CHAPTER IX

Assignment Problems

9.1 ASSIGNMENT PROBLEMS.

In practical field we are sometimes faced with a type of problem which consists in assigning men to offices, jobs to machines, classes in a school to rooms, drivers to trucks, delivery trucks to different routes or problems to different research teams etc. in which the assignees possess varying degree of efficiency, called *cost* or *effectiveness*. The

FACILITIES

JOBS	1	2	3	...	...	...	...	m	a_i
1	c_{11}	c_{12}	c_{13}	...	...	...	...	c_{1m}	1
2	c_{21}	c_{22}	c_{23}	...	...	...	...	c_{2m}	1
3	c_{31}	c_{32}	c_{33}	...	...	...	...	c_{3m}	1
⋮	...	...	...	...	...	...	...	...	⋮
⋮	...	...	...	...	...	...	...	...	⋮
⋮	...	...	...	...	...	...	...	...	⋮
m	c_{m1}	c_{m2}	c_{m3}	...	...	...	...	c_{mm}	1
b_j	1	1	1	...	...	...	...	1	m

basic assumption of this type of problem is that one person can perform one job at a time. An assignment plan is optimal if it minimizes the total cost or effectiveness or maximizes the profit of performing all the jobs. For example, the manager of a firm may be interested in finding the best assignment of m jobs to his m employees. Thus the assignment problem is a special type of transportation problem in which the objective is to optimize the effect of allocating a number of jobs to an equal number facilities ; the jobs and facilities in assignment problem represent origins and destinations in transportation problem. As only one job is assigned to one facility, the cost matrix is always a square (in general) ; but in transportation, the matrix is of any order. In transportation, the variable may be any non-negative integer (including zero) but in assignment, the variable must be either zero or unity.

If c_{ij} be the cost of assigning the i-th job to the j-th facility, then we can represent the cost or effectiveness matrix in the tableau given in the previous page.

The tableau represents that only one unit of job is available for one facility. The assignment is to be made in such a way that each job can be associated with one and only one facility. The problem is to determine our assignment of job to facilities so as to minimize the over all cost.

9.2 MATHEMATICAL FORMULATION OF THE PROBLEM.

Assuming that c_{ij} is the cost of assigning the i-th job to the j-th facility, we state the assignment problem mathematically as :

Determine $x_{ij} \geq 0,\ i, j = 1, 2, \ldots\ldots, m$

which optimizes the total cost $z = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij}$

subject to $\sum_{j=1}^{m} x_{ij} = 1,\ i = 1, 2, \ldots\ldots, m$ ... (1)

and $\sum_{i=1}^{m} x_{ij} = 1,\ j = 1, 2, \ldots\ldots, m$ ... (2)

The requirement $x_{ij} \geq 0$ in the assignment problem has the explicit form

$$x_{ij} = \begin{cases} 1, & \text{if the } i\text{-th job be assigned to the } j\text{-th facility} \\ 0, & \text{otherwise ;} \end{cases}$$

and as such the assignment problem is not a linear programming problem as the variable x_{ij} can assume only 0 and 1.

The constraints (1) insure that only one job is assigned to one person and the constraints (2) insure that only one person should be assigned with one job and as such they restrict the possible integral values of each variable to either 0 or 1.

9.3 SOLUTION OF THE ASSIGNMENT PROBLEM.

From the formulation of an assignment problem, we see that it is a special class of transportation problem in which

$m = n,\ a_i = 1,\ i = 1, 2, \ldots\ldots, m$ and $b_j = 1,\ j = 1, 2, \ldots\ldots, m.$

At first consideration we may be tempted to solve the assignment problem by the transportation problem algorithm. But for the reason given below we shall develop a special algorithm for the solution of these problems instead of trying to use transportation problem algorithm directly.

In assignment problem we observe that a basic feasible solution for the constraint equations will consist of $(2m - 1)$ variables. But as is seen from the constraints that every basic solution will consist of m basic variables equal to 1 and $(m - 1)$ basic variables equal to 0 and as such the basic feasible solution will have a high level of degeneracy. Hence, if we apply the transportation problem algorithm to solve the assignment problem, then we shall have to perform a large number of iterations for the resolution of degeneracy until the optimal solution is obtained.

