Consider the set $A = \{x \in \mathbb{R} / 1 < x < 2\}$.

(a) Show that A is bounded from ~~by~~ above. Find the supremum. Is this supremum a ~~no~~ max^m of A?

(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Sol^n:- (a) 2 is an upper bound of A.

Let $M > 1$ be an upper bound of A.

We will show that $2 \leq M$.

Suppose it is not true.

That is suppose $1 < M < 2$.

Let n be a national number such that $M < n < 2$.

Then $n \in A$ and $M < n$ which contradicts the fact that M is an upper bound of A.

Hence, we must have $2 \leq M$, so that $\sup\{A\} = 2$.

Since the supremum is not an element of A we conclude that 2 is not a maximum of A.

(b) Clearly, 1 is a lower bound of A.

Let m be a lower bound of A.

We will show that $m \leq 1$. Suppose it is not true.

That is, suppose that $1 < m < 2$.

Let n be a national number such that $1 < n < m$. Then $n \in A$ and $n < m$ which contradicts the fact that m is a lower bound of A. Thus we must have ~~8~~ $1 \leq m$ so that

$\inf\{A\} = 1$.
Since 1 is not in A, it is not a ~~max~~ minimum
~~sup~~ of A.

For each of the following sets S find
$\sup\{S\}$ and $\inf\{S\}$ if they exist.
~~You do not need to justify your answer~~

(a) $S = \{x \in \mathbb{R} : x^2 < 5\}$
(b) $S = \{x \in \mathbb{R} : x^2 > 7\}$
(c) $S = \{-\frac{1}{n} : n \in \mathbb{N}\}$

$\Rightarrow$ (a) $S = \{x \in \mathbb{R} : x^2 < 5\}$
$= \{x \in \mathbb{R} \mid -\sqrt{5} < x < \sqrt{5}\}$

So, $\sqrt{5}$ is an upper bound of S.
Let M be an upper bound S.
Suppose that $M < \sqrt{5}$.
Let r be a rational number such that $M < r < \sqrt{5}$. Then $r \in S$ and $M < r$.
But this contradicts the fact that M is an upper bound of S.
Thus, $\sqrt{5} \le M$ so that $\sup\{S\} = \sqrt{5}$.
Similarly one can show that $\inf\{S\} = -\sqrt{5}$.

(b) ~~sup S = ...~~ ~~inf S = ...~~
$\inf S = -\infty$ $\sup S = \infty$.

(c) $\sup\{S\} = 0$, ~~sup~~ $\inf\{S\} = -1$.

$\inf\{A\} = 1$.

Since 1 is not in A, it is not a ~~min~~ minimum ~~sup~~ of A.

For each of the following sets S find $\sup\{S\}$ and $\inf\{S\}$ if they exist. ~~You do not need to justify your answer~~

(a) $S = \{x \in \mathbb{R} : x^2 < 5\}$

(b) $S = \{x \in \mathbb{R} : x^2 > 7\}$

(c) $S = \{-\frac{1}{n} : n \in \mathbb{N}\}$

$\Rightarrow$ (a) $S = \{x \in \mathbb{R} : x^2 < 5\}$
$= \{x \in \mathbb{R} \mid -\sqrt{5} < x < \sqrt{5}\}$

So, $\sqrt{5}$ is an upper bound of S.

Let M be an upper bound S.

Suppose that $M < \sqrt{5}$.

Let r be a rational number such that $M < r < \sqrt{5}$. Then $r \in S$ and $M < r$.

But this contradicts the fact that M is an upper bound of S.

Thus, $\sqrt{5} \leq M$ so that $\sup\{S\} = \sqrt{5}$.

Similarly one can show that $\inf\{S\} = -\sqrt{5}$

(b) ~~sup S = ...~~ ~~inf S = ...~~

$\inf S = -\infty \quad \sup S = \infty$.

(c) $\sup\{S\} = 0$, $\inf\{S\} = -1$.

