nonnegative constant M exists such that $|f(z)| < M$ for all z; and, because the constant M_R in inequality (2) is always less than or equal to M, it follows that

(3) $$|f'(z_0)| \le \frac{M}{R},$$

where z_0 is any fixed point in the plane and R is arbitrarily large. Now the number M in inequality (3) is independent of the value of R that is taken. Hence that inequality can hold for arbitrarily large values of R only if $f'(z_0) = 0$. Since the choice of z_0 was arbitrary, this means that $f'(z) = 0$ everywhere in the complex plane. Consequently, f is a constant function, according to the theorem in Sec. 23.

The following theorem, known as the *fundamental theorem of algebra*, follows readily from Liouville's theorem.

Theorem 2. *Any polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \ne 0)$$

of degree n $(n \ge 1)$ has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.

The proof here is by contradiction. Suppose that $P(z)$ is *not* zero for any value of z. Then the reciprocal

$$f(z) = \frac{1}{P(z)}$$

is clearly entire, and it is also bounded in the complex plane.

To show that it is bounded, we first write

(4) $$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z},$$

so that $P(z) = (a_n + w)z^n$. We then observe that a sufficiently large positive number R can be found such that the modulus of each of the quotients in expression (4) is less than the number $|a_n|/(2n)$ when $|z| \ge R$. The generalized triangle inequality, applied to n complex numbers, thus shows that $|w| < |a_n|/2$ for such values of z. Consequently, when $|z| \ge R$,

$$|a_n + w| \ge ||a_n| - |w|| > \frac{|a_n|}{2};$$

and this enables us to write

$$(5) \qquad |P(z)| = |a_n + w||z^n| > \frac{|a_n|}{2}|z|^n \geq \frac{|a_n|}{2}R^n \quad \text{whenever} \quad |z| \geq R.$$

Evidently, then,

$$|f(z)| = \frac{1}{|P(z)|} < \frac{2}{|a_n|R^n} \quad \text{whenever} \quad |z| > R.$$

So f is bounded in the region *exterior to* the disk $|z| \leq R$. But f is continuous in that closed disk, and this means that f is bounded there too. Hence f is bounded in the entire plane.

It now follows from Liouville's theorem that $f(z)$, and consequently $P(z)$, is constant. But $P(z)$ is not constant, and we have reached a contradiction.*

The fundamental theorem tells us that any polynomial $P(z)$ of degree n ($n \geq 1$) can be expressed as a product of linear factors:

$$(6) \qquad P(z) = c(z - z_1)(z - z_2) \cdots (z - z_n),$$

where c and z_k ($k = 1, 2, \ldots, n$) are complex constants. More precisely, the theorem ensures that $P(z)$ has a zero z_1. Then, according to Exercise 10, Sec. 50,

$$P(z) = (z - z_1)Q_1(z),$$

where $Q_1(z)$ is a polynomial of degree $n - 1$. The same argument, applied to $Q_1(z)$, reveals that there is a number z_2 such that

$$P(z) = (z - z_1)(z - z_2)Q_2(z),$$

where $Q_2(z)$ is a polynomial of degree $n - 2$. Continuing in this way, we arrive at expression (6). Some of the constants z_k in expression (6) may, of course, appear more than once, and it is clear that $P(z)$ can have no more than n *distinct* zeros.

50. MAXIMUM MODULUS PRINCIPLE

In this section, we derive an important result involving maximum values of the moduli of analytic functions. We begin with a needed lemma.

Lemma. *Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \varepsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.*

---

*For an interesting proof of the fundamental theorem using the Cauchy–Goursat theorem, see R. P. Boas, Jr., *Amer. Math. Monthly*, Vol. 71, No. 2, p. 180, 1964.