Suppose in a given assignment problem all $c_{ij} \geq 0$ and a feasible assignment exists for which all corresponding c_{ij} are equal to zero; then this assignment will make the objective function value zero and the solution will be optimal. With this in mind, our object will be to develop a convenient algorithm for the solution of such problems which is based on the next two theorems.

Theorem 1. *If a constant be added to any row and / or any column of the cost matrix of an assignment problem, then the resulting assignment problem has the same optimal solution as the original problem.*

Let the cost matrix be $\mathbf{C} = [\,c_{ij}\,]$ and suppose that we add α_i to row $i, i = 1, 2, \ldots\ldots, m$ and β_j to column $j, j = 1, 2, \ldots\ldots, m.$

Thus the new cost matrix is $\overline{\mathbf{C}} = [\,\overline{c}_{ij}\,]$, where

$$\overline{c}_{ij} = c_{ij} + \alpha_i + \beta_j.$$

If we denote by z and $\bar{z}$ the values of the objective function of the original and the new problems respectively, then

$$\bar{z} = \sum_{i=1}^{m} \sum_{j=1}^{m} \bar{c}_{ij} x_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} (c_{ij} + \alpha_i + \beta_j) x_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i x_{ij} + \sum_{j=1}^{m} \sum_{i=1}^{m} \beta_j x_{ij}$$

$$= z + \sum_{i=1}^{m} \alpha_i \left(\sum_{j=1}^{m} x_{ij} \right) + \sum_{j=1}^{m} \beta_j \left(\sum_{i=1}^{m} x_{ij} \right)$$

$$= z + \sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{m} \beta_j, \qquad \text{by the constraints } \sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{m} x_{ij} = 1.$$

Thus we see that z and $\bar{z}$ differ by a constant which is independent of values of the variables x_{ij} and hence the optimal solution of the original problem must be the optimal solution of the new problem and vice-versa.

Note. From this theorem we see that if in an assignment problem some of the costs be negative, then we can form a new assignment problem to deal with, such that those costs are all non-negative by simply adding a large enough constant to each and every cost.

Theorem 2. *If all $c_{ij} \geq 0$ and we can find a set $x_{ij} = x_{ij}^*$ such that*

$$\sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij}^* = 0 \; (\textit{minimization}),$$

then this solution is optimal.

We note that as $c_{ij} \geq 0$ and $x_{ij} \geq 0$, then

$$z = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} x_{ij} \text{ cannot be negative.}$$

Hence its minimum value is zero which is attained for $x_{ij} = x_{ij}^*$.

Thus the present solution is an optimal solution.

From these two theorems we see that we can introduce a good number of zeros into the cost matrix by proper choice of α_i and β_j. Then we shall find a feasible assignment for which the corresponding c_{ij} are equal to zero. This will be our optimal solution. Thus our desired algorithm will be such that it will provide a systematic procedure for introducing zeros into the cost matrix.

process as soon as enough zeros are introduced to get a complete assignment.

Theorem 4. *If k be the maximum number of zeros which can be assigned, then there exists a set of k lines which will cover all the zeros.*

If k be the maximum number of zeros which can be assigned, then each unassigned zero must lie in either the same row or the same column as an assigned zero. Thus with a proper choice of lines, there must be at least one set of k lines which will cover all the zeros.

9.4 COMPUTATIONAL PROCEDURE.

The method of solution being so explained, let us give the steps of procedure for the computation of an algorithm. If the original elements of the cost matrix be not already non-negative, then we are to make them so by applying theorem 1 of the previous article.

Step 1. Subtract the minimum element of each row in the cost matrix from all elements of the respective row. The matrix has now at least one zero in every row. Then subtract the minimum element of each column, which does not have a zero, from all elements of the respective column, to get the *starting cost matrix.*

These operations introduce more zeros in the matrix. Assignments are made in terms of zeros. The assignment will be optimal if now it be possible to assign only in the cells with zero cost. In the following step we device a method to check whether this will be a complete assignment.

Note. The operation in step 1 could have been interchanged in order. Although the results will be different but both are equally acceptable. The alternatives usually generate different intermediate steps but always provide same or equivalent result.

