Metric Spaces

Second Edition

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This book intended to serve as textbook for honours and postgraduate students of universities in India and abroad begins with a chapter on Preliminaries discussing basic concepts and results followed by chapters on Introductory Concepts, Completeness, Continuous Functions, Compactness, Connectedness, and Fixed Point Theorems and Their Applications. Two appendices have been given on Control Set and Limits in Metric Spaces.

Encouraged by the response to the first edition the authors have thoroughly revised this edition.

New to this edition:

- The concept of neighborhood, open sets and interior points are defined via neighborhood and the proofs of results relating to open sets and interior points modified accordingly. Closed sets are defined as the complement of an open set and necessary changes in the statements of the results and their proofs on closed sets are made
- The concept of equivalent metrics introduced in Chapter 2
- Motivation for homeomorphism
- Geometrical interpretation of open spheres in R and their figures added
- A number of results, examples and counter examples along with examples of non-Metric Spaces
- Numerous solved and unsolved problems

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(iv) The set $\mathcal{A}$ of all algebraic numbers.
(v) The set of all polynomials with rational coefficients.
(vi) The set of all complex numbers which are algebraic over the field of rational numbers.
(vii) The set of all finite sequences whose terms are algebraic numbers.
(vii) The set of all straight lines in a plane each of which passes through (at least) two different points with rational coordinates.
(ix) The set of all rational points in $\mathbb{R}^n$.

7. The family of all finite subsets of a countable set is countable.
8. Each of the following is an uncountable set:

(i) An open interval $]a,b[$, a closed interval $[a, b]$, where $a \neq b$; more generally, any interval I which do not degenerate to a single point.
(ii) The set of irrational numbers, the set of transcendental numbers.
(iii) $\mathbb{R}$, the set of all real numbers.
(iv) The set of all sequences of natural numbers.
(v) The set of all points of a plane.
(vi) The family of all subsets of a denumerable set.

1.6 INEQUALITIES

We now give some inequalities, which will be freely used in the subsequent work. Throughout, denote by $\mathbb{K}$, the field $\mathbb{R}$ (or $\mathbb{C}$) of real (or complex) numbers.

1.6.1 The Triangle Inequality

Let $\alpha, \beta \in \mathbb{K}$. Then

$$|\alpha+\beta| \leq |\alpha| + |\beta|$$

1.6.2 Let $\alpha, \beta \in \mathbb{K}$. Then

$$\frac{|\alpha+\beta|}{1+|\alpha+\beta|} \leq \frac{|\alpha|}{1+|\alpha|} + \frac{|\beta|}{1+|\beta|}$$

1.6.3 Let $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. If $\alpha, \beta \in \mathbb{K}$, then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

with equality if and only if $\alpha^p = \beta^q$.

1.6.4 Holder's Inequality (Finite form)

Let $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. If $\alpha_i, \beta_i \in \mathbb{K}$ $(i = 1, 2, \ldots, n)$, then

$$\sum_{i=1}^{n} |\alpha_i\beta_i| \leq \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |\beta_i|^q\right)^{1/q}$$

Also $$\sum_{i=1}^{n} |\alpha_i\beta_i| \leq \left(\sum_{i=1}^{n} |\alpha_i|\right) . \max_{1\leq i\leq n} |\beta_i|$$

1.6.5 Cauchy-Schwarz Inequality (Finite form)

Note the inequality (1.6.4) for the case when $p = q = 2$.

$$\sum_{i=1}^{n} |\alpha_i\beta_i| \leq \left(\sum_{i=1}^{n} |\alpha_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |\beta_i|^2\right)^{1/2}$$

1.6.6 Minkowski's Inequality (Finite form)

Let $1 \leq p < \infty$. If $\alpha_i, \beta_i \in \mathbb{K}$ $(i = 1, 2, \ldots, n)$, then

$$\left(\sum_{i=1}^{n} |\alpha_i + \beta_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |\alpha_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |\beta_i|^p\right)^{1/p}$$

1.6.7 Let $0 \leq p \leq 1$. If $\alpha_i, \beta_i \in \mathbb{K}$ $(i = 1, 2, \ldots, n)$, then

$$\sum_{i=1}^{n} |\alpha_i + \beta_i|^p \leq \sum_{i=1}^{n} |\alpha_i|^p + \sum_{i=1}^{n} |\beta_i|^p$$

1.6.8 Holder's Inequality (Infinite form)

Let $1 \leq p < \infty$ and q is conjugate to p. If $(\alpha_1, \alpha_2, \ldots) \in l^p$, $(\beta_1, \beta_2, \ldots) \in l^q$; i.e.,

$$\sum_{i=1}^{\infty} |\alpha_i|^p < \infty, \sum_{i=1}^{\infty} |\beta_i|^q < \infty,$$

then $$\sum_{i=1}^{\infty} |\alpha_i\beta_i| \leq \left(\sum_{i=1}^{\infty} |\alpha_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |\beta_i|^q\right)^{1/q}$$

1.6.9 Cauchy Schwarz Inequality (Infinite form)

Note the inequality (1.6.8) for the case when $p = q = 2$.

