8.5 INITIAL BASIC FEASIBLE SOLUTION.

In a transportation table it is always possible to assign allocations to different cells to have an initial basic feasible solution which will satisfy all the availabilities and requirements of the problem. This can be achieved either by random allocation or by following some simple rules. We discuss some of the methods for getting the initial basic feasible solution. These methods are best explained by numerical examples.

(a) North-West Corner Method.

This method consists in allocating the maximum amount allowable by the availability a_i and requirement b_j to the cell at the north-west corner of the table.

We start with the top-most left corner of the following table (north-west corner) and allocate maximum units possible there, that is,

	D_1	D_2	D_3	D_4	a_i
O_1	2	1	3	4	30
O_2	3	2	1	4	50
O_3	5	2	3	8	20
b_j	20	40	30	10	

$$x_{11} = \text{Min}\,(a_1,\ b_1)$$

$$= \text{Min}\,(30,\ 20) = 20.$$

In the table, 30 is the maximum availability at O_1 and 20 is the maximum requirement at D_1.

This allocation exhausts either the availability of O_1 or requirement of D_1. In this problem requirement of D_1 is exhausted by allocating 20 in the cell (1, 1), but availability of O_1 is not exhausted and hence we move to the right hand cell (1, 2) and allocate there $x_{12} = \text{Min}\,(a_1 - x_{11},\ b_2) = \text{Min}\,(30 - 20,\ 40) = \text{Min}\,(10,\ 40) = 10.$

[We are to move vertically down, if $b_1 > a_1$, that is, if the capacity of O_1 be exhausted but the requirement of D_1 is not satisfied. The second allocation will be of magnitude

$$x_{21} = \text{Min}\,(a_2,\ b_1 - x_{11}) \text{ in the cell } (2,\ 1).$$

If $b_1 = a_1$, then there is a tie for the second allocation and we shall have $x_{12} = 0$ in (1, 2) and $x_{21} = 0$ in (2, 1), that is, the next variable to be added to the basic solution will necessarily be at the zero level.]

With this allocation at (1, 2), the availability of O_1 is exhausted but the requirement of D_2 is not met and hence we go down to (2, 2) and allocate there

$$\text{Min}\,(a_2,\ b_2 - x_{12}) = \text{Min}\,(50,\ 40 - 10) = 30.$$

The above process is continued until all the available quantities are exhausted and all the requirements are satisfied as in the adjacent table. This should be checked at the end.

	D_1	D_2	D_3	D_4	a_i
O_1	20 (2)	10 (1)			30
O_2		30 (2)	20 (1)		50
O_3			10 (3)	10 (8)	20
b_j	20	40	30	10	

As usual the feasible solution x_{ij} is displayed at the upper left hand corner of each cell and the cost of the associated cell is indicated at the right hand bottom.

The resulting starting basic solution is

$$x_{11} = 20,\ x_{12} = 10,\ x_{22} = 30,\ x_{23} = 20,\ x_{33} = 10 \text{ and } x_{34} = 10.$$

The cost corresponding to this feasible solution is

$$20 \times 2 + 10 \times 1 + 30 \times 2 + 20 \times 1 + 10 \times 3 + 10 \times 8 = 240.$$

The cells corresponding to the feasible solution or a sub-set of them do not form a loop and hence it is a basic feasible solution.

The cells which get allocations are called *basic cells*.

The initial basic feasible solution obtained by this method is, in general, not optimal as the costs were not taken into account.

Note. Here $m + n - 1 = 3 + 4 - 1 = 6$ allocations have been made. In the north-west corner procedure as we move to the right or down no close loop can form of the allotted cells. Also in each allocation at least one row or column is discarded from further consideration, but the last allocation discards both a column and a row. Thus the solution obtained in this procedure is a basic feasible solution with $(m + n - 1)$ variables although it may not be optimal.

(b) Matrix Minima Method. (*Least Cost Entry Method*)

In this method the cost matrix is carefully examined and the cell with the minimum cost is chosen. As much as possible allocation is made in this cell. The row (or column) whose capacity (or requirement) is exhausted (or met) is discarded. Adjustment is made for the new availability and requirement in the new tableau and the process is repeated with the shrunken matrix so obtained.

If the cell with minimum cost be not unique, then any one of these cells is selected for allotment arbitrarily.

In the transportation problem considered in the north-west corner method we see that the minimum cost is in the cells (1, 2) and (2, 3). We choose the (1, 2) cell arbitrarily and allocate there Min (30, 40) = 30 as the maximum availability and maximum requirement corresponding to the cell (1, 2) are 30 and 40 respectively. Then in the shrunken matrix obtained by discarding the first row (as the capacity of O_1 is exhausted) we adjust the availability and requirement as shown in the adjacent table. (Requirement for new D_2 is $40 - 30 = 10$).

	D_1	D_2	D_3	D_4	
O_2	3	2	1	4	50
O_3	5	2	3	8	20
	20	10	30	10	

The cell (2, 3) in the new tableau has the minimum cost and we allocate Min (50, 30) = 30 to it and repeat the process of discarding

	D_1	D_2	D_4	
O_2	3	2	4	20
O_3	5	2	8	20
	20	10	10	

(a)

	D_1	D_4	
O_2	3	4	10
O_3	5	8	20
	20	10	

(b)

the column D_3 whose requirement is met. The new tableau, after discarding column D_3, is (a).

In (a) we allocate 10 in the cell (2, 2) and then we get the table (b) as the new shrunken matrix. Finally, as is obvious, following the same procedure, 10 units are allocated to the cell (2, 1), then 10 units in the cell (3, 1) and finally 10 units in the cell (3, 4). Thus the final tableau is as shown on the right.

