CHAPTER VIII	Transportation Problems

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8.1 INTRODUCTION.

This section presents a special class of the general linear programming problem. It is called the *transportation type problem* (*T-T* problem) and has its own solution method independent of the general problem and completely detached from the simplex method. This model seeks the minimization of the cost of transporting a single commodity from a number of origins to different destinations. The capacity of production at each origin, the demand at each destination and the transportation cost of the commodity from different sources to various destinations are known. It is assumed that the cost of

transportation on a given route is directly proportional to the number of units transported.

A transportation problem is completely defined by a tableau of the following type :

DESTINATIONS

		D_1	D_2	....	D_n		
	O_1	c_{11}	c_{12}	....	c_{1n}	a_1	
	O_2	c_{21}	c_{22}	....	c_{2n}	a_2	
ORIGINS	⋮	.....	.....	.....	.....	⋮	... (1)
	O_m	c_{m1}	c_{m2}	....	c_{mn}	a_m	
		b_1	b_2	....	b_n		

The statement of the problem is as follows :

There are m origins O_i, $i = 1, 2, \ldots\ldots, m$. It is assumed that at each origin there are in storage a_i (shown on the right) units of some product, and are available for shipment. There are also n destinations D_j, $j = 1, 2, \ldots\ldots, n$. Each destination requires a shipment b_j units (shown at the bottom) of the product. The a_i's are called the *row requirements* (availability) and the b_j's are called the *column requirements.* These quantities are positive as non-positive row availability or column requirements have no physical meaning. Since there is only one commodity, a destination can receive its demand from one or more sources. In order for this problem to have feasible solution, the total supply from the origins must equal the total demand at the destinations and we assume

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j . \qquad \ldots \quad (2)$$

The cost of shipping a unit of the product from the origin O_i to the destination D_j is given by c_{ij}, called the *cost coefficient* assumed to be non-negative and are given in the matrix (1).

A solution to the transportation problem can be written as following solution matrix :

$$\mathbf{X} = \begin{array}{|c|c|c|c|} \hline x_{11} & x_{12} & \cdots\cdots & x_{1n} \\ \hline x_{21} & x_{22} & \cdots\cdots & x_{2n} \\ \hline \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \hline x_{m1} & x_{m2} & \cdots\cdots & x_{mn} \\ \hline \end{array} \quad \ldots$$

Each x_{ij} is constrained to be non-negative by

$$x_{ij} \geq 0, \text{ for all } i \text{ and } j \quad \ldots$$

and denotes the quantity to be shipped from the origin O_i to destination D_j.

Since there are a_i units to be shipped from each origin, ea solution matrix must satisfy

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots\ldots, m. \quad \ldots \quad ($$

Since b_j units must be shipped to each destination, each soluti matrix must satisfy

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots\ldots, n. \quad \ldots \quad ($$

In any given problem there are many solution matrices whic satisfy (5) and (6) , called the *rim requirements*, the cost of each bein given by

$$z = c_{11}\, x_{11} + c_{12}\, x_{12} + \cdots\cdots + c_{mn}\, x_{mn}$$

or written in a compact form it is

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij} \quad \ldots \quad (7)$$

The object will be to find the solution matrix that satisfies (5) and (6) and also minimizes z in (7) .

The transportation problem has $(m + n)$ constraints (5) and (6) in mn variables x_{ij} .

In view of (2), this transportation problem is called a *balanced transportation problem.*

If the total supply exceeds total demand or the total demand exceeds the total supply, then (5) and (6) are replaced by inequations.

If the sum of the shipments from a source cannot exceed its supply, then

$$\sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, 2, \ldots\ldots, m.$$

Similarly if the sum of the shipments to a destination cannot exceed its demand, then

$$\sum_{i=1}^{m} x_{ij} \leq b_j, \quad j = 1, 2, \ldots\ldots, n.$$

8.2 THE TRANSPORTATION TYPE PROBLEMS IN STANDARD L. P. FORM.

A general transportation problem involving m origins and n destinations can now be stated as below.