$$\sum_{i=1}^{\infty} |\alpha_i\beta_i| \leq \left(\sum_{i=1}^{\infty} |\alpha_i|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} |\beta_i|^2\right)^{1/2}$$

1.6.10 Minkowski's Inequality (Infinite form)

Let $1 \le p < \infty$. If $(\alpha_1, \alpha_2, \ldots), (\beta_1, \beta_2, \ldots) \in l^p$; i.e.,

$$\sum_{i=1}^{\infty} |\alpha_i|^p < \infty, \sum_{i=1}^{\infty} |\beta_i|^p < \infty,$$

then
$$\left(\sum_{i=1}^{\infty} |\alpha_i + \beta_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{\infty} |\alpha_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |\beta_i|^p\right)^{1/p}.$$

2

INTRODUCTORY CONCEPTS

In analysis, we are concerned mainly with two elementary concepts: (i) convergent sequences in $\mathbb{K}$ (real or complex numbers) and (ii) continuous functions with domains and ranges in $\mathbb{K}$. We note that each of these notions depend precisely on the concept of the absolute value $|x - x_0|$ of the difference between the numbers x and x_0 in $\mathbb{K}$. Many of the properties of convergent sequences and continuous functions depend only on the properties of this distance, and not directly on the algebraic nature of the real (or complex) number systems. We write $d(x, y) = |x - y|$, the distance between x and y. The following properties of the distance are well known:

(i) $|x - y| \ge 0$
(ii) $|x - y| = 0 \Leftrightarrow x = y$
(iii) $|x - y| = |y - x|$
(iv) $|x - y| \le |x - z| + |z - y|$

We want to generalize the concept of distance by taking any non-empty set (instead of $\mathbb{R}$ or $\mathbb{C}$) in such a way that we can generalize the notions of convergence of sequences and continuity of functions (real and complex) to a more general situation.

2.1 DEFINITION AND EXAMPLES OF METRIC SPACES

2.1.1 Definition

Let X be a non-empty set. A function

$$d: X \times X \to \mathbb{R}$$

is said to be a *metric* on X if it satisfies the following conditions:

(i) $d(x,y) \ge 0, \quad \forall\, x, y \in X$ (Positiveness) +ve
(ii) $d(x, y) = 0 \Leftrightarrow x = y, \quad x, y \in X$
(iii) $d(x, y) = d(y, x), \quad \forall\, x, y \in X$ (symmetry)
(iv) $d(x, y) \le d(x, z) + d(z, y), \quad \forall\, x, y, z \in X$ (triangle inequality)

The ordered pair (X, d) is called a *metric space*. If there is no confusion likely to occur we, sometimes, denote the metric space (X, d) by X.

Note When we say that d is *a* metric on X or (X, d) is a metric space, it is understood throughout that X is *a* non-empty set.

Semi-metric/semi-metric space - if satisfies all (i),(ii),(iv).

Remarks

1. The triangle inequality may be interpreted as that "*the length of one side of a triangle can not exceed the sum of the lengths of the other two sides*". Equivalently, the distance from x to y via any intermediate point z can not be shorter than the direct distance from x to y.

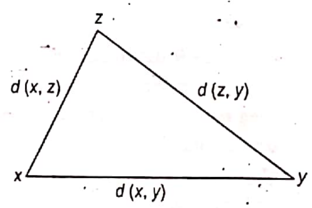

Fig. 2.1

2. The triangle inequality can be generalised for any number of additional points $z_1, z_2, \ldots, z_n$ in X; i.e.,

$$d(x, y) \le d(x, z_1) + d(z_1, z_2) + \ldots + d(z_n, y)$$

2.1.2 Examples

1. Let $X = \mathbb{R}$, the set of all real numbers. For $x, y \in X$, define

$$d(x, y) = |x - y|$$

Then (X, d) is a metric space. This is called the metric space $\mathbb{R}$ with the usual metric and we denote it by $\mathbb{R}_u$.

VH-12

2. Let $X = \mathbb{C}$, the set of all complex numbers. For $x, y \in X$, define

$$d(x, y) = |x - y|$$

Then (X, d) is a metric space. This is called the metric space $\mathbb{C}$ with the usual metric and we denote it by $\mathbb{C}_u$.

Note If there is no confusion likely to occur, we may simply write $\mathbb{R}$ (resp., $\mathbb{C}$) in place of $\mathbb{R}_u$ (resp., $\mathbb{C}_u$).

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3. Let X be an arbitrary non-empty set. For $x, y \in X$, define d by

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \ne y \end{cases}$$

Then (X, d) is a metric space. The metric d is called the *discrete metric* and the space (X, d) is called *discrete metric space* and is denoted by X_d.

4. Let $X = \mathbb{Q}$, the set of all rational numbers. For $x, y \in X$, define

$$d(x, y) = |x - y|$$

Then (X, d) is a metric space.

5. Let $X = [0, 1[$. For $x, y \in X$, define

$$d(x, y) = |x - y|$$

Then (X, d) is a metric space.