	D_1	D_2	D_3	D_4	a_i
O_1		[30] 1			30
O_2	[10] 3	[10] 2	[30] 1		50
O_3	[10] 5			[10] 8	20
b_j	20	40	30	10	

The resulting starting basic feasible solution is

$x_{12} = 30,\ x_{21} = 10,\ x_{22} = 10,\ x_{23} = 30,\ x_{31} = 10,\ x_{34} = 10.$

The number of allocations is 6, as was expected. The cost corresponding to this feasible solution is

$$30 \times 1 + 10 \times 3 + 10 \times 2 + 30 \times 1 + 10 \times 5 + 10 \times 8 = 240.$$

Here too the cells corresponding to the feasible solution or a sub-set of them do not form a loop. Accidentally the cost has become the same in the two methods. This does not happen always.

Note. This method attempts to locate a good starting solution by utilising the cheap routes in the transportation model.

(c) **Vogel's Approximation Method** (*VAM*). (*Unit Penalty Method*).

In this method we take into account the least cost c_{ij} in each row and in each column and the cost that is just next to c_{ij} in the respective row and column. We determine the *non-negative difference* (or *penalty*) between the smallest and the second smallest cost in each row and exhibit it against the respective row in parenthesis in the transportation tableau by the side of the availabilities. The same procedure is followed for the column as shown in the matrix in the next page.

Let us consider the transportation problem as given by the cost matrix in the next page and compute the penalties.

The maximum of these differences for all the rows and columns is 3 in the fourth column. If this maximum be not unique, then we choose any one arbitrarily. From this we see that if we take the next-to-minimum cost cell instead of the minimum cost cell, then the cost will increase at least by 3, that is, if we do not allocate in the cell (2, 4) with minimum cost, then we shall increase the cost. Thus we shall have to allocate as much as possible in this cell. This will be Min $(a_i, b_j) =$ Min $(50, 10) = 10$. We shall allocate 0 to the remaining cell of this column after allocating 10 in the cell (2, 4). The demand of this column being met, we shall give up this column in the next tableau and adjust the availability and the requirement of the remaining origins and destinations accordingly.

	D_1	D_2	D_3	D_4	a_i
O_1	1	2	1	4	30(0)
O_2	3	3	2	1	50(1)
O_3	4	2	5	9	20(2)
b_j	20	40	30	10	
	(2)	(0)	(1)	(3)	

With the shrunken matrix so obtained we follow the previous steps.

The next tableau is as shown on the left.

Penalties are computed as before.

	D_1	D_2	D_3	
O_1	1	2	1	30(0)
O_2	3	3	2	40(1)
O_3	4	2	5	20(2)
	20	40	30	
	(2)	(0)	(1)	

The greatest penalty in it is (2) for both the first column and the third row. We choose arbitrarily the third row and allocate 20 in the (3, 2) cell which is the Min (a_i, b_j) of that cell. The origin O_3 being thus exhausted we allocate zero to all other cells of this row and delete this row from the next tableau to write the shrunken matrix.

Proceeding in this way, we get the following two matrices successively :

	D_1	D_2	D_3	
O_1	1	2	1	30(0)
O_2	3	3	2	40(1)
	20	20	30	
	(2)	(1)	(1)	

	D_2	D_3	
O_1	2	1	10(1)
O_2	3	2	40(1)
	20	30	
	(1)	(1)	

In the first column of the first matrix we are to allocate 20 in the (1, 1) cell and zero to the other cell and write the second matrix discarding the first column. In the second matrix we allocate 10 at (1, 3) cell and then to meet the requirement of the destination D_3, we are to allocate 20 in the (2, 3) cell. The balance 20 of the origin O_2 is then allocated to the cell (2, 2). Thus the initial feasible solution is

	D_1	D_2	D_3	D_4	
O_1	[20] 1		[10] 1		30
O_2		[20] 3	[20] 2	[10] 1	50
O_3		[20] 2			20
	20	40	30	10	

$x_{11} = 20,\ x_{13} = 10,\ x_{22} = 20,\ x_{23} = 20,\ x_{24} = 10,\ x_{32} = 20.$

The allocated cells are basic cells.

The cost corresponding to this basic feasible solution is

$20 \times 1 + 10 \times 1 + 20 \times 3 + 20 \times 2 + 10 \times 1 + 20 \times 2 = 180.$

To save labour and time, we can put the above computation (of *VAM*) in a single table instead of different shrunken matrices, as follows :

	D_1	D_2	D_3	D_4	a_i			
O_1	[20] 1	2	[10] 1	4	30 (0)	30 (0)	30 (0)	10 (1)
O_2	3	[20] 3	[20] 2	[10] 1	50 (1)	40 (1)	40 (1)	40 (1)
O_3	4	[20] 2	5	9	20 (2)	20 (2)		
b_j	20 (2)	40 (0)	30 (1)	10 (3)				
	20 (2)	40 (0)	30 (1)					
	20 (2)	20 (1)	30 (1)					
		20 (1)	30 (1)					

In the above table the penalties in each row and in each column have been displayed in parenthesis in each row and in each column with the respective availabilities and requirements. These have been shown in the first compartment. Maximum penalty being 3 in the column D_4 we allocate Min (50, 10) = 10 in the cell (2, 4) which exhibits the least cost 1. The requirement of D_4 being satisfied, we shade the column D_4 as shown in the table and give up further consideration for D_4. The resulting cost matrix is made up of the rows O_1, O_2, O_3 and the columns D_1, D_2, D_3. Applying the same technique to the new matrix with necessary adjustment of availability of the row O_2, we compute the new penalties for the rows and columns of the new matrix and put them in the second compartment. Proceeding as above, we get ultimately the initial basic feasible solution in which all availabilities will be exhausted and all requirements will be met.

(d) Row Minima Method and Column Minima Method.

In these methods, instead of finding the minimum cost cell as we did in the matrix minima method, we find the minimum cost cell in the first row or in the first column respectively. Then we allocate the maximum possible unit to that cell and proceed step by step deleting either a row or a column to get the shrunken matrix until all the availabilities are exhausted and all the requirements are met.