Minimize $z = \mathbf{cx}$, subject to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,

in which

$\mathbf{c} = (c_{11}, c_{12}, \ldots, c_{ij}, \ldots, c_{mn})$ is an mn-component row vector,

$\mathbf{x} = [x_{11}, x_{12}, \ldots, x_{ij}, \ldots, x_{mn}]$ is an mn-component column vector,

$\mathbf{b} = [a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n]$ is an $(m+n)$-component column vector,

$\mathbf{A} = [\mathbf{a}_{11}, \mathbf{a}_{12}, \ldots, \mathbf{a}_{ij}, \ldots, \mathbf{a}_{mn}]$ is the coefficient matrix in which $\mathbf{a}_{ij}$ is the column vector associated with the variable x_{ij}.

Rewriting the constraints $\mathbf{Ax} = \mathbf{b}$ in details, we have

$$\begin{array}{lllll}
x_{11} + x_{12} + \cdots\cdots + x_{1n} & & & & = a_1 \\
 & x_{21} + x_{22} + \cdots\cdots + x_{2n} & & & = a_2 \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
 & & x_{m1} + x_{m2} + \cdots\cdots + x_{mn} & & = a_m \\
x_{11} & + x_{21} + \cdots\cdots + x_{m1} & & & = b_1 \\
x_{12} & & + x_{22} + \cdots\cdots + x_{m2} & & = b_2 \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
x_{1n} & & + x_{2n} + \cdots\cdots + x_{mn} & & = b_n
\end{array}$$

from which the coefficient matrix **A** can be written easily. It is observed from the formation of **A** that each of the column vectors is an $(m+n)$-component vector. It is further observed that the variable x_{ij} appears only in the two constraints, which correspond to the i-th and $(m+j)$-th constraints respectively.

Thus, in each of the vectors $\mathbf{a}_{ij}$, there will be two 'one's and $(m+n-2)$ 'zero's as its elements.

As a matter of fact

$$\mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j}$$

where $\mathbf{e}_k$ denotes the k-th column of the identity matrix of order $(m+n)$.

A set of variables $x_{ij} \geq 0$ is feasible, if and only if

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots\ldots, m$$

and

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots\ldots, n.$$

8.3 SPECIAL FEATURE OF TRANSPORTATION TYPE PROBLEM.

In this section we discuss some of the special features as to the solution of a transportation problem with the help of some theorems.

Theorem 1. *The number of basic variables in a transportation problem is at most* $(m+n-1)$.

We shall now show that a basis for the transportation problem consists at most of $(m+n-1)$ variables although there are $(m+n)$ constraints given by

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \cdots\cdots, m \qquad \ldots (1)$$

and

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \cdots\cdots, n. \qquad \ldots (2)$$

If we sum the m constraints (1), then we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \text{(balanced problem).}$$

Summing the first $(n-1)$ constraints (2), we get

$$\sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j .$$

Subtracting the second from the former, we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j$$

or, $$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij} \right) = b_n$$

or, $$\sum_{i=1}^{m} x_{in} = b_n$$

which is precisely the n-th constraint of (2). Thus one of the $(m+n)$ constraints is redundant and may be removed from the set of constraints. As a result a basis consists of at most $(m+n-1)$ variables.

Note. From this we conclude that a non-degenerate basic feasible solution to the problem will consist of at most $(m+n-1)$ positive variables, the rest being zero. By the fundamental theorem of linear programming, one of these basic feasible solutions will be optimal.

Theorem 2. *The transportation problem always has a feasible solution.*

If $T = \sum_i a_i = \sum_j b_j$, then we shall show that $x_{ij} = \dfrac{a_i b_j}{T}$ will be such a feasible solution.

Since all a_i and b_j are positive quantities, therefore $x_{ij} > 0$, for all i and j.