Step 2. Draw the *least possible* number of horizontal and vertical lines to cover all the zeros of the starting cost matrix.

Now two cases may arise :

(i) The number of lines so drawn may be equal to the order of the cost matrix, in this case an optimal assignment has been reached.

(ii) The number of lines so drawn may be less than the order of the cost matrix.

If the number of lines be equal to the order of the cost matrix, then we pass on to step 3, to find the assigned zeros.

Step 3. Starting with the first row of the starting matrix, examine all the rows of this matrix which contain only one zero in it (*row operation*). Mark this zero with □ as an assignment will be made there. Draw vertical lines along the columns containing these assigned zeros. This eliminates the possibility of making further assignments in those columns. Examine all the rows in this way.

When all the rows have thus been completely examined, apply similar procedure to the columns successively. In this case start from the first column and examine all the uncrossed columns to find columns containing exactly one remaining zero (*column operation*). Mark these zeros by □ where an assignment will be made and draw horizontal lines through these marked zeros.

We illustrate below the case (i), under step 2.

Let us consider the assignment problem represented by the adjacent cost matrix in which the elements represent the times in hours required by a machine to perform the corresponding job. The problem is to allocate the jobs to the machines so as to minimize the total time.

MACHINES

JOBS	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

The minimum elements of the rows *A*, *B*, *C*, *D* are respectively 8, 4, 15, 10. We subtract these elements from all elements of the respective rows, yielding the first matrix below. This matrix has at least one zero in every row and also at least one zero in every column except column II. We can easily remedy this by

	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0

	I	II	III	IV
A	0	14	9	3
B	9	20	0	22
C	23	0	3	0
D	9	12	14	0

subtracting 4, the minimum element of the column II, from all elements of that column yielding the second matrix. This is the starting matrix from which we shall have to find out the required assignment.

We have zeros in the cells (1, 1), (2, 3), (3, 2), (3, 4) and (4, 4).

We draw minimum number of vertical and horizontal lines to cover all the zeros of the starting matrix. The number of lines so required is four which is equal to the order of the cost matrix.

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Thus the total number of lines (vertical and horizontal) so drawn being equal to the order of the matrix, a complete assignment will be possible from this matrix. Then, following step 3, assignments are made in the cells (1, 1), (2, 3) and (4, 4), for the first, second and fourth rows which contain only one zero in each. These are marked with □. There being two zeros in the third row, no assignment is made in that row. Vertical lines are drawn through the assigned cells which eliminate the possibility of assignment in (3, 4). The only untouched column is the second column which contains a zero only in the cell (3, 2). Then assignment is made in the cell (3, 2) which is also marked by □. The optimal assignment will be

	I	II	III	IV
A	[0]	14	9	3
B	9	20	[0]	22
C	23	[0]	3	0
D	9	12	14	[0]

$A \to \text{I},\ B \to \text{III},\ C \to \text{II},\ D \to \text{IV}$

as there is no unmarked zero left.

The minimum cost (time) is, from the original matrix,

sum of cost of the cells (1, 1), (2, 3), (4, 4) and (3, 2)

$= (8 + 4 + 19 + 10)$ hours $= 41$ hours.

Note. Notice that the number of lines so drawn is the smallest number possibl such that all zero elements are covered, each line covering one row or one column.

Notice further that an alternative way to obtain the minimum cost is by summing up all the subtractions performed upto this point, in rows and in columns, which is 8 from row A, 4 from row B, 15 from row C, 10 from row D and 4 from column II only.

Thus (8 + 4 + 15 + 10 + 0 + 4 + 0 + 0) hours = 41 hours.

Following step 2, if the number of lines so drawn be less than the order of the matrix, we pass on to step 4. The lines should be so drawn that a minimum number of them will pass through all the zeros of the matrix.

Step. 4. Find the smallest element in the starting tableau among the uncovered elements left after drawing the lines as given in step 2. Subtract this element from all the uncovered elements of the current matrix and add the same element to the elements lying at the intersection of the horizontal and vertical lines. Do not alter the elements through which only one line passes. This gives the modified matrix with more zeros.

In fact the two operations of adding and subtracting do not alter the optimum solution as these operations are the resultant of the operations of subtracting the above smallest element from all the uncrossed rows and adding it to all the crossed columns.