Let us apply the Column Minima Method to find the initial basic feasible solution of the following transportation problem. In a similar way the Row Minima Method can be applied. Let the problem be

	W_1	W_2	W_3	W_4	a_i
F_1	[5] 19	30	50	[2] 10	7
F_2	70	30	[7] 40	[2] 60	9
F_3	40	[8] 8	70	[10] 20	18
b_j	5	8	7	14	

We first consider the first column W_1 in which the minimum cost 19 is in the cell (1, 1). We allocate 5 = Min (7, 5) there. By similar consideration, we allocate Min (18, 8) = 8 in the minimum cost cell (3, 2) of W_2 and Min (9, 7) = 7 in (2, 3) of W_3 and Min (2, 14) in the cell (1, 4) of W_4. Row availability 2 of the row F_1 is the availability left after 5 is allocated in (1, 1). The remaining availabilities 2 of F_2 and

10 of F_3, are allocated in the cells (2, 4) and (3, 4) to meet the requirement of the column W_4. Thus the initial basic solution of the problem is

$$x_{11} = 5,\ x_{14} = 2,\ x_{23} = 7,\ x_{24} = 2,\ x_{32} = 8 \text{ and } x_{34} = 10.$$

The number of allocations being 6, the solution is basic.

The cost for this allocation is

$$5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20 = 779.$$

8.6 OPTIMALITY TEST OF THE BASIC FEASIBLE SOLUTION.

After the determination of an initial basic feasible solution to a transportation problem, our object will be to see how to improve the solution to arrive at the optimal solution. For that, we shall have to compute the effect of allocating a unit in an unoccupied cell after making adjustment in the solution to maintain the availability and requirement conditions intact. This net change in the total cost resulting from the unit allocation in the (i, j) cell is called the *cell evaluation* of that cell and is denoted by Δ_{ij}. If Δ_{ij} be positive for some cell, then the new solution increases the total cost and if Δ_{ij} be negative, then the new solution reduces the total cost which implies that we can reduce the total cost by allocating some quantity in the cells whose evaluation is negative.

Thus, if the cell evaluation for all the unoccupied cells be greater than or equal to zero, then there is no scope for decreasing the total cost any further and hence the solution under test will be optimal.

Now the number of unoccupied cells is

$$mn - (m + n - 1) = (m - 1)(n - 1)$$

and hence their cell evaluation will be a huge job by the above method. To avoid this difficulty, we device a method for the cell evaluation in a simpler way through the following theorem :

... solution having $(m + n - 1)$

(i) First we find a basic feasible solution of the given transportation problem by any one of the methods discussed previously. This solution will provide $(m + n - 1)$ independent positive allocations (for a non-degenerate problem).

Enter the solution in the upper left corners of the basic cells.

(ii) Then, for all the occupied cells (i, j), we determine a set of $(m + n)$ numbers u_i and v_j, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ such that

$$c_{ij} = u_i + v_j .$$

In practice this is achieved by choosing arbitrarily any one of u_i or v_j equal to zero. This choice is made for that u_i or v_j for which the corresponding row or column contains maximum number of occupied cells.

Enter them in circles in the upper right corners of the corresponding unoccupied cells.

(iii) Then the cell evaluations for the unoccupied (or empty) cells are made by the formula

$$\Delta_{ij} = c_{ij} - (u_i + v_j), \text{ for the cell } (i, j).$$

It is better to construct a table with these cell evaluations.

Three cases can occur :

(a) If all $\Delta_{ij} > 0$, then the solution is optimal and unique.

(b) If all $\Delta_{ij} > 0$ with at least one $\Delta_{ij} = 0$, then the solution is optimal but not unique.

(c) If at least one $\Delta_{ij} < 0$, then the solution is not optimal and we are to seek a new basic feasible solution and pass on to the next step.

(iv) To find a new basic feasible solution, we include in it that cell for which Δ_{ij} is minimum (negative). We allocate maximum amount possible in this cell and make one previously occupied cell empty. For this, new adjustments are made as explained before. By allocating in a cell with minimum cell evaluation, we decrease the total cost.

(v) Then the steps (ii) and (iii) are repeated until an optimal basic feasible solution is obtained.

This is a trial and error method and is known as *Modi test for optimality.*

The method is best explained by an example.

Without testing the optimality of the solution, one cannot be sure of the optimality of the solution, by whichever method it is found.

We test the optimality of the problem solved earlier by *VAM* in the final tableau (page 321) with the initial basic feasible solution as below :

As before the allocations are displayed at the upper left hand corner of each cell while the costs are shown at the right hand bottom, being covered in small rectangles at the corners of the cells.

	D_1	D_2	D_3	D_4	a_i	u_i
O_1	20 / 1	(2) / 2	10 / 1	(0) / 4	30	-1
O_2	(2) / 3	20 / 3	20 / 2	10 / 1	50	0
O_3	(1) / 4	20 / 2	(1) / 5	(0) / 9	20	-1
b_j	20	40	30	10		
v_j	2	3	2	1		

We choose $u_2 = 0$, as the row associated with u_2 has the maximum number of allocations. To compute $u_1, u_3, v_1, v_2, v_3, v_4$, we make use of the relation $c_{ij} = u_i + v_j$ for the occupied cells.

Thus $c_{24} = u_2 + v_4$ gives $1 = 0 + v_4$, that is, $v_4 = 1$.

Similarly, we compute $u_1 = -1$, $u_3 = -1$, $v_1 = 2$, $v_2 = 3$, $v_3 = 2$ and set in the tableau along the corresponding rows and columns.

CELL EVALUATIONS

	0		4
1			
3		4	9

The quantities $(u_i + v_j)$ for the unoccupied cells are displayed in circles.

Then we compute $\Delta_{ij} = c_{ij} - (u_i + v_j)$, the cell evaluations, for the unoccupied cells. Thus

$$\Delta_{12} = 0,\ \Delta_{14} = 4,\ \Delta_{21} = 1,\ \Delta_{31} = 3,\ \Delta_{33} = 4,\ \Delta_{34} = 9.$$

Since all Δ_{ij} are non-negative, the solution under test is optimal though not unique as at least one Δ_{ij} is zero.

Thus the optimal solution is $x_{11} = 20$, $x_{13} = 10$, $x_{22} = 20$, $x_{23} = 20$, $x_{24} = 10$, $x_{32} = 20$ and the minimum cost is 180 as before.