6. Let $X = \mathbb{R}^2$, the set of all points in the coordinate plane. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X, define

(i) $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ $= \left\{\sum_{i=1}^{2} (x_i - y_i)^2\right\}^{1/2}$

(ii) $d^*(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ $= \max_{1 \le i \le 2} \{|x_i - y_i|\}$

(iii) $d^{**}(x, y) = |x_1 - y_1| + |x_2 - y_2|$ $= \sum_{i=1}^{2} |x_i - y_i|$

Then each of the spaces (X, d), (X, d^*) and (X, d^{**}) is a metric space.

7. Let $X = \mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X, define

$$d(x, y) = \begin{cases} |x_1 - y_1|, & x_2 = y_2 \\ |x_1| + |y_1| + |x_2 - y_2|, & x_2 \ne y_2 \end{cases}$$

Then (X, d) is a metric space.

8. Let $X = \mathbb{R}^n$, the set of all ordered n-tuples of real numbers. For $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $y = (\beta_1, \beta_2, \ldots, \beta_n)$ in X, define

$$d(x, y) = \left\{\sum_{i=1}^{n} (\alpha_i - \beta_i)^2\right\}^{\frac{1}{2}}$$

Then (X, d) is *a* metric space.

In view of Minkowski's Inequality 1.6.6, for $p = 2$, we note that

$$d(x, y) = \left\{\sum_{i=1}^{n} (\alpha_i - \beta_i)^2\right\}^{\frac{1}{2}}$$

$$= \left\{\sum_{i=1}^{n} (\alpha_i - \gamma_i + \gamma_i - \beta_i)^2\right\}^{\frac{1}{2}}$$

$$\le \left\{\sum_{i=1}^{n} (\alpha_i - \gamma_i)^2\right\}^{\frac{1}{2}} + \left\{\sum_{i=1}^{n} (\gamma_i - \beta_i)^2\right\}^{\frac{1}{2}}$$

$$= d(x, z) + d(z, y),$$

where $z = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is in X. This verifies the triangle inequality. The other conditions for d to be a metric on $\mathbb{R}^n$ are quite easy to check.

9. Let $X = \mathbb{C}^n$, the set of all ordered n-tuples of complex numbers. For $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $y = (\beta_1, \beta_2, \ldots, \beta_n)$ in X, define

$$d(x, y) = \left\{\sum_{i=1}^{n} |\alpha_i - \beta_i|^2\right\}^{\frac{1}{2}}$$

Then (X, d) is a metric space. δ is called Euclidean metric on $\mathbb{R}^n$

Note The metric space in Example 8 is called the Euclidean n-space and is denoted by $\mathbb{R}^n$ while the metric space in Example 9 is called the unitary n-space (or Complex Euclidean n-space) and is denoted by $\mathbb{C}^n$.

10. Let $X = \mathbb{K}^n$ ($\mathbb{R}^n$ or $\mathbb{C}^n$). Define

(i) $d^*(x, y) = \max\limits_{1 \le i \le n} |\alpha_i - \beta_i|$

VH-II (ii) $d^{**}(x, y) = \sum\limits_{i=1}^{n} |\alpha_i - \beta_i|$ [δ^{**} is called rectangular metric on $\mathbb{R}^n$]

Then, each of the spaces (X, d^*) and (X, d^{**}) is a metric space.

11. Let $X = \mathbb{K}^n$ ($\mathbb{R}^n$ or $\mathbb{C}^n$). Define

$$d_p(x, y) = \left\{\sum_{i=1}^{n} |\alpha_i - \beta_i|^p\right\}^{\frac{1}{p}}, 1 \le p < \infty$$

Then (X, d_p) is a metric space (verify). We denote the metric space $(\mathbb{K}^n, d_p)$ by l_n^p.

Note l_n^2 is the Euclidean metric n-space $\mathbb{R}^n$ or the unitary n-space $\mathbb{C}^n$ according as $X = \mathbb{R}^n$ or $\mathbb{C}^n$.

Let us now generalize $\mathbb{R}^n$ (and $\mathbb{C}^n$), in Example 11 above, to 'infinite-tuples' which, in fact, are the sequences in $\mathbb{K}$. While doing so, we shall be putting suitable restrictions on the sequences to be considered so as to make them meaningful.

12. Let $1 \le p < \infty$. Consider the set of all sequences $\{\alpha_n\}$ in $\mathbb{K}$ such that

$$\sum_{n=1}^{\infty} |\alpha_n|^p < \infty$$

Denote this set by l^p. For $x = \{\alpha_n\}$ and $y = \{\beta_n\}$ in l^p, define

$$d_p(x, y) = \left\{\sum_{n=1}^{\infty} |\alpha_n - \beta_n|^p\right\}^{\frac{1}{p}}$$

Then (X, d_p) is a metric space.

In order to establish that d_p defines a metric on l^p, it is enough to verify the triangle inequality since the other axioms are easy to check.