Summing $x_{ij} = \dfrac{a_i b_j}{T}$ over j and i respectively, we get

$$\sum_j x_{ij} = \frac{a_i}{T} \sum_j b_j = \frac{a_i}{T} T = a_i .$$

and $$\sum_i x_{ij} = \frac{b_j}{T} \sum_i a_i = \frac{b_j}{T} \cdot T = b_j .$$

These are the constraints of the transportation problem and are satisfied by

$$x_{ij} = \frac{a_i b_j}{T}, \quad i = 1, 2, \ldots\ldots, m; \; j = 1, 2, \ldots\ldots, n.$$

Hence there exists a feasible solution.

Theorem 3. *The solution of the transportation problem is never unbounded.*

Each variable x_{ij} appears exactly in the two constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad \sum_{i=1}^{m} x_{ij} = b_j,$$

both times with a coefficient (+ 1).

Hence it is easily seen that x_{ij} is bounded and

$$0 \leq x_{ij} \leq \text{Max } (a_i, b_j).$$

Thus no x_{ij} can assume an arbitrarily large value, as a_i and b_j are finite.

From these we observe that the transportation problem must have a basic feasible solution, as it has a feasible solution. Furthermore as all c_{ij} and x_{ij} are finite, therefore z will be bounded from below and above and hence the objective function will have some optimal value. There will be a basic feasible solution to the problem which will be optimal.

8.4 A SET OF BASIC FEASIBLE SOLUTIONS.

We have seen earlier that a basic feasible solution will contain at most $(m + n - 1)$ positive variables, with the remaining variables zero. In order that a set of feasible solutions is basic, the $(m + n - 1)$ columns of the coefficient matrix **A**, corresponding to these variables must be linearly independent, that is to say, the vectors $\mathbf{a}_{ij}$ associated to those variables must be linearly independent. To derive a criterion whether a given feasible solution is basic, we proceed with the following definitions and properties associated with the format of the tableau.

We have seen before that a transportation tableau has been sub-divided into mn boxes arranged in m rows and n columns. The box in the i-th row and j-th column of the tableau is called the (i, j)

cell. In the transportation tableau, a sequence of cells is said to form a *loop*, if

(i) each adjacent pair of cells either lies in the same column or in the same row ;

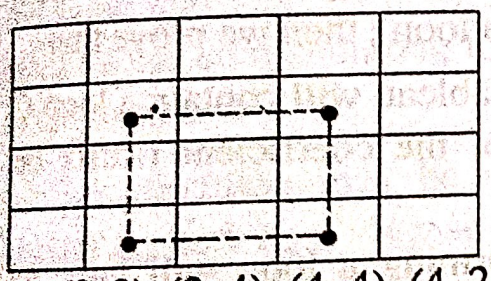
(ii) not more than two consecutive cells in the sequence lie in the same row or in the same column ;

(iii) the first and the last cells in the sequence lie either in the same row or in the same column ;

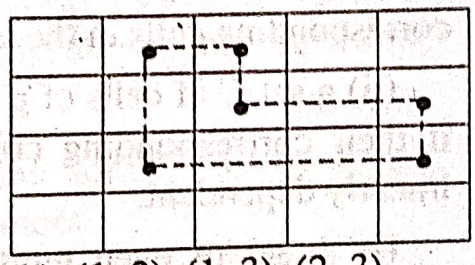
(iv) the sequence must involve at least two rows or two columns of the table.

Loops are closed which have neither 'beginning' nor 'end'.

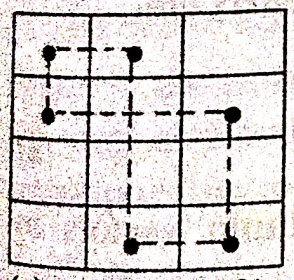
We give below a few illustrations of loops. The cells forming the loops are shown with the respective diagram.