In this example we see that the initial basic feasible solution as obtained by *VAM* is the optimal solution. No further improvement was necessary.

To find the alternative optimal solution, we allocate maximum quantity to the cell (1, 2) where the cell evaluation is zero and make one of the cells empty. For this, readjustment of allocation is made by adding and subtracting 10 units so that the row and column requirements are not disturbed and the allocated cells do not form a loop. This is performed by adding 10 units to the cell (1, 2), subtracting 10 units from the cell (1, 3), adding 10 units to the cell (2, 3) and subtracting 10 units from the cell (2, 2). A new table is constructed with the new allocations. Then the same procedure is followed in the following iterations and the final table is obtained in which the corresponding $(u_i + v_j)$ are shown in circles of the unoccupied cells.

	+ 10	− 10	
	− 10	+ 10	

	D_1	D_2	D_3	D_4	a_i	u_i
O_1	20 (1)	10 (2)	⓵ (1)	⓪ (4)	30	− 1
O_2	⓶ (3)	10 (3)	30 (2)	10 (1)	50	0
O_3	⓵ (4)	20 (2)	⓵ (5)	⓪ (9)	20	− 1
b_j	20	40	30	10		
v_j	2	3	2	1		

Then the cell evaluations $\Delta_{ij} = c_{ij} - (u_i + v_j)$ are computed for all the unoccupied cells which are given in the table below.

CELL EVALUATIONS

		0	4
1			
3		4	9

The cell evaluations are non-negative and hence the solution $x_{11} = 20$, $x_{12} = 10$, $x_{22} = 10$, $x_{23} = 30$, $x_{24} = 10$, $x_{32} = 20$ which gives the minimum cost

$$20 \times 1 + 10 \times 2 + 10 \times 3 + 30 \times 2 + 10 \times 1 + 20 \times 2 = 180 \text{ (as before)}$$

is another optimal solution.

Notice that the cell evaluation zero in the cell (1, 3) indicates the presence of alternate optimal solution which has been obtained earlier.

In the adjacent example we shall see how the initial solution is improved when some cell evaluations become negative.

Consider the adjacent transportation problem :

	D_1	D_2	D_3	a_i
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
b_j	7	9	18	

By Vogel's approximation method we first find out the initial feasible solution as below :

	D_1	D_2	D_3	a_i
O_1	[5] 2	7	4	5 (2)
O_2	3	3	1	8 (2)
O_3	5	4	7	7 (1)
O_4	1	6	2	14 (1)
b_j	7 (1)	9 (1)	18 (1)	

	D_1	D_2	D_3	
O_2	3	3	[8] 1	8 (2)
O_3	5	4	7	7 (1)
O_4	1	6	2	14 (1)
b_j	2 (2)	9 (1)	18 (1)	

	D_1	D_2	D_3	
O_3	5	4	7	7 (1)
O_4	1	6	[10] 2	14 (1)
	2 (4)	9 (2)	10 (5)	

	D_1	D_2	
O_3	5	[7] 4	7(1)
O_4	[2] 1	[2] 6	4(5)
	2 (4)	9 (2)	

The solution is thus

$$x_{11} = 5,\ x_{23} = 8,\ x_{32} = 7,\ x_{41} = 2,\ x_{42} = 2,\ x_{43} = 10.$$

The total cost is

$$5 \times 2 + 8 \times 1 + 7 \times 4 + 2 \times 1 + 2 \times 6 + 10 \times 2 = 80.$$

The number of basic variables is $m + n - 1 = 4 + 3 - 1 = 6$.

As before we construct a table in which the quantity at the left hand upper corner of the cell represents the cell allocation, the quantity at the right hand bottom represents the cost.

Then, for the occupied cells, we determine a set of $(m + n)$ numbers u_i and v_j as before. The quantities in circles are $(u_i + v_j)$ of the unoccupied cells. The cell evaluations $\{c_{ij} - (u_i + v_j)\}$ of the unoccupied cells are shown on the right hand table below.

	D_1	D_2	D_3		u_i
O_1	[5] 2	(7) 7	(3) 4	5	1
O_2	(0) 3	(5) 3	[8] 1	8	−1
O_3	(−1) 5	[7] 4	(0) 7	7	−2
O_4	[2] 1	[2] 6	[10] 2	14	0
	7	9	18		
v_j	1	6	2		

CELL EVALUATIONS

•	0	1
3	−2	•
6	•	7
•	•	•

(•) marked cells are occupied cells

We observe that at least one Δ_{ij} $(= c_{ij} - u_i - v_j)$ *i.e.* $\Delta_{22} < 0$. Hence the solution is not optimal. There being only one negative cell evaluation, we allocate maximum quantity to this cell to make one of the occupied cells empty. For this, readjustment of allocation is made by adding and subtracting 2 units so that the row and column requirements are maintained along with non-negativity restriction. For that, we see that we can allocate 2 units to the cell (2, 2). Subtract 2 units from (2, 3), add 2 units to (4, 3) and subtract 2 units from (4, 2). This makes the cell (4, 2) empty.

This is shown in the table given in the next page.

A new table is constructed with the new allocations. Then in the next iteration the same procedure is followed until all the cell evaluations are non-negative.

We see that all the cell evaluations of the unoccupied cells are non-negative and thus the unique optimal solution is

$$x_{11} = 5,\ x_{22} = 2,\ x_{23} = 6,$$

$$x_{32} = 7,\ x_{41} = 2,\ x_{43} = 12.$$

Notice that the allocated cells are basic cells.

	+2	8 − 2
	2 − 2	10 + 2

	D_1	D_2	D_3		u_i
O_1	5 (cost 2)	(5) (cost 7)	(3) (cost 4)	5	1
O_2	(0) (cost 3)	2 (cost 3)	6 (cost 1)	8	−1
O_3	(1) (cost 5)	7 (cost 4)	(2) (cost 7)	7	0
O_4	2 (cost 1)	(4) (cost 6)	12 (cost 2)	14	0
	7	9	18		
v_j	1	4	2		

CELL EVALUATIONS

•	2	1
3	•	•
4	•	5
•	2	•

The minimum cost is thus

$$5 \times 2 + 2 \times 3 + 6 \times 1 + 7 \times 4 + 2 \times 1 + 12 \times 2 = 76.$$

It should be noted that the cost with the initial basic solution as obtained by *VAM* was 80.