Let $z = \{\gamma_n\} \in l^p$ be arbitrary. Then,

$$d(x, y) = \left\{\sum_{n=1}^{\infty} |\alpha_n - \beta_n|^p\right\}^{1/p}$$

$$= \left\{\sum_{n=1}^{\infty} |\alpha_n - \gamma_n + \gamma_n - \beta_n|^p\right\}^{1/p}$$

$$\le \left\{\sum_{n=1}^{\infty} |\alpha_n - \gamma_n|^p\right\}^{1/p} + \left\{\sum_{n=1}^{\infty} |\gamma_n - \beta_n|^p\right\}^{1/p}$$

(by Minkowski's Inequality 1.6.10)

$$= d(x, z) + d(z, y)$$

13. Let l^∞ be the set of all bounded sequences in $\mathbb{K}$; i.e.,

$$l^\infty = \left\{\{\alpha_n\} \subset \mathbb{K} : \sup_{1 \le n < \infty} |\alpha_n| < \infty\right\}$$

For $x = \{\alpha_n\}$ and $y = \{\beta_n\}$ in l^∞, define

$$d_\infty(x, y) = \sup_{1 \le n < \infty} |\alpha_n - \beta_n|$$

P-9 PYQ Then, d_∞ defines a metric on l^∞.

14. Let $B[a, b]$ be the set of all real valued functions defined and bounded on $[a, b]$. For $x, y \in B[a, b]$, define

$$d_\infty(x, y) = \sup_{t \in [a, b]} |x(t) - y(t)|$$

P-9 Then $(B[a, b], d_\infty)$ is a metric space.

15. Let $C[a, b]$ be the set of all real valued continuous functions defined on $[a, b]$. For $x, y \in C[a, b]$, define

$$d_\infty(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|$$

Then $(C[a, b], d_\infty)$ is a metric space.

Note The metric d_∞ defined on $B[a, b]$ and $C[a, b]$, in Examples 14 and 15, is called the *uniform metric*. This metric, in fact, corresponds to the uniform convergence of functions in the space $C[a, b]$.

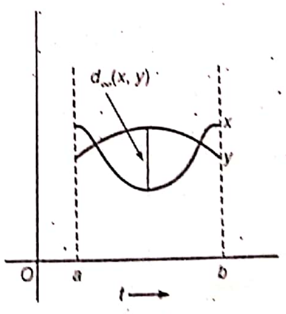

Fig. 2.2

VH-09

Ex) Let X be a non-empty set and a function $\delta : X \times X \to \mathbb{R}$ satisfy (i) $\delta(x, y) = 0$ iff $x = y$ and (ii) $\delta(x, y) \le \delta(x, z) + \delta(y, z)$, $\forall x, y, z \in X$. Prove that (X, δ) is a metric space.

16. For $x, y \in C[a, b]$, define

$$d_1(x, y) = \int_a^b |x(t) - y(t)| \, dt$$

the integral on the right being taken in the sense of Riemann which is possible since the functions x and y are continuous on $[a, b]$. Then $(C[a, b], d_1)$ is a metric space.

Note $d_1(x, y)$ represents the absolute area between the functions x and y as a measure of the distance between these two functions.

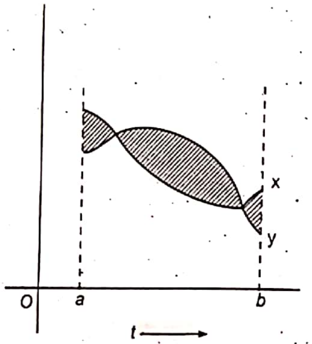

Fig. 2.3

17. For $x, y \in C[a, b]$, define

$$d_p(x, y) = \left\{ \int_a^b |x(t) - y(t)|^p \right\}^{1/p}, \quad 1 \le p < \infty$$

Then $(C[a, b], d_p)$ is a metric space.

18. Let $X = \mathbb{R}$. For $x, y \in X$, define

$$d(x, y) = |x^2 - y^2|$$

Then (X, d) is not a metric space.

19. Let $X = \mathbb{R}$. For $x, y \in X$, define

$$d(x, y) = |\sin(x - y)|$$

Then (X, d) is not a metric space.

Problems

1. For $x, y \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), define

$$d(x, y) = \min\{1, |x - y|\}$$

Prove that d is a metric on $\mathbb{K}$.

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2. Let (X, d) be a metric space and let k be a fixed positive real number. For $x, y \in X$, define

$$d^*(x, y) = kd(x, y)$$

Prove that d^* is a metric on X.

3. Let (X, d) be a metric space. For $x, y \in X$, define

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$ VH-13

and

$$d^{**}(x, y) = \min\{1, d(x, y)\}$$

Prove that d and d^* are metrics on X.
[Hint: The triangle inequality for d^* follows by using the inequalities

$$\frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \le \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)}$$

and

$$\frac{1}{1 + d(x, y) + d(y, z)} \le \frac{1}{1 + d(y, z)}].$$

4. For $x, y \in \mathbb{K}$, define

$$d(x, y) = \begin{cases} 0, & x = y \\ |x| + |y|, & x \ne y \end{cases}$$

Prove that d is a metric on $\mathbb{K}$.

5. Let ω denote the space of all sequences in $\mathbb{K}$. Define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n (1 + |x_n - y_n|)}$$ – (Frechet space 1906)

where $x = \{x_n\}$ and $y = \{y_n\}$ are in ω. Prove that d defines a metric on ω.
[Hint: The factor 2^{-n} in the terms of the series ensures the convergence of the series. Use the Inequality 1.6.2].