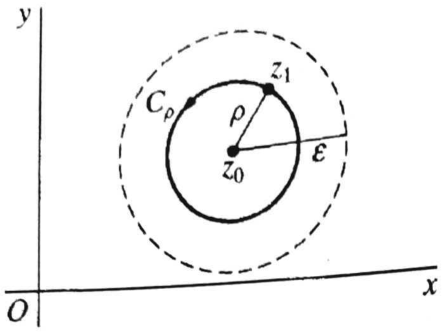

FIGURE 68

To prove this, we assume that f satisfies the stated conditions and let z_1 be any point other than z_0 in the given neighborhood. We then let ρ be the distance between z_1 and z_0. If C_ρ denotes the positively oriented circle $|z - z_0| = \rho$, centered at z_0 and passing through z_1 (Fig. 68), the Cauchy integral formula tells us that

(1) $$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)\,dz}{z - z_0};$$

and the parametric representation

$$z = z_0 + \rho e^{i\theta} \qquad (0 \le \theta \le 2\pi)$$

for C_ρ enables us to write equation (1) as

(2) $$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta.$$

We note from expression (2) that when a function is analytic within and on a given circle, its value at the center is the arithmetic mean of its values on the circle. This result is called *Gauss's mean value theorem.*

From equation (2), we obtain the inequality

(3) $$|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta.$$

On the other hand, since

(4) $$|f(z_0 + \rho e^{i\theta})| \le |f(z_0)| \qquad (0 \le \theta \le 2\pi),$$

we find that

$$\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta \le \int_0^{2\pi} |f(z_0)|\,d\theta = 2\pi |f(z_0)|.$$

Thus

(5) $$|f(z_0)| \ge \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\,d\theta.$$

It is now evident from inequalities (3) and (5) that

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta,$$

or

$$\int_0^{2\pi} [|f(z_0)| - |f(z_0 + \rho e^{i\theta})|]\, d\theta = 0.$$

The integrand in this last integral is continuous in the variable θ; and, in view of condition (4), it is greater than or equal to zero on the entire interval $0 \leq \theta \leq 2\pi$. Because the value of the integral is zero, then, the integrand must be identically equal to zero. That is,

(6) $$|f(z_0 + \rho e^{i\theta})| = |f(z_0)| \qquad (0 \leq \theta \leq 2\pi).$$

This shows that $|f(z)| = |f(z_0)|$ *for all points* z *on the circle* $|z - z_0| = \rho$.

Finally, since z_1 is any point in the deleted neighborhood $0 < |z - z_0| < \varepsilon$, we see that the equation $|f(z)| = |f(z_0)|$ is, in fact, satisfied by all points z lying on any circle $|z - z_0| = \rho$, where $0 < \rho < \varepsilon$. Consequently, $|f(z)| = |f(z_0)|$ everywhere in the neighborhood $|z - z_0| < \varepsilon$. But we know from Exercise 7(*b*), Sec. 24, that when the modulus of an analytic function is constant in a domain, the function itself is constant there. Thus $f(z) = f(z_0)$ for each point z in the neighborhood, and the proof of the lemma is complete.

This lemma can be used to prove the following theorem, which is known as the *maximum modulus principle*.

Theorem. *If a function* f *is analytic and not constant in a given domain* D*, then* $|f(z)|$ *has no maximum value in* D*. That is, there is no point* z_0 *in the domain such that* $|f(z)| \leq |f(z_0)|$ *for all points* z *in it.*

Given that f is analytic in D, we shall prove the theorem by assuming that $|f(z)|$ *does* have a maximum value at some point z_0 in D and then showing that $f(z)$ must be constant throughout D.

The general approach here is similar to that taken in the proof of the lemma in Sec. 26. We draw a polygonal line L lying in D and extending from z_0 to any other point P in D. Also, d represents the shortest distance from points on L to the boundary of D. When D is the entire plane, d may have any positive value. Next, we observe that there is a finite sequence of points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n$$

along L such that z_n coincides with the point P and

$$|z_k - z_{k-1}| < d \qquad (k = 1, 2, \ldots, n).$$

Corollary. *Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.*

EXAMPLE. Let R denote the rectangular region $0 \le x \le \pi$, $0 \le y \le 1$. The corollary tells us that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R that occurs somewhere on the boundary, and not in the interior, of R. This can be verified directly by writing (see Sec. 33)

$$|f(z)| = \sqrt{\sin^2 x + \sinh^2 y}$$

and noting that, in R, the term $\sin^2 x$ is greatest when $x = \pi/2$ and that the increasing function $\sinh^2 y$ is greatest when $y = 1$. Thus the maximum value of $|f(z)|$ in R occurs at the boundary point $z = (\pi/2, 1)$ and at no other point in R (Fig. 70).