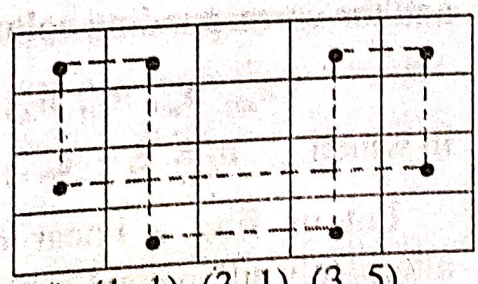
(a) (2, 2), (2, 4), (4, 4), (4, 2).



(b) (1, 2), (1, 3), (2, 3), (2, 5), (3, 5), (3, 2).



(c) (1, 1), (1, 2), (4, 2), (4, 3), (2, 3), (2, 1).



(d) (1, 1), (3, 1), (3, 5), (1, 5), (1, 4), (4, 4), (4, 2), (1, 2).

The loop (d) contains four cells in the first row, but no three of them are consecutive members of the loop. We observe that every loop contains an even number of cells ; every row or column either contains no cell or an even number of cells in the loop. In forming a loop, the cells are ordered as (a, b), (a, c), (d, c), (d, e), . . . , (p, q), (p, b),

that is, we alternate between moving to a cell in the same column and between moving to a cell in the same row when the cells form a loop. A set of cells S of a transportation table is said to contain a loop if the cells S or a sub-set of S can be sequenced to form a loop. Then we prove a very important property related to linear independence of a set of $(m + n - 1)$ columns of basis vectors of the coefficient matrix.

Theorem. *A sub-set of the columns of the coefficient matrix of a transportation problem are linearly dependent, if and only if, the corresponding cells or a sub-set of them can be sequenced to form a loop.*

We prove this theorem in two parts. First we prove that

(i) a sub-set S of column vectors of the coefficient matrix of a transportation problem will be linearly dependent, if their corresponding cells in the tableau contain a loop ; then we prove that

(ii) a set C of cells of a transportation tableau will contain a loop C if their corresponding column vectors of the coefficient matrix be linearly dependent.

(i) A set of even number of cells forming a loop will have its sequence of the form

$$(a, b), (a, c), (d, c), (d, e), \cdots\cdots, (p, q), (p, b)$$

and the corresponding columns $\mathbf{a}_{ij}$ of the coefficient matrix are

$$\mathbf{a}_{ab}, \mathbf{a}_{ac}, \mathbf{a}_{dc}, \mathbf{a}_{de}, \cdots\cdots, \mathbf{a}_{pq}, \mathbf{a}_{pb} \qquad \ldots (1)$$

in which $\quad \mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j}$. $\qquad \ldots (2)$

Let us find a linear combination of these column vectors by alternately adding and subtracting the vectors. Thus

$$\begin{aligned}
&\mathbf{a}_{ab} - \mathbf{a}_{ac} + \mathbf{a}_{dc} - \mathbf{a}_{de} + \cdots\cdots + \mathbf{a}_{pq} - \mathbf{a}_{pb} \\
&= (\mathbf{e}_a + \mathbf{e}_{m+b}) - (\mathbf{e}_a + \mathbf{e}_{m+c}) + (\mathbf{e}_d + \mathbf{e}_{m+c}) \\
&\qquad - (\mathbf{e}_d + \mathbf{e}_{m+e}) + \cdots\cdots + (\mathbf{e}_p + \mathbf{e}_{m+q}) - (\mathbf{e}_p + \mathbf{e}_{m+b}) \\
&= (\mathbf{e}_a - \mathbf{e}_a) + (\mathbf{e}_{m+b} - \mathbf{e}_{m+b}) + \cdots\cdots + (\mathbf{e}_p - \mathbf{e}_p) \\
&= \mathbf{0}, \text{ since the number of cells is even.}
\end{aligned}$$

Thus the set of column vectors (1) equals the null vector and is thus linearly dependent.