6. Let d and d^* be metrics defined on a set X. Prove that:
   (*a*) $d + d^*$ is a metric on X. In particular, $2d$ is a metric on X.
   (*b*) $\max\{d, d^*\}$ is a metric on X.

7. Let (X, d) be a metric space. Prove that

$$|d(x, z) - d(z, y)| \le d(x, y)$$

for all $x, y, z \in X$.

8. For a given set A, let X denote the family of all finite subsets of A. Let $x \Delta y$ denote the symmetric difference of the two subsets x and y of A. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \#(x \Delta y), \text{ the number of elements in } x \Delta y.$$

Prove that d defines a metric on X.

Ex! Is ρ defined by $\rho(x, y) = \lim_{n\to\infty} |x_n - y_n|$ a metric on c (set of all conv. seq.)

Hint! at $x_n = \frac{1}{n}$, $y_n = 0$ $\forall n \in \mathbb{N}$

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9. In $C[0,1]$, determine the value of $d_\infty(x, y)$, when

(a) $x(t) = [t]$ and $y(t) = t^2$,
where [ ] denotes the greatest integer function.

(b) $x(t) = t$ and $y(t) = t^2$.

10. In $C[0, 1]$, determine the values of $d_\infty(x, y)$ and $d_1(x, y)$, when

(a) $x(t) = t^3 + t + 1$ and $y(t) = t^3 + t^2 + \frac{1}{2}t + 1$

(b) $x(t) = \sin t$ and $y(t) = t$

(c) $x(t) = \sin t$ and $y(t) = t - \frac{t^3}{6}$

(d) $x(t) = \exp(t)$ and $y(t) = \sum_{m=0}^{n} \frac{t^m}{m!}$

2.2 OPEN SPHERES AND CLOSED SPHERES

or OPEN BALL-$B_r(x,r)$ OR CLOSED BALL- $B_r[x,r]$

2.2.1 Definition

Let (X, d) be a metric space. Let $x \in X$ and $r > 0$ be a real number. The *open shpere* with centre x and radius r, denoted by $S_r(x)$, is the subset of X given by

$$S_r(x) = \{y \in X : d(x, y) < r\}$$

Note An open sphere is always non-empty since it contains its centre at least.

2.2.2 Examples

1. In the usual metric space $\mathbb{R}_u$, the open sphere $S_r(x_0)$ is the interval $]x_0 - r, x_0 + r[$, $x_0 \in \mathbb{R}$ and $r > 0$.

Remark Every open shpere in the usual metric space $\mathbb{R}_u$ is an open interval. But the converse is not true; for instance, $]-\infty, \infty[$ is an open interval in $\mathbb{R}$ but not an open sphere.

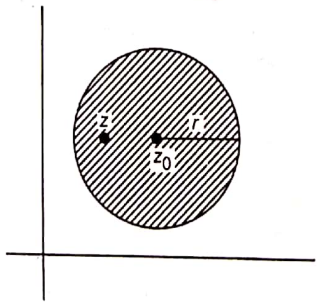

Fig. 2.4

2. In the usual metric space $\mathbb{C}_u$ the open sphere $S_r(z_0)$ is the circular disc $|z - z_0| < r$, $z_0 \in \mathbb{C}$ and $r > 0$.

3. Let x_0 be any point in the discrete metric space X_d (Example 2.1.2(3)). Then

$$S_r(x_0) = \begin{cases} \{x_0\}, & 0 < r \le 1 \\ X, & r > 1 \end{cases}$$

4. In the metric space in Example 2.1.2(5),

$$S_r(0) = \begin{cases} [0, r[, & r \le 1 \\ [0, 1[, & r > 1 \end{cases}$$

5. In the metric space $(\mathbb{R}^2, d)$ of Example 2.1.2(6), the unit sphere centred at the origin is given by

$$S_1((0,0)) = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$$

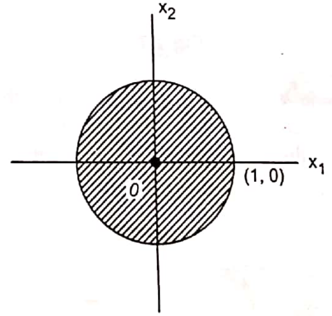

Fig. 2.5

6. In the metric space $(\mathbb{R}^2, d^*)$ of Example 2.1.2(6), the unit sphere centred at the origin is given by

$$S_1((0, 0)) = \{(x_1, x_2) : |x_1| < 1, |x_2| < 1\}$$

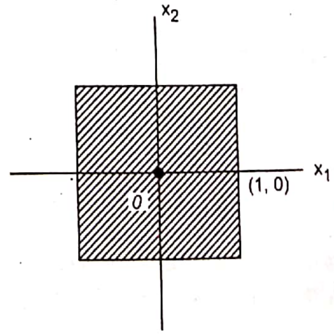

Fig. 2.6

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7. In the metric space $(\mathbb{R}^2, d^{**})$ of Example 2.1.2(6), the unit sphere centred at the origin is given by

$$S_1((0,0)) = \{(x_1, x_2) : |x_1| + |x_2| < 1\}$$

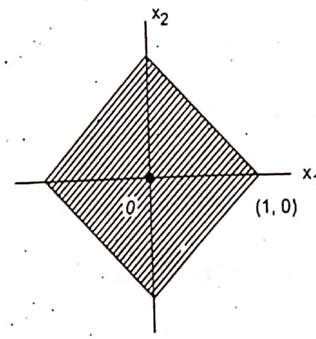


Fig. 2.7

8. In the metric space $C[a, b]$ (Example 2.1.2(15)) the open sphere $S_r(x_0)$ consists of all functions x in $C[a, b]$ whose graphs lie within the shaded band of vertical width $2r$ centred on the graph of x_0.