Then go to step 2 with this modified matrix. If a complete assignment be not still available, then repeat the steps 4 and 2 iteratively and finally apply the step 3.

Step. 5. Repeat the two operations of step 3 (row and column operations) successively until one of the two following cases arise :

(i) There will be no unmarked zero left ;

(ii) There lie more than one unmarked zero in one row or column.

In the first case the algorithm stops and we have exactly one marked zero in each row and in each column of the given matrix. The assignment corresponding to these zeros is the optimal assignment.

In the second case mark with ☐ one of the unmarked zeros arbitrarily and ignore the remaining zero in that row or column. Repeat the process until no unmarked zero is left in the matrix.

As an example, we consider the assignment problem for which the cost matrix is given below. Following step 1, we get the second matrix on the right from the first.

	M_1	M_2	M_3	M_4	M_5
J_1	160	130	175	190	200
J_2	135	120	130	160	175
J_3	140	110	155	170	185
J_4	50	50	80	80	110
J_5	55	35	70	80	105

	M_1	M_2	M_3	M_4	M_5
J_1	30	0	35	30	15
J_2	15	0	0	10	0
J_3	30	0	35	30	20
J_4	0	0	20	0	5
J_5	20	0	25	15	15

This is the starting cost matrix and we pass on to step 2. We draw least number of horizontal and vertical lines to cover all the zeros of the starting table.

Here the least number of lines so required is 3 which is less than 5, the order of the matrix. Hence we pass on to step 4.

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

The smallest of the uncrossed elements is 15. We add 15 to the elements lying at the intersection of the vertical and horizontal lines drawn before and subtract 15 from all the uncrossed elements and we get the modified matrix. [It can easily be verified that the theorem 3 is satisfied for these two matrices ; for, here $m = 5$, $p = 15$, $k = 3$ and $mp(m-k) = 150 =$ net total decrease of the sum of the elements $= 350 - 200$.]

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

	M_1	M_2	M_3	M_4	M_5
J_1	15	0	20	15	$\boxed{0}$
J_2	15	15	$\boxed{0}$	10	0
J_3	15	$\boxed{0}$	20	15	5
J_4	$\boxed{0}$	15	20	0	5
J_5	5	0	10	$\boxed{0}$	0

Then we follow the step 2 and draw minimum number of lines to cover all the zeros.

The least number of lines required is 5 which is equal to the order of the matrix.

Then following step 3, we get the required assignment, by row and column operations.

The assignments are made in the cells in the following order : (3, 2) by row operation and (4, 1), (2, 3), (5, 4) and (1, 5) by column operation.

There is no zero left and every row has an assignment.

Thus the optimal solution will be

$$J_1 \to M_5,\ J_2 \to M_3,\ J_3 \to M_2,\ J_4 \to M_1,\ J_5 \to M_4\ ;$$

the total cost is 570 being the sum of the costs of the cells mentioned above as computed from the original matrix.

The algorithm described by the above steps is called the *Hungarian Method,* since it is based on the works of two Hungarian Mathematicians Konig and Egervary.

Note. An assignment problem may have more than one solution having the same minimum cost.

9.5 VARIATIONS IN ASSIGNMENT PROBLEMS.

The structure of the assignment problem may be broadened in the following cases :

(i) *Maximization problem*

If the problem be a maximization problem, say with a profit matrix, then a conversion is made first.

The largest element of the profit matrix is selected. A new cost matrix is formed whose elements are each the largest element minus the profit element of the corresponding cell. Such as

3	9
6	4

→

6	0
3	5

Here the largest element is 9. The operation is performed as stated and we get the modified cost matrix.

Then we find the solution to find the minimum cost which will provide the maximum profit of the original problem.

Alternatively since the maximum profit is equivalent to the minimum cost, we may attempt these problems by multiplying each element of the profit matrix by (– 1) to get the modified matrix. Then we proceed as usual.

(ii) *Unbalanced problem*

If the number of jobs and the number of facilities be not equal, then the problem is unbalanced. In this case we add a fictitious job or facility, whichever has the defficiency, with zero cost as we did in the case of unbalanced transportation problem. Then we apply the assignment algorithm to this resulting balanced problem.