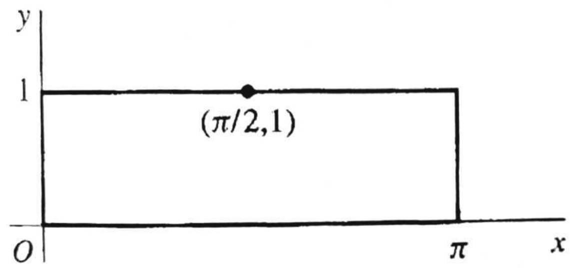

FIGURE 70

When the function f in the corollary is written $f(z) = u(x, y) + iv(x, y)$, *the component function $u(x, y)$ also has a maximum value in R which is assumed on the boundary of R and never in the interior,* where it is harmonic (Sec. 25). For the composite function $g(z) = \exp[f(z)]$ is continuous in R and analytic and not constant in the interior. Consequently, its modulus $|g(z)| = \exp[u(x, y)]$, which is continuous in R, must assume its maximum value in R on the boundary. Because of the increasing nature of the exponential function, it follows that the maximum value of $u(x, y)$ also occurs on the boundary.

Properties of *minimum* values of $|f(z)|$ and $u(x, y)$ are treated in the exercises.

EXERCISES

1. Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 49) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

2. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \text{Re}[f(z)]$ has an upper bound u_0; that is, $u(x, y) \le u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.

   Suggestion: Apply Liouville's theorem (Sec. 49) to the function $g(z) = \exp[f(z)]$.

3. Show that, for R sufficiently large, the polynomial $P(z)$ in Theorem 2, Sec. 49, satisfies the inequality

$$|P(z)| < 2|a_n||z|^n \quad \text{whenever} \quad |z| \ge R.$$

   [Compare the first of inequalities (5), Sec. 49.]

   Suggestion: Observe that there is a positive number R such that the modulus of each quotient in expression (4), Sec. 49, is less than $|a_n|/n$ when $|z| \ge R$.

4. Let a function f be continuous in a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \ne 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value m* in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values (Sec. 50) to the function $g(z) = 1/f(z)$.

5. Use the function $f(z) = z$ to show that in Exercise 4 the condition $f(z) \ne 0$ anywhere in R is necessary in order to obtain the result of that exercise. That is, show that $|f(z)|$ *can* reach its minimum value at an interior point when that minimum value is zero.

6. Consider the function $f(z) = (z + 1)^2$ and the closed triangular region R with vertices at the points $z = 0$, $z = 2$, and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values, thus illustrating results in Sec. 50 and Exercise 4.

   Suggestion: Interpret $|f(z)|$ as the square of the distance between z and -1.

   Ans. $z = 2$, $z = 0$.

7. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function $u(x, y)$ has a minimum value in R which occurs on the boundary of R and never in the interior. (See Exercise 4.)

8. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \le x \le 1$, $0 \le y \le \pi$. Illustrate results in Sec. 50 and Exercise 7 by finding points in R where the component function $u(x, y) = \text{Re}[f(z)]$ reaches its maximum and minimum values.

   Ans. $z = 1$, $z = 1 + \pi i$.

9. Let the function $f(z) = u(x, y) + iv(x, y)$ be continuous on a closed bounded region R, and suppose that it is analytic and not constant in the interior of R. Show that the component function $v(x, y)$ has maximum and minimum values in R which are reached on the boundary of R and never in the interior, where it is harmonic.

   Suggestion: Apply results in Sec. 50 and Exercise 7 to the function $g(z) = -if(z)$.

10. Let z_0 be a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \ne 0)$$