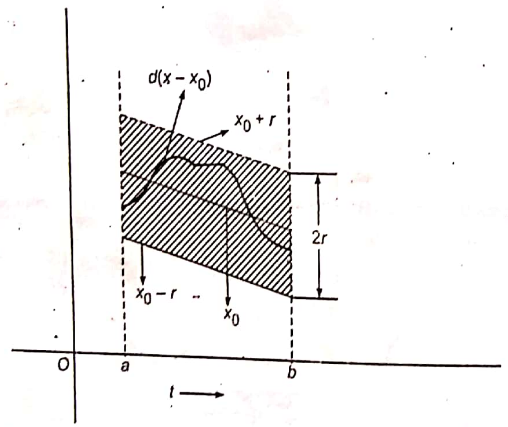


Fig. 2.8

2.2.3 Definition

Let (X, d) be a metric space. Let $x \in X$ and $r > 0$. The *closed sphere* with centre x and radius r, denoted by $S_r[x]$, is the subset of X given by

$$S_r[x] = \{y \in X : d(x, y) \leq r\}$$

2.2.4 Examples

1. In the usual metric space $\mathbb{R}_u$, the closed sphere $S_r[x_0]$ is the closed interval $[x_0 - r, x_0 + r]$.
2. In the usual metric space $\mathbb{C}_u$, the closed sphere $S_r[z_0]$ is the closed disc $|z - z_0| \leq r$, $z_0 \in \mathbb{C}$ and $r > 0$.
3. In the discrete metric space X_d (Example 2.1.2(3)), the closed sphere $S_r[x_0]$ is given by

$$S_r[x_0] = \begin{cases} \{x_0\}, & 0 < r < 1 \\ X, & r \geq 1 \end{cases}$$

Remarks In the metric space X_d, observe that

(i) For $0 < r < 1$,

$$S_r(x_0) = S_r[x_0] = \{x_0\}$$

(ii) For $r > 1$,

$$S_r(x_0) = S_r[x_0] = X$$

(iii) For $r = 1$,

$$S_r(x_0) = \{x_0\}, S_r[x_0] = X$$ $\therefore$ $S_r(x_0) \subset S_r[x_0]$, $\forall r > 0$, $\forall x_0 \in X$

2.3 NEIGHBOURHOODS

2.3.1 Definition

Let (X, d) be a metric space and $x \in X$. A set $N \subset X$ is said to be a *neighbourhood* (nbd) of x if $\exists$ an open sphere centred at x and contained in N, i.e., if $S_r(x) \subset N$, for some $r > 0$.

2.3.2 Examples

1. The open interval $]a, b[$ is a nbd of each of its points.
2. The set $\mathbb{R}$ of real numbers is a nbd of each of its points.
3. The closed interval $[a, b]$ is a nbd of each point of $]a, b[$ but is not a nbd of the end points a and b.
4. The set $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ ($\mathbb{R}-\mathbb{Q}$) is not a nbd of any of its points.
5. In a discrete metric space X_d, a subset $Y \subset X$ is a nbd of each of its points.
6. $S = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ is not a nbd of any of its points.

2.3.3 Theorem

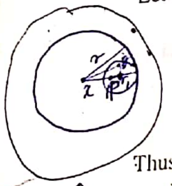
Let (X, d) be a metric space. A set $N \subset X$ is a nbd of a point $p \in X$ if and only if $\exists$ an open sphere $S_r(x)$ such that $p \in S_r(x) \subset N$.

Proof First we assume that N is a nbd of a point $p \in X$. Then, $\exists$ an $r > 0$ such that $p \in S_r(p) \subset N$. This proves that $\exists$ an open sphere $S_r(p)$ containing p and contained in N.

Conversely, assume that $p \in N$ and $\exists$ an open sphere $S_r(x)$ such that $p \in S_r(x) \subset N$. We shall prove that $\exists$ an open sphere centred at p and contained in N. Now

$$p \in S_r(x) \Rightarrow d(x, p) < r$$

Let $r_1 = r - d(x, p)$. Then $r_1 > 0$. Let $y \in S_{r_1}(p)$. Then $d(y, p) < r_1$. Now, we have



$$d(y, p) < r_1 \Rightarrow d(y, p) < r - d(x, p)$$
$$\Rightarrow d(x, p) + d(y, p) < r \quad [\text{By triangle inequality} -$$
$$\Rightarrow d(x, y) < r \qquad (\because d(x, y) \le d(x, p) + d(y, p))$$
$$\Rightarrow y \in S_r(x)$$