(iii) *Impossible assignment*

If some assignment be impossible, that is, if some job cannot be performed by some particular facility, then we avoid this effectively by putting a large cost in that cell which prevents that particular assignment from being effective in the optimal solution.

(iv) *Negative cost*

If the cost matrix contains some negative cost, then we add to each element of the rows or columns a quantity, sufficient to make all the cell-elements non-negative. Then we proceed with the usual assignment algorithm.

9.6 ILLUSTRATIVE EXAMPLES.

Ex. 1. *Find the optimal assignment for a problem with the following cost matrix :*

	M_1	M_2	M_3	M_4	M_5
J_1	8	4	2	6	1
J_2	0	9	5	5	4
J_3	3	8	9	2	6
J_4	4	3	1	0	3
J_5	9	5	8	9	5

[*Jadavpur M. Sc., 1980*]

To get the starting matrix, we subtract the minimum element of each row from all elements of the respective row. We get the left hand matrix below. Each row and each column excepting the third has a zero. Then subtract the minimum element of the third column, from all elements of the respective column to get the starting matrix.

Then we draw the least number of vertical and horizontal lines to cover all the zeros.

7	3	1	5	0
0	9	5	5	4
1	6	7	0	4
4	3	1	0	3
4	0	3	4	0

	M_1	M_2	M_3	M_4	M_5
J_1	7	3	0	5	0
J_2	0	9	4	5	4
J_3	1	6	6	0	4
J_4	4	3	0	0	3
J_5	4	0	2	4	0

The least number of lines so required is 5, which is equal to the order of the cost matrix.

Hence this gives the optimal assignment. Then following step 3, we perform the row and column operations as shown for assignment and we get the optimal solution. We make assignments in the cells (2, 1), (3, 4) and (4, 3) by row operation and then in (5, 2) and (1, 5) by column operation. The optimal solution is

$$J_1 \to M_5,\ J_2 \to M_1,\ J_3 \to M_4,\ J_4 \to M_3,\ J_5 \to M_2\ ;$$

Minimum cost = sum of the costs of the assigned cells

$$= 1 + 0 + 2 + 1 + 5 = 9.$$

Ex. 2. *The Head of the department has five jobs A, B, C, D, E and five sub-ordinates V, W, X, Y, Z. The number of hours each man would take to perform each job is as follows :*

	V	W	X	Y	Z
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

How would the jobs be allocated to minimize the total time ? [*Bombay,* 1977]

Following step 1, we get the two matrices below. The second being the starting cost matrix, we follow the step 2 by drawing the least possible number of vertical and horizontal lines to cover all the zeros. But we see that the number of lines so required is 3 which is less than the order of the matrix, which is 5 in this case.

	V	*W*	*X*	*Y*	*Z*
A	0	2	7	12	5
B	0	3	11	14	4
C	0	4	12	12	4
D	0	0	3	5	1
E	0	0	5	15	0

0	2	4	7	5
0	3	8	9	4
0	4	9	7	4
0	0	0	0	1
0	0	2	10	0

Then we pass on to step 4. We see that 2 is the minimum element among the uncovered elements.

We subtract 2 from all the uncovered elements and add 2 to the elements lying at the intersection of the horizontal and vertical lines. We get the modified matrix as given below on the left hand side. Then use the step 2 to this modified matrix and see that the least number of lines to cover all the zeros of this modified matrix is 4 which is again less than the order of the matrix.

	V	*W*	*X*	*Y*	*Z*
A	0	0	2	5	3
B	0	1	6	7	2
C	0	2	7	5	2
D	2	0	0	0	1
E	2	0	2	10	0

1	0	2	5	3
0	0	5	6	1
0	1	6	4	1
3	0	0	0	1
3	0	2	10	0

Again following step 4 we subtract 1, the minimum element among the uncovered elements, from all the uncovered elements and add 1 to the elements lying at the intersection of the horizontal and vertical lines. We get the modified matrix (given on the right above) with a larger number of zeros. We draw vertical and horizontal lines again to cover all the zeros. But here too the least number of the required lines, which is 4, is less than the order of the matrix. Hence we repeat the steps 4 and 2 iteratively.