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Exercise

Let $A \subseteq \mathbb{R}$. Let $f, g : A \to \mathbb{R}$ be such that $|f(x)| \le M_1$ and $|g(x)| \le M_2$ $\forall x \in A$. Show the following

(a) $\sup\{f(x)+g(x) / x \in A\} \le \sup\{f(x)/x \in A\} + \sup\{g(x)/x \in A\}$

(b) $\inf\{f(x)+g(x)/x \in A\} \ge \inf\{f(x)/x \in A\} + \inf\{g(x)/x \in A\}$

(c) $\sup\{-f(x)/x \in A\} = -\inf\{f(x)/x \in A\}$

(d) $\sup\{f(x)-g(x)/x \in A\} \le \sup\{f(x)/x \in A\} - \inf\{g(x)/x \in A\}$

Solⁿ:- (a) For all $x \in A$, we have

$f(x)+g(x) \le \sup\{f(x)/x \in A\} + \sup\{g(x)/x \in A\}$

Thus, $\sup\{f(x)/x \in A\} + \sup\{g(x)/x \in A\}$ is an upper bound of $\{f(x)+g(x)/x \in A\}$

But $\sup\{f(x)+g(x)/x \in A\}$ is the smallest upper bound of $\{f(x)+g(x)/x \in A\}$ so that

$\sup\{f(x)+g(x)/x \in A\} \le \sup\{f(x)/x \in A\} + \sup\{g(x)/x \in A\}$

(b) ~~Similar~~

$f(x)+g(x) \ge \inf\{f(x)\} + \inf\{g(x)\}$

$\Rightarrow \inf\{f(x)\} + \inf\{g(x)\}$ is a lower bound of $\{f(x)+g(x)/x \in A\}$

But $\inf\{f(x)+g(x)/x \in A\}$ is the greatest lower bound of $\{f(x)+g(x)/x \in A\}$ so

Therefore
$$\inf\{f(x)+g(x)\} \geq \inf\{f(x) : x \in A\} + \inf\{g(x) : x \in A\}.$$

(c) We have
$$f(x) \geq \inf\{f(x)\}$$
$$\Rightarrow -f(x) \le -\inf\{f(x)\} \quad \forall x \in A.$$
$\Rightarrow -\inf\{f(x) / x \in A\}$ is an upper bound of $\{-f(x) / x \in A\}$.

Hence, $\sup\{-f(x) / x \in A\} \le -\inf\{f(x) / x \in A\}$

Suppose that,
$$\sup\{-f(x) / x \in A\} < -\inf\{f(x) / x \in A\}.$$
Let $\varepsilon = -\inf\{f(x)/x \in A\} - \sup\{-f(x)/x \in A\} > 0$

~~Therefore~~ By the defn of inf
$\exists\, a \in A$ s.t.
$$\inf\{f(x) / x \in A\} + \varepsilon > f(a).$$
$$\Rightarrow -\inf\{f(x) / x \in A\} - \varepsilon < -f(a) \le \sup\{-f(x) / x \in A\}$$

This leads to the contradiction
$$\sup\{-f(x) / x \in A\} < \sup\{-f(x) / x \in A\}.$$
Hence $\sup\{-f(x)\} = -\inf\{f(x) / x \in A\}$

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(d) We have,

$$\sup\{f(x)-g(x)/x\in A\} = \sup\{f(x)+(-g(x))/x\in A\}$$
$$\le \sup\{f(x)/x\in A\} + \sup\{-g(x)/x\in A\}$$
$$= \sup\{f(x)/x\in A\} - \inf\{g(x)/x\in A\}$$

For each of the following sets, compute the supremum & infimum

a) $A_1 = \{n\in\mathbb{N} / n^2<10\}$

b) $A_2 = \{\frac{n}{m+n} / m,n\in\mathbb{N}\}$

c) $A_3 = \{\frac{n}{2n+1} / n\in\mathbb{N}\}$

(d) $A_4 = \{n/m / m,n\in\mathbb{N} \text{ \& } m+n\le 10\}$.