$< d(x,p) + r_1$, $= r$

Thus $S_{r_1}(p) \subset S_r(x) \subset N$. This verifies that N is a nbd of p. $\square$

$p \in S_{r_1}(p) \subset N$ $\Rightarrow y \in S_r(x)$

2.3.4 Theorem

Let (X, d) be a metric space and $x \in X$. Let $\mathbb{N}_x$ be the collection of all nbds of x. Then:

(a) $M, N \in \mathbb{N}_x \Rightarrow M \cap N \in \mathbb{N}_x$
(b) $N \in \mathbb{N}_x$ and $M \supset N \Rightarrow M \in \mathbb{N}_x$

Proof

(a) We have

$$M, N \in \mathbb{N}_x \Rightarrow \exists\, r_1, r_2 > 0 \text{ such that } S_{r_1}(x) \subset M \text{ and } S_{r_2}(x) \subset N$$
$$\Rightarrow S_r(x) \subset M \text{ and } S_r(x) \subset N, \text{ where } r = \min\{r_1, r_2\}$$
$$\Rightarrow S_r(x) \subset M \cap N$$
$$\Rightarrow M \cap N \text{ is a nbd of } x$$
$$\Rightarrow M \cap N \in \mathbb{N}x$$

(b) We have

$$N \in \mathbb{N}_x \Rightarrow \exists \text{ an } r > 0 \text{ such that } S_r(x) \subset N$$
$$\Rightarrow S_r(x) \subset M \qquad (\because M \supset N)$$
$$\Rightarrow M \in \mathbb{N}_x. \ \square$$

2.4 OPEN SETS

2.4.1 Definition

Let (X, d) be a metric space. A set $G \subset X$ is said to be an *open set* if it is a nbd of each of its points.

(Equivalently, a set $G \subset X$ is said to be an open set if for each $x \in G$, $\exists$ an $r > 0$ such that $S_r(x) \subset G$).

ie,

2.4.2 Examples

1. In the usual metric space $\mathbb{R}_u$

(a) $\mathbb{R}$ is an open set.

(b) $]0, 1[$ is an open set.
(c) $[0, 1[$ is not an open set.
(d) The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are not open sets.
(e) The set of all irrational numbers is not an open set.
(f) $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$ is not an open set.
(g) $\{x\}$, $x \in \mathbb{R}$, is not an open set.
(h) The Cantor set C is not an open set.

2. In the metric space of Example 2.1.2(5), $[0, \alpha[$, $\alpha \le 1$, is an open set.
3. In the metric space of Example 2.1.2(4), the set

(a) $\mathbb{Q}$ is an open set.

(b) $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$ is not an open set.

4. In the discrete metric space X_d, every set $G \subset X$ is an open set. In particular, every singleton set in X_d is open.

2.4.3 Theorem

Let (X, d) be a metric space. Then, the empty set ϕ and the whole space X are open sets.

Proof ~~In order~~ To prove that ϕ is an open set, we need to verify that each point of ϕ is the centre of some open sphere contained in ϕ. But ϕ contains no point. Therefore, the requirement is automatically satisfied.

For the second part, let $x \in X$ be arbitrary. Then, $\exists$ an open sphere $S_r(x) \subset X$. for all r

$\because$ [This is possible since any open sphere centred on a point of X cannot go beyond X]. Thus X is a nbd of x. But $x \in X$ is arbitrary. Therefore X is a nbd of each of its points. Hence X is an open set. $\square$

2.4.4 Theorem

Let (X, d) be a metric space. Then, each open sphere in X is an open set.

VH-12

Proof Let

$$S_r(x_0) = \{x \in X: d(x, x_0) < r\}$$

be an open sphere in (X, d). Let $y_0 \in S_r(x_0)$ be arbitrary but fixed. Then, $d(x_0, y_0) < r$. Write

$$r_1 = r - d(x_0, y_0)$$

Clearly, $r_1 > 0$. Consider

$$S_{r_1}(y_0) = \{y \in X: d(y, y_0) < r_1\}$$

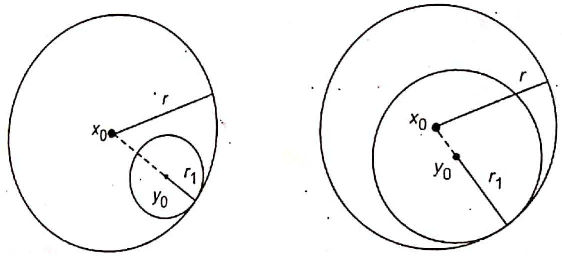
Let $y \in S_{r_1}(y_0)$ be arbitrary. Then $d(y, y_0) < r_1$. Now

$$d(x_0, y) \le d(x_0, y_0) + d(y_0, y) \quad \text{(by triangle inequality)}$$

$\Rightarrow \qquad y \in S_r(x_0)$

Consequently $\qquad S_{r_1}(y_0) \subset S_r(x_0)$ or, $y_0 \in S_{r_1}(y_0) \subset S_r(x_0)$

This proves that $S_r(x_0)$ is a nbd of y_0. But $y_0 \in S_r(x_0)$ is arbitrary. Therefore, $S_r(x_0)$ is a nbd of each of its points. Hence $S_r(x_0)$ is an open set. □



$r_1 < d(x_0, y_0)$ (a) $\qquad r_1 > d(x_0, y_0)$ (b)

Fig. 2.9

Remark Converse of the above theorem need not be true. For instance, the interval $]-\infty, \infty[$ in the usual metric space $\mathbb{R}_u$ is an open set while it is not open sphere.