1	0	1	4	2
0	0	4	5	0
0	1	5	3	0
4	1	0	0	1
4	1	2	10	0

	V	W	X	Y	Z
A	1	0	0	3	2
B	0	0	3	4	0
C	0	1	4	2	0
D	5	2	0	0	2
E	4	1	1	9	0

In the final matrix the number of lines required to cover all the zeros is 5, which is equal to the order of the cost matrix. Hence optimality has been realised. Then following step 3, we find the optimal assignment by row and column operations. The assignment is made by row operation in (5, 5) and then for the other assignments, column operation is made in the order (4, 4), (1, 3), (2, 2) and (3, 1). The optimal assignment will be

$$A \to X,\ B \to W,\ C \to V,\ D \to Y,\ E \to Z;$$

Minimum time = 10 + 7 + 8 + 10 + 10 = 45 hours.

Ex. 3. *Three persons are being considered for three open positions. Each person has been given a rating for each position as shown in the following table :*

PERSON \ POSITION	I	II	III
1	7	5	6
2	8	4	7
3	9	6	4

Assign each person to one and only one position in such a way that the sum of ratings for all three persons is maximum.

[*Calcutta Hons.*, *1982*]

Here the problem is a maximization problem with a profit matrix. The modified matrix will be obtained by subtracting the profit elements of the respective cells from the largest element 9 in the cell (3, 1). Then we are to proceed as usual. The modified matrix is shown on the extreme left below.

Then following step 1, we get successively the following two matrices :

2	4	3
1	5	2
0	3	5

0	2	1
0	4	1
0	3	5

0	0	0
0	2	0
0	1	4

We draw horizontal and vertical lines to cover all the zeros of the starting matrix.

The minimum number of lines to cover all the zeros is 3, which is equal to the order of the matrix. Then, by row operation, assignments are made successively in the cells (3, 1), (2, 3) and in (1, 2) by column operation. Hence the optimal solution is

	I	II	III
1	0	[0]	0
2	0	2	[0]
3	[0]	1	4

$$1 \to \text{II},\ 2 \to \text{III},\ 3 \to \text{I}$$

and the maximum profit, that is, sum of the ratings as computed from the given table is $9 + 5 + 7 = 21$.

Ex. 4. *Find the minimum cost solution for the* 4×4 *assignment problem whose cost coefficients are as given below :*

	I	II	III	IV
1	4	5	3	2
2	1	4	− 2	3
3	4	2	1	− 5

[*North Bengal Hons., 2005 ; Calcutta Hons., 2006*]

The given cost matrix contains some negative cost at the same time it is an unbalanced problem having three jobs and four machines. First of all we make all the cell elements non-negative by adding 5 to each element of the matrix and then make the problem a balanced one by adding a fictitious job 4 with zero cost. The transformed cost matrix is given here.

	I	II	III	IV
1	9	10	8	7
2	6	9	3	8
3	9	7	6	0
4	0	0	0	0

Then we subtract the minimum element of each row from all elements of the respective row. Similar operation need not be performed for the columns as there is at least one zero in each column.

The starting matrix so obtained is given below.

	I	II	III	IV
1	2	3	1	0
2	3	6	0	5
3	9	7	6	0
4	0	0	0	0

We now draw the minimum number of horizontal and vertical lines to cover all the zeros. This minimum number is 3 which is less than the order of the matrix. Hence we follow next the steps 4 and 2 iteratively which yields successively the following matrices :

	I	II	III	IV
1	1	2	0	0
2	3	6	0	6
3	8	6	5	0
4	0	0	0	1

	I	II	III	IV
1	[0]	1	0	0
2	2	5	[0]	6
3	7	5	5	[0]
4	0	[0]	1	2

The last modified matrix needs 4 lines to cover all the zeros. Hence optimal assignment will be obtained from this matrix.

Following step 3 by row and column operations as shown in the last table, we get the complete assignment which is

$1 \to I$, $2 \to III$, $3 \to IV$ and the minimum cost is $4 - 2 - 5 = -3$.

Examples IX (A)

Find the optimal assignments to find the minimum cost for the assignment problems with the following cost matrices (**1** – **10**) :

1.