8.8 DEGENERACY IN TRANSPORTATION PROBLEM.

If at the very initial stage or in any subsequent iteration the number of individual independent allocations be less than $(m + n - 1)$, then we have a case of degeneracy. This degeneracy may occur at the very initial stage or at any intermediate stage.

To resolve this degeneracy, we allocate a very small positive quantity ε to one or more (as many allocations are required to have $(m + n - 1)$ independent set of allocations) of the empty cells (generally the lowest cost cells) and consider these cells to be the occupied cells.

The quantity ε is so chosen that

$0 < \varepsilon < x_{ij}$, $\varepsilon + 0 = \varepsilon$, $x_{ij} \pm \varepsilon = x_{ij}$ (ultimately), $x_{ij} > 0$.

With this choice of ε, the original problem is not changed, since it does not violate the row and column restrictions.

ε is considered as a real positive allocation so long as is required, ultimately it is to be omitted in the optimal solution.

Let us consider the following cost matrix as an example :

By *VAM* we see that the initial solution is (adjacent table)

$x_{13} = 60,\ x_{21} = 50,\ x_{23} = 20,$
$x_{32} = 80.$

TO

FROM				
	8	7	3	60
	3	8	9	70
	11	3	5	80
	50	80	80	

TO

FROM					
	8	7	[60] 3	60 (4)	60 (4)
	[50] 3	8	[20] 9	70 (5)	20 (1)
	11	[80] 3	5	80 (2)	80 (2)
	50 (5)	80 (4)	80 (2)		
		80 (4)	80 (2)		

Now the number of occupied cells is four, which is not equal to $m + n - 1 = 5$. Thus there is degeneracy in the initial stage.

We now add a small positive quantity ε to a cell such that this does not result in forming a loop among some or all of the occupied cells and make them dependent. For a dependent set of cells, unique determination of u_i and v_j will not be possible. With this in view we allocate ε to the cell (1, 2) and construct the following new tableaux computing u_i and v_j for the new allocation. Then we get the cell evaluations and observe that Δ_{22} is negative. The numbers in the circles of the unoccupied cells are $(u_i + v_j)$ of that cell. Then the cell evaluations are $\Delta_{ij} = c_{ij} - (u_i + v_j)$, which are all positive except that of the cell (2, 2).

The following tables are self-explanatory :

					u_i
	(−3) 8	ε 7	60 3	60	0
	50 3	(13) 8	20 9	70	6
	(−7) 11	80 3	(−1) 5	80	−4
	50	80	80		
v_j	−3	7	3		

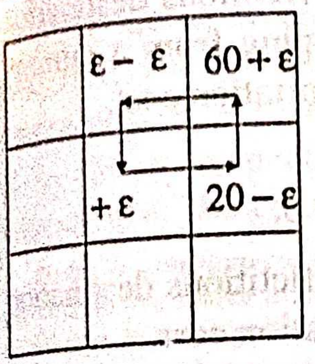
CELL EVALUATIONS

11	•	•
•	−5	•
18	•	6

$\Delta_{22} = -5 < 0$

Then we allocate maximum possible unit to (2, 2) cell, since this contains the negative cell-evaluation and repeat the iterations as shown in the next tableaux.

By the following adjustment, we make the (1, 2) cell empty and allocate ε to the cell (2, 2), which becomes occupied.



					u_i
(−3) [8]	(2) [7]	60 [3]		60	−6
50 [3]	ε [8]	20 [9]		70	0
(−2) [11]	80 [3]	(4) [5]		80	−5
50	80	80			
v_j 3	8	9			

CELL EVALUATIONS

11	5	•
•	•	•
13	•	1

All the cell evaluations are made and they are non-negative. So we have obtained the optimal solution as

$x_{13} = 60,\ x_{21} = 50,\ x_{23} = 20,$

$x_{32} = 80,$ which is basic too.

The infinitesimal quantity ε plays only an auxiliary role as far as number of allocations are concerned and is removed ultimately while writing the optimal solution.

The minimum cost is 750.

8.9 VARIATIONS IN TRANSPORTATION PROBLEM.

(a) *Unbalanced transportation problem.*

If in a transportation problem the sum of the availabilities or capacities of the origins be not equal to the sum of the requirements of the destinations, that is, if

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j ,$$

then we have an *unbalanced transportation problem.*

total cost, we allocate at the minimum cost cells and hence due to very largeness of c_{ij} no allocation will ever be made in the (i, j)-th cell.

In a maximizing problem, this is achieved by assigning a very large negative number for c_{ij} in the (i, j)-th cell.

(d) *Some positive allocation in a particular cell.*

If we are to necessarily allocate at least a quantity p in the (i, j)-th cell, then we are to consider $(a_i - p)$ as the availability in the i-th source and $(b_j - p)$ as the requirement at the j-th destination in place of a_i and b_j respectively. Then the problem is solved in the usual manner with the changed rim requirements.

Note. A *transhipment problem,* unlike a transportation problem, is such that the material moves from one source to other sources and / or other destinations before reaching the specific destination. In these problems, the costs of despatch from one source to another or from one destination to another are required. The solution method is not discussed here.

8.10 ILLUSTRATIVE EXAMPLES.

Ex. 1. ***For the following transportation problem obtain the different starting solutions by adopting the North-West corner method and Vogel's approximation method and find out which solution is better ?***

	D_1	D_2	D_3	a_i
O_1	5	1	8	12
O_2	2	4	0	14
O_3	3	6	7	4
b_j	9	10	11	

We first adopt the North-West corner method for the allocations in the given transportation problem.

Beginning with the North-West corner of the table, the allocations are made as shown in the table. The allocations are 9 in (1, 1), 3 in (1, 2), 7 in (2, 2), 7 in (2, 3), 4 in (3, 3).