Sol^n: (a) $\sup A_1 = 3$, $\inf A_1 = 1$ (d) $\inf A_4 = \frac{1}{9}$, $\sup A_4 = 9$.

(b) ~~$\sup A_2 =$~~

Fix m and very n

then $\lim_{n\to\infty}\frac{n}{m+n} = \lim_{n\to\infty}\frac{1}{\frac{m}{n}+1} = 1$

Fix n & very m

then $\lim_{m\to\infty}\frac{n}{m+n} = 0$

So, $\sup A_2 = 1$, $\inf A_2 = 0$.

(c) $\inf A_3 = \frac{1}{3}$

$\sup A_3 = 1/2$

$\boxed{2n > n,\ 2n+1 > n+1 \ge 2n,\ \frac{n}{2n+1} <}$ $\lim_{n\to\infty}\frac{n}{2n+1} = \lim_{n\to\infty}\frac{1}{2+1/n} = 1/2$

3. SETS IN R

3.1. Intervals.

Let $a, b \in \mathbb{R}$ and $a < b$.

The subset $\{x \in \mathbb{R} : a < x < b\}$ is said to be an *open interval*. The points a and b are called the *end points* of the interval. a and b are not points in the open interval. This open interval is denoted by (a, b).

The subset $\{x \in \mathbb{R} : a \leq x \leq b\}$ is said to be a *closed interval*. The end points a and b are points in the closed interval. This closed interval is denoted by $[a, b]$.

The subsets $\{x \in \mathbb{R} : a < x \leq b\}$ and $\{x \in \mathbb{R} : a \leq x < b\}$ are said to be *half open* (or *half closed*) intervals. One of the end points is a point in the interval. These half open intervals are denoted by $(a, b]$ and $[a, b)$ respectively.

The subset $\{x \in \mathbb{R} : x > a\}$ is an *infinite open interval*. This is denoted by (a, ∞).

The subset $\{x \in \mathbb{R} : x \geq a\}$ is an *infinite closed interval*. This is denoted by $[a, \infty)$

The subset $\{x \in \mathbb{R} : x < a\}$ is an *infinite open interval*. This is denoted by $(-\infty, a)$.

The subset $\{x \in \mathbb{R} : x \leq a\}$ is an *infinite closed interval*. This is denoted by $(-\infty, a]$.

When both the end points of an interval belong to $\mathbb{R}$, the interval is said to be a *bounded interval*.

Therefore the intervals $(a, b), [a, b], (a, b], [a, b)$ are all bounded intervals.

The intervals $(a, \infty), [a, \infty), (-\infty, a), (-\infty, a]$ are *unbounded intervals*.

If $a = b$, the closed interval $[a, a]$ is the singleton set $\{a\}$.

The set $\mathbb{R}$ is also denoted by $(-\infty, \infty)$. This is an unbounded interval without end points.

3.2. Neighbourhood.

Let $c \in \mathbb{R}$. A subset $S \subset \mathbb{R}$ is said to be a *neighbourhood* of c if there exists an open interval (a, b) such that $c \in (a, b) \subset S$.

Clearly, an open bounded interval containing the point c is a neighbourhood of c. Such a neighbourhood of c is denoted by $N(c)$.

A closed bounded interval containing the point c may not be a neighbourhood of c. For example, $1 \in [1, 3]$ but $[1, 3]$ is not a neighbourhood of 1.

Let $c \in \mathbb{R}$ and $\delta > 0$. The open interval $(c - \delta, c + \delta)$ is said to the δ-neighbourhood of c and is denoted by $N(c, \delta)$. Clearly, the δ-neighbourhood of c is an open interval symmetric about c.