	J_1	J_2	J_3
P_1	12	24	15
P_2	23	18	24
P_3	30	14	28

[**Ans.** $P_1 \to J_1,\ P_2 \to J_3,\ P_3 \to J_2$; Min. cost = 50.]

2.

	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

[*Kalyani Hons., 1994*]

[**Ans.** $A \to 2,\ B \to 3,\ C \to 4,\ D \to 1$; Min. cost = 38.]

3.

	1	2	3	4
1	1	4	6	3
2	9	7	10	9
3	4	5	11	7
4	8	7	8	5

[**Ans.** $1 \to 1,\ 2 \to 3,\ 3 \to 2,\ 4 \to 4$; Min. cost = 21.]

4.

	I	II	III	IV	V
A	6	5	8	11	16
B	1	13	16	1	10
C	16	11	8	8	8
D	9	14	12	10	16
E	10	13	11	8	16

[**Ans.** $A \to II,\ B \to I,\ C \to V,\ D \to III,\ E \to IV$;
Or $A \to II,\ B \to IV,\ C \to V,\ D \to I,\ E \to III$; Min. Cost = 34.]

5.

	I	II	III	IV
A	5	3	1	8
B	7	9	2	6
C	6	4	5	7
D	5	7	7	6

[**Ans.** $A \to III$, $B \to IV$, $C \to II$, $D \to I$; Min. cost = 16.]

6.

	M_1	M_2	M_3	M_4
J_1	10	24	30	15
J_2	16	22	28	12
J_3	12	20	32	10
J_4	9	26	34	16

[**Ans.** $J_1 \to M_3$, $J_2 \to M_2$, $J_3 \to M_4$, $J_4 \to M_1$;
$J_1 \to M_3$, $J_2 \to M_4$, $J_3 \to M_2$, $J_4 \to M_1$;
$J_1 \to M_2$, $J_2 \to M_3$, $J_3 \to M_4$, $J_4 \to M_1$;
Min. cost = 71.]

7.

	a	b	c	d	e
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

[**Ans.** $1 \to a$, $2 \to d$, $3 \to e$, $4 \to c$, $5 \to b$; Min. cost = 60.]

8.

JOBS \ MEN	I	II	III	IV	V
A	1	3	2	3	6
B	2	4	3	1	5
C	5	6	3	4	6
D	3	1	4	2	2
E	1	5	6	5	4

[*Calcutta Hons.*, 2006]

[**Ans.** $A \to I$, $B \to IV$, $C \to III$, $D \to II$, $E \to V$;
or $A \to II$, $B \to IV$, $C \to III$, $D \to V$, $E \to I$;
Min. cost = 10.]

9.

	a	b	c	d	e
1	2	9	2	7	1
2	6	8	7	6	1
3	4	6	5	3	1
4	4	2	7	3	1
5	5	3	9	5	1

[*North Bengal Hons., 2007*]

[Ans. $1 \to c, 2 \to e, 3 \to a, 4 \to d, 5 \to b$;
$1 \to c, 2 \to e, 3 \to d, 4 \to b, 5 \to a$;
$1 \to c, 2 \to e, 3 \to d, 4 \to a, 5 \to b$;
Min. cost = 13.]

10.

	a	b	c	d
1	18	26	17	11
2	13	28	14	26
3	38	19	18	15
4	19	26	24	10

[*Calcutta Hons., 1987, 2002*]

[**Ans.** $1 \to c, 2 \to a, 3 \to b, 4 \to d$;
Min. cost = 59.]

11. One car is available at each of the stations 1, 2, 3, 4, 5, 6 and one car is required at each of the stations 7, 8, 9, 10, 11, 12. The distances between the various stations are given in the adjacent matrix. How should the cars be despatched so as to minimize the total mileage coverage ?

	7	8	9	10	11	12
1	41	72	39	52	25	51
2	22	29	49	65	81	50
3	27	39	60	51	32	32
4	45	50	48	52	37	43
5	29	40	39	26	30	33
6	82	40	40	60	51	30

[**Ans.** $1 \to 11, 2 \to 8, 3 \to 7$,
$4 \to 9, 5 \to 10, 6 \to 12$;
Minimum mileage = 185.]

12. A team of five horses and five riders has entered a jumping show contest. The number of penalty points to be expected when each rider rides any horse is given in the adjacent tableau. How should the horses be allotted to the riders so as to minimize the expected loss of the team ?