	D_1	D_2	D_3	a_i
O_1	[9] 5	[3] 1	8	12
O_2	2	[7] 4	[7] 0	14
O_3	3	6	[4] 7	4
b_j	9	10	11	

Total cost in this method is $45 + 3 + 28 + 0 + 28$

$= 104.$

Then we apply Vogel's approximation method to the problem.

We first compute the penalties.

These penalties as shown in the brackets are the difference between the smallest and second smallest cost in each row and in each column.

	D_1	D_2	D_3	a_i	
O_1	5	1	8	12	(4)
O_2	2	4	[11] 0	14	(2)
O_3	3	6	7	4	(3)
b_j	9	10	11		
	(1)	(3)	(7)		

The maximum penalty being 7 in the third column, we allocate Min $(a_2, b_3) =$ Min $(14, 11) = 11$ in the cell (2, 3), with minimum cost, 0. Then the shrunken matrix becomes, after discarding the third column whose requirement is fulfilled as shown below.

We again compute the penalties in this shrunken matrix as given in the table below. 4 being the maximum penalty, we allocate in the cell (1, 2), with minimum cost, 1.

	D_1	D_2	a_i	
O_1	5	[10] 1	12	(4)
O_2	2	4	3	(2)
O_3	3	6	4	(3)
b_j	9	10		
	(1)	(3)		

Min $(a_1, b_2) =$ Min $(12, 10) = 10$, by which the requirement of the second column is met and we are left with the shrunken matrix given below.

Then the allocations of 2, 3, 4 are made in the cells (1, 1), (2, 1), (3, 1).

Thus the starting solution in this method will be

	D_1	a_i
O_1	[2] 5	2
O_2	[3] 2	3
O_3	[4] 3	4
b_j	9	

11 in (2, 3), 10 in (1, 2), 2 in (1, 1), 3 in (2, 1) and 4 in (3, 1).

The total cost in this method is

$11 \times 0 + 1 \times 10 + 5 \times 2 + 2 \times 3 + 3 \times 4$

$= 10 + 10 + 6 + 12 = 38.$

Thus we see that Vogel's approximation method gives better result in this problem.

In both the methods the number of allocations is $m + n - 1 = 3 + 3 - 1 = 5$. The corresponding cells do not form a loop and hence the allocations are independent.

Ex. 2 *Solve the following transportation problem :*

	D_1	D_2	D_3	D_4	a_i
O_1	10	7	3	6	3
O_2	1	6	8	3	5
O_3	7	4	5	3	7
b_j	3	2	6	4	

We use *VAM* to find the initial basic feasible solution of the problem. For that, we take into account the smallest cost c_{ij} in each row and each column and the second smallest cost, that is, just next to c_{ij} in the respective row and column. We then compute the non-negative difference between the smallest and the second smallest cost in each row and column and exhibit them in parenthesis. The maximum of these differences is 6 in the first column and we allocate

	D_1	D_2	D_3	D_4	a_i	
O_1	10	7	3	6	3	(3)
O_2	[3] 1	6	8	3	5	(2)
O_3	7	4	5	3	7	(1)
b_j	3	2	6	4		
	(6)	(2)	(2)	(0)		

	D_2	D_3	D_4	a_i	
O_1	7	[3] 3	6	3	(3)
O_2	6	8	3	2	(3)
O_3	4	5	3	7	(1)
b_j	2	6	4		
	(2)	(2)	(0)		

3 in the least cost cell (2, 1) of the first column, since Min (5, 3) = 3. This satisfies the column requirement of the first column and we pass on to the next table deleting the first column. In the second table we again compute the penalties and allocate 3 in the cell (1, 3) and delete the row O_1 by the same consideration. Same procedure is followed for the next tables and we get an initial basic solution as

$$x_{13} = 3,\ x_{21} = 3,\ x_{24} = 2,\ x_{32} = 2,\ x_{33} = 3,\ x_{34} = 2.$$

	D_2	D_3	D_4	a_i
O_2	6	8	[2] 3	2 (3)
O_3	[2] 4	[3] 5	[2] 3	7 (1)
b_j	2 (2)	3 (3)	4 (0)	

	D_1	D_2	D_3	D_4	a_i	u_i
O_1	(11) 10	(5) 7	[3] 3	(5) 6	3	−2
O_2	[3] 1	(2) 6	(3) 8	[2] 3	5	0
O_3	(6) 7	[2] 4	[3] 5	[2] 3	7	0
b_j	3	2	6	4		
v_j	1	4	5	3		

Here the number of allocations is 6 which is equal to $m+n-1 = 3+4-1$. Hence the solution is non-degenerate.

Then we test for the optimality of the solution, since the solution of the problem means the optimal solution. With this in view, we compute in the last table u_i and v_j as shown there such that for the occupied cells $c_{ij} = u_i + v_j$. Then, for the non-occupied cells, we compute the cell evaluations $\Delta_{ij} = c_{ij} - (u_i + v_j)$ and put them in circles in the unoccupied cells for saving space. We see that the cell evaluations are all positive and hence the above solution is optimal and unique.

The minimum cost of transportation will be

$3\times 3 + 3\times 1 + 2\times 3 + 2\times 4 + 3\times 5 + 2\times 3 = 47.$

Ex. 3. *Solve the following transportation problem :*

	A	B	C	a_i
F_1	10	9	8	8
F_2	10	7	10	7
F_3	11	9	7	9
F_4	12	14	10	4
b_j	10	10	8	

[*Kalyani Hons., 1982 ; Calcutta Hons., 2007*]

We apply the matrix minima method to find the initial basic feasible solution. There is a tie of minimum cost co-efficients in (2, 2) and (3, 3) cells. We choose (2, 2) arbitrarily and allocate there Min (a_2, b_2) = Min (7, 10) = 7 and discard the empty row F_2.

Then we allocate Min (9, 8) = 8 in (3, 3) and discard the column C whose demand is met. In the shrunken matrix the tie in minimum costs are in (1, 2) and (3, 2). We choose arbitrarily (3, 2) to allocate 1 unit there which is the left out capacity of F_3 after the allocation of 8 units in (3, 3). Proceeding in this way, we finally get the following complete allocation table in which the number of allocations is 6, which is $m + n - 1 = 4 + 3 - 1$.