Theorem 3.2.1. Let $c \in \mathbb{R}$. The union of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exist open intervals $(a_1, b_1), (a_2, b_2)$ such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1, a_2 < b_1; a_1 < b_2, a_2 < b_2$. Let $a_3 = \min\{a_1, a_2\}$, $b_3 = \max\{b_1, b_2\}$. Then $(a_1, b_1) \cup (a_2, b_2) = (a_3, b_3)$ and $c \in (a_3, b_3)$.

Now $(a_1, b_1) \subset S_1 \cup S_2$ and $(a_2, b_2) \subset S_1 \cup S_2$

$\Rightarrow (a_3, b_3) = (a_1, b_1) \cup (a_2, b_2) \subset S_1 \cup S_2$.

Thus $c \in (a_3, b_3) \subset S_1 \cup S_2$.

This proves that $S_1 \cup S_2$ is a neighbourhood of c.

Note. The union of a finite number of neighbourhoods of c is a neighbourhood of c.

Theorem 3.2.2. Let $c \in \mathbb{R}$. The intersection of two neighbourhoods of c is a neighbourhood of c.

Proof. Let $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then there exist open intervals $(a_1, b_1), (a_2, b_2)$ such that $c \in (a_1, b_1) \subset S_1$ and $c \in (a_2, b_2) \subset S_2$.

Then $a_1 < b_1, a_2 < b_1; a_1 < b_2, a_2 < b_2$.

Let $a_3 = \max\{a_1, a_2\}$, $b_3 = \min\{b_1, b_2\}$. Then $(a_1, b_1) \cap (a_2, b_2) = (a_3, b_3)$ and $c \in (a_3, b_3)$.

$$\begin{aligned} \text{Now } (a_3, b_3) &= (a_1, b_1) \cap (a_2, b_2) \subset (a_1, b_1) \subset S_1 \\ \text{and } (a_3, b_3) &= (a_1, b_1) \cap (a_2, b_2) \subset (a_2, b_2) \subset S_2 \\ &\Rightarrow (a_3, b_3) \subset S_1 \cap S_2. \end{aligned}$$

Thus $c \in (a_3, b_3) \subset S_1 \cap S_2$.

This proves that $S_1 \cap S_2$ is a neighbourhood of c.

Note. The intersection of a finite number of neighbourhoods of a point

c is a neighbourhood of c.

The intersection of an infinite number of neighbourhoods of a point c may not be a neighbourhood of c.

For example, for every $n \in \mathbb{N}$, $(-\frac{1}{n}, \frac{1}{n})$ is a neighbouhood of 0. $\bigcap\limits_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$. This is not a neighbourhood of 0.

3.3. Interior point.

Let S be a subset of $\mathbb{R}$. A point x in S is said to be an *interior point* of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$.

The set of all interior points of S is said to be the *interior* of S and is denoted by int S (or by S^o).

From definition it follows that $S^o \subset S$ for any set $S \subset \mathbb{R}$.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots \cdots\}$.
   Let $x \in S$. Every neighbourhood of x contains some points not in S. So x can not be an interior point of S. Therefore int $S = \phi$.

2. Let $S = \mathbb{N}$.
   Let $x \in S$. Every neighbourhood of x contains points not belonging to S. So x can not be an interior point of S. Therefore int $S = \phi$.

3. Let $S = \mathbb{Q}$.
   Let $x \in \mathbb{Q}$. Every neighbourhood of x contains rational as well as irrational points. So x can not be an interior point of $\mathbb{Q}$. So $S^o = \phi$.

4. Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Each point of S is an interior point of S. So int $S = S$.

5. Let $S = \mathbb{R}$. Each point of S is an interior point of S. Therefore $S^o = S$.

6. Let $S = \phi$. S has no interior point. Therefore int $S = \phi$.

3.4. Open set.

Let $S \subset \mathbb{R}$. S is said to be an *open set* if each point of S is an interior point of S.

Examples.

1. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\cdots\}$. No point of S is an interior point of S. S is not an open set.

2. Let $S = \mathbb{Z}$. No point of S is an interior point of S. S is not an open set.

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