RIDERS

HORSES	R_1	R_2	R_3	R_4	R_5
H_1	5	3	4	7	1
H_2	2	3	7	6	5
H_3	4	1	5	2	4
H_4	6	8	1	2	3
H_5	4	2	5	7	1

[**Ans.** $H_1 \to R_5, H_2 \to R_1, H_3 \to R_4$,
$H_4 \to R_3, H_5 \to R_2$;
Min. loss = 8.]

13. Solve the following assignment problem :

PROJECT	LOCATION				
	I	II	III	IV	V
A	15	21	6	4	9
B	3	40	21	10	7
C	9	6	5	8	10
D	14	8	6	9	3
E	21	16	18	7	4

[**Ans.** $A \to IV$, $B \to I$, $C \to II$, $D \to III$, $E \to V$; Min. cost = 23.]

14. A computer centre has three expert programmers. The centre needs three application programmes to be developed. The head of the computer centre, after studying carefully the programmes to be developed, estimates the computer time in minutes required by the experts to the application programmes as follows :

		PROGRAMMES		
		A	*B*	*C*
PROGRAMMERS	1	120	100	80
	2	80	90	110
	3	110	140	120

Assign the programmers to the programmes in such a way that the total computer time is least. [*Kalyani Hons., 1988*]

[**Ans.** $1 \to C$, $2 \to B$, $3 \to A$; Min. time = 280 minutes.]

15. A car hire company has one car at each of the five depots *a, b, c, d* and *e*. A customer in each of the five towns *A, B, C, D* and *E* requires a car. The distance (in miles) between the depots (origins) and the towns (destinations) where the customers are, is given in the adjacent distance matrix.

	a	*b*	*c*	*d*	*e*
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	125	170	185
D	50	50	80	80	110
E	55	35	80	80	105

How should the cars be assigned to the customers so as to minimize the distance travelled. [*Calcutta M. Sc., 1976*]

[**Ans.** $A \to b$, $B \to e$, $C \to c$, $D \to a$, $E \to d$; Min. dist. = 560 miles.]

(c) A company has five jobs (1, 2, 3, 4, 5) to be done. The adjacent matrix shows the return in rupees on assigning the jobs to the machines *a, b, c, d, e*. Assign the five jobs to the machines to maximize the expected return.

	a	*b*	*c*	*d*	*e*
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

[**Ans.** $1 \to b,\ 2 \to a,\ 3 \to e,\ 4 \to c,\ 5 \to d$; Max. profit = 191.]

19. Find the minimum cost solution for the 4×4 assignment problem whose cost coefficients are as given in the adjacent table :

	I	II	III	IV
1	2	−1	−1	−2
2	1	0	−2	−1
3	1	−1	−2	0
4	2	2	1	1

[Add 2 to all the elements and then proceed as usual.]

[**Ans.** $1 \to IV,\ 2 \to III,\ 3 \to II,\ 4 \to I$; Min. cost = −3.]

20. Find the minimum cost solution for the 5×5 assignment problem whose cost coefficients are as given in the adjacent table.

[*Kharagpur, 1978*]

	1	2	3	4	5
1	−2	−4	−8	−6	−1
2	0	−9	−5	−5	−4
3	−3	−8	−9	−2	−6
4	−4	−3	−1	0	−3
5	−9	−5	−8	−9	−5

[**Ans.** $1 \to 3,\ 2 \to 2,\ 3 \to 5,\ 4 \to 1,\ 5 \to 4$
or $1 \to 4,\ 2 \to 2,\ 3 \to 3,\ 4 \to 5,\ 5 \to 1$;
Min. cost = − 36.]

21. Consider the problem of assigning five operators to five machines. The assignment costs in rupees are given in the adjacent table. Operator *B* cannot be assigned to machine 2 and operator *E* cannot be assigned to machine 4. Find the optimal cost of assignment.

	1	2	3	4	5
A	8	4	2	6	1
B	0	—	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	—	5

[*North Bengal Hons., 2006*]

[**Ans.** $A \to 5,\ B \to 1,\ C \to 4,\ D \to 3,\ E \to 2$; Min. cost = 9.]