	A	B	C	a_i
F_1	[6] 10	[2] 9	8	8
F_2	10	[7] 7	10	7
F_3	11	[1] 9	[8] 7	9
F_4	[4] 12	14	10	4
b_j	10	10	8	

The allocations are independent. The initial basic solution is thus $x_{11} = 6$, $x_{12} = 2$, $x_{22} = 7$, $x_{32} = 1$, $x_{33} = 8$, $x_{41} = 4$ and the corresponding cost is

$$6 \times 10 + 2 \times 9 + 7 \times 7 + 1 \times 9 + 8 \times 7 + 4 \times 12 = 240.$$

To test the optimality, we construct u_i and v_j as usual and proceed to find the cell evaluations, that is, $\{c_{ij} - (u_i + v_j)\}$ for the unoccupied cells (shown in circles) as below :

	A	B	C	u_i
F_1	[6] 10	[2] 9	(1) 8	0
F_2	(2) 10	[7] 7	(5) 10	−2
F_3	(1) 11	[1] 9	[8] 7	0
F_4	[4] 12	(3) 14	(1) 10	2
v_j	10	9	7	

Here the cell evaluations are all positive. Hence the initial basic solution obtained is optimal and unique.

Ex. 4. *Obtain an optimal basic feasible solution to the following transportation problem :*

	W_1	W_2	W_3	W_4	
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
	5	8	7	14	

[*Kalyani M. Sc., 1982; Vidyasagar Hons., 2002*]

We use *VAM* to find as usual the initial basic feasible solution. As seen from the table we see that the initial allocations are

$x_{11} = 5,\ x_{14} = 2,\ x_{23} = 7,\ x_{24} = 2,\ x_{32} = 8,\ x_{34} = 10,$

whose number is $m + n - 1 = 6$ and whose cost comes out as 779.

To test the optimality, we compute u_i and v_j and have the cell evaluations in circles of the unoccupied cells.

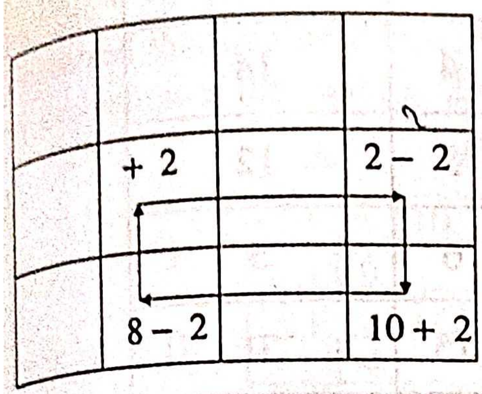
[The numbers in circles are the cell evaluations, that is, $c_{ij} - (u_i + v_j)$.]

	W_1	W_2	W_3	W_4	u_i
F_1	5 / 19	(32) / 30	(60) / 50	2 / 10	0
F_2	(1) / 70	(−18) / 30	7 / 40	2 / 60	50
F_3	(11) / 40	8 / 8	(70) / 70	10 / 20	10
v_j	19	− 20	− 10	10	

This shows that $\Delta_{22} = -18$, a negative quantity. Hence this allocation does not give an optimal solution.

We allocate maximum quantity (+ 2) to this cell (2, 2) and make the cell (2, 4) empty. Re-adjustment of allocations are made to maintain the row capacities and column requirements. For that, we add 2 to the cell (2, 2), subtract 2 from (2, 4), add 2 to (3, 4) and subtract 2 from (3, 2), (shown in the tables in the next page).

With these revised allocations, the new transportation table is constructed and shown on the right below.



	5 / 19	30	50	2 / 10	7
	70	2 / 30	7 / 40	60	9
	40	6 / 8	70	12 / 20	18
	5	8	7	14	

In the next iteration the same procedure is followed. We compute u_i and v_j for the table and get all cell evaluations non-negative in the unoccupied cells. The allocated cells or any sub-set of them do not form a loop and hence the solution is basic.

The elements in circles are cell evaluations, that is, $\{c_{ij} - (u_i + v_j)\}$ and they are all positive.

	W_1	W_2	W_3	W_4	u_i
F_1	5 / 19	(32) 30	(42) 50	2 / 10	0
F_2	(19) 70	2 / 30	7 / 40	(18) 60	32
F_3	(11) 40	6 / 8	(52) 70	12 / 20	10
v_j	19	− 2	8	10	

Thus the unique optimal solution is

$$x_{11} = 5,\ x_{14} = 2,\ x_{22} = 2,\ x_{23} = 7,\ x_{32} = 6,\ x_{34} = 12$$

and the optimal cost is

$$5 \times 19 + 2 \times 10 + 2 \times 30 + 7 \times 40 + 6 \times 8 + 12 \times 20 = 743.$$

Examples VIII

1. Find the basic feasible solution of the following transportation problems by North-West-Corner rule :

(i)

	D_1	D_2	D_3	D_4	a_i
O_1	19	20	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	14	

(ii)

FROM \ TO				a_i
	2	7	4	5
	3	3	1	8
	5	4	7	7
	1	6	2	14
b_j	7	9	18	

(iii)

ORIGINS \ DESTINATIONS						a_i
	2	11	10	3	7	4
	1	4	7	2	1	8
	3	9	4	8	12	9
b_j	3	3	4	5	6	

[*Jadavpur M.Sc.*, 1992]

[**Ans.** (i) $x_{11}=5,\ x_{12}=2,\ x_{22}=6,\ x_{23}=3,\ x_{33}=4,\ x_{34}=14.$

(ii) $x_{11}=5,\ x_{21}=2,\ x_{22}=6,\ x_{32}=3,\ x_{34}=4,\ x_{43}=14.$

(iii) $x_{11}=3,\ x_{12}=1,\ x_{22}=2,\ x_{23}=4,\ x_{24}=2,\ x_{34}=3,\ x_{35}=6.$]

2. Determine the basic feasible solution of Ex. 1 (iii) by Vogel's approximation method.

[**Ans.** $x_{14}=4,\ x_{22}=2,\ x_{25}=6,\ x_{31}=3,\ x_{32}=1,\ x_{33}=4,\ x_{34}=1.$]

3. Apply (*a*) Vogel's approximation method and

(*b*) Row minima method to find the basic feasible solution of the following transportation problem :

	1	2	3	4	a_i
1	21	16	25	13	11
2	17	18	14	23	13
3	32	27	18	41	19
b_j	6	10	12	15	

[**Ans.** (*a*) $x_{14}=11,\ x_{21}=6,\ x_{22}=3,\ x_{24}=4,\ x_{32}=7,\ x_{33}=12.$

(*b*) $x_{14}=11,\ x_{21}=1,\ x_{22}=12,\ x_{31}=5,\ x_{32}=10,\ x_{34}=4.$]

14. In the previous problems apply the matrix minima method to find the basic feasible solution in each.

15. Find the optimal (minimum) solution of the following transportation problems :

(i)

	D_1	D_2	D_3	D_4	a_i
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
b_j	20	40	30	10	

(ii)

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
b_j	16	18	31	25	

[*Tripura M. Sc., 1981*]

(iii)

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	3	4	6	8	8	20
O_2	2	10	0	5	8	30
O_3	7	11	20	40	3	15
O_4	1	0	9	14	16	13
b_j	40	6	8	18	6	

[*Kalyani M.Sc., 1991*]

(iv)

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	4	30
O_2	3	4	7	8	15
O_3	9	6	5	8	15
b_j	10	25	18	7	

[*Jadavpur M.Sc., 1983*]

(v)

	D_1	D_2	D_3	D_4	a_i
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
b_j	22	35	25	41	

(vi)

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25	

[**Ans.** (i) $x_{11} = 20$, $x_{13} = 10$, $x_{22} = 20$, $x_{23} = 20$, $x_{24} = 10$, $x_{32} = 20$;
Min. cost = 180.

(ii) $x_{12} = 18$, $x_{13} = 1$, $x_{21} = 12$, $x_{24} = 25$, $x_{31} = 4$, $x_{33} = 30$;
Min. cost = 355.

(iii) $x_{11} = 20$, $x_{21} = 4$, $x_{23} = 8$, $x_{24} = 18$, $x_{31} = 9$, $x_{35} = 6$, $x_{41} = 7$, $x_{42} = 6$
Min. cost = 246.

(iv) $x_{12} = 23$, $x_{14} = 7$, $x_{21} = 10$, $x_{22} = 2$, $x_{23} = 3$, $x_{33} = 15$;
Min. cost = 231.

(v) $x_{14} = 30$, $x_{21} = 5$, $x_{22} = 35$, $x_{31} = 17$, $x_{33} = 25$, $x_{34} = 11$;
Min. cost = 2221.

(vi) $x_{11} = 20$, $x_{13} = 10$, $x_{23} = 20$, $x_{24} = 10$, $x_{25} = 20$, $x_{32} = 40$, $x_{35} = 10$, $x_{36} = 25$, $x_{45} = 20$; Min. cost = 430.]

6. Determine the optimal basic solution to the following transportation problems :

(i)

	1	2	3	4	a_i
1	2	3	11	7	6
2	1	0	6	1	1
3	5	8	15	9	10
b_j	7	5	3	2	

[*Kalyani Hons., 1987*]

(ii)

	1	2	3	4	5	6	a_i
1	9	12	9	6	9	10	5
2	7	3	7	7	5	5	6
3	6	5	9	11	3	11	2
4	6	8	11	2	2	10	9
b_j	4	4	6	2	4	2	

[*Kalyani M. Sc., 1983*]

[**Ans.** (i) $x_{12} = 5$, $x_{13} = 1$, $x_{23} = 1$, $x_{31} = 7$, $x_{33} = 1$, $x_{34} = 2$;
Min. cost = 100.

(ii) $x_{13} = 5$, $x_{22} = 4$, $x_{26} = 2$, $x_{31} = 1$, $x_{33} = 1$, $x_{41} = 3$, $x_{44} = 2$, $x_{45} = 4$; Min. cost = 112.]

7. Solve the following transportation problems :

(i)

	A	B	C	a_i
I	50	30	220	1
II	90	45	170	3
III	270	200	50	4
b_j	4	2	2	

(ii)

	D_1	D_2	D_3	a_i
O_1	0	2	1	5
O_2	2	1	5	10
O_3	2	4	3	5
b_j	5	5	10	

[*Calcutta Hons., 1993*]

[**Ans.** (i) $x_{11} = 1$, $x_{21} = 3$, $x_{32} = 2$, $x_{33} = 2$; cost = 820.
(ii) $x_{13} = x_{21} = x_{22} = x_{33} = 5$; Min. cost = 35.]

8. Solve the following transportation problem :

	1	2	3	4	5	a_i
1	73	40	9	79	20	8
2	62	93	96	8	13	7
3	96	65	80	50	55	9
4	57	58	29	12	87	3
5	56	23	87	18	12	5
b_j	6	8	10	4	4	

[**Ans.** $x_{13} = 8$, $x_{24} = 4$, $x_{25} = 3$, $x_{31} = 5$, $x_{32} = 4$, $x_{41} = 1$, $x_{43} = 2$,
$x_{52} = 4$, $x_{55} = 1$; cost = 1102.]

9. Solve the following transportation problem :

	a	b	c	d	e	f	a_i
A	5	3	7	3	8	5	3
B	5	6	12	5	7	11	4
C	2	1	3	4	8	2	2
D	9	6	10	5	10	9	8
b_j	3	3	6	2	1	2	17

[**Ans.** $x_{12} = 1$, $x_{16} = 2$, $x_{21} = 3$, $x_{25} = 1$, $x_{33} = 2$, $x_{42} = 2$, $x_{43} = 4$,
$x_{44} = 2$; cost = 103.] [*Calcutta Hons., 2001*]