2.4.5 THEOREM

Let (X, d) be a metric space and $G \subset X$. Then, G is an open set if and only if it is the union of open spheres.

Proof Let G be an open set. then G is a nbd of each of its points. ~~Then~~ each point of G is the centre of some open sphere which is contained in G. Clearly, union of all such open spheres is precisely the set G.

Conversely, let G be the union of open spheres. (~~Let P be the family of these~~ open spheres). Let $x \in G$ be arbitrary. Then, x belongs to some open sphere, say, $S_r(p) \in G$. Since every open sphere is an open set, x is the centre of an open sphere $S_{r_1}(x)$ such that

But $\qquad x \in S_{r_1}(x) \subset S_r(p)$

Therefore $\qquad S_r(p) \subset G$

$\qquad x \in S_{r_1}(x) \subset G$

This proves that G is a nbd of x. But $x \in G$ is arbitrary. Therefore G is a nbd of each of its points. Hence G is an open set. □

2.4.6 Theorem

Let (X, d) be a metric space. Then:

(a) Arbitrary union of open sets in X is open. V H-11, 13

(b) Finite intersection of open sets in X is open.

Proof

(a) Let $\{G_\alpha\}_{\alpha \in \Lambda}$ be a family of open sets in X. We shall prove that $G = \bigcup_{\alpha \in \Lambda} G_\alpha$ is open. Since each G_α is open, it is union of open spheres for each $\alpha \in \Lambda$. Then $G = \bigcup_{\alpha \in \Lambda} G_\alpha$ is the union of unions of open spheres. Hence, by Theorem 2.4.5, it is open.

(b) Let $\{G_i; i = 1, 2, \ldots, n\}$ be the finite family of open sets in X. We shall prove that $G = \bigcap_{i=1}^{n} G_i$ is open. Let $x \in \bigcap_{i=1}^{n} G_i$ be arbitrary. Then

$$x \in G_i, \text{ for each } i = 1, 2, \ldots, n$$

$\Rightarrow \qquad \exists$ an $r_i > 0$ such that $S_{r_i}(x) \subset G_i, (i = 1, 2, \ldots, n)$

$\qquad (\because$ Each G_i is open)

Write

$$r = \min_{1 \le i \le n} r_i$$

Then

$$S_r(x) \subset S_{r_i}(x) \subset G_i, \qquad (i = 1, 2, \ldots, n)$$

$\Rightarrow \qquad S_r(x) \subset \bigcap_{i=1}^{n} G_i$

Hence $\bigcap_{i=1}^{n} G_i$ is an open set. □

Remark Arbitrary intersection of open sets need not be open.

2.4.7 Example

In the usual metric space $\mathbb{R}_u$, consider the family $\left\{\left]-\frac{1}{n}, \frac{1}{n}\right[: n \in \mathbb{N}\right\}$ of open sets. Then $\cap \left\{\left]-\frac{1}{n}, \frac{1}{n}\right[: n \in \mathbb{N}\right\} = \{0\}$ which is not an open set.

Remark Open sets are among the most important concepts in geometry and analysis. One may note how often new ideas are defined in terms of open sets and there are results and theorems which give necessary and sufficient conditions for these new ideas in terms of open sets. Furthermore, the basic foundation of topology lies on the concept of open sets.

2.4.8 THEOREM

Every non-empty open set in the usual metric space $\mathbb{R}_u$ is the union of a countable disjoint class of open intervals.

Proof Let G be a non-empty open set in $\mathbb{R}_u$. Let $x \in G$. Since G is open, x is the centre of a bounded open interval contained in G. Let I_x be the union of all open intervals which contain x and is contained in G. We note the following:

(i) I_x is an open interval which contains x such that $I_x \subset G$. (If $\{I_i\}$ is a non empty class of intervals on the real line such that $\bigcap_{i=1}^{\infty} I_i \neq \phi$, then $\bigcup_{i=1}^{\infty} I_i$ is an interval.)

(ii) I_x contains every open interval which contains x and is contained in G.

(iii) If $y \in I_x$, then $I_x = I_y$.

(iv) If x and y are two distinct points in G, then either $I_x = I_y$ or $I_x \cap I_y = \phi$; for if there is a point $z \in I_x \cap I_y$ then

$$z \in I_x \quad \text{and} \quad z \in I_y$$

$$\Rightarrow \quad I_x = I_z \quad \text{and} \quad I_y = I_z$$

$$\Rightarrow \quad I_x = I_y$$

Now, consider the class $\mathcal{I}$ of all distinct sets of the form I_x for points $x \in G$. Clearly, $\mathcal{I}$ is a disjoint class of open intervals such that

$$G = \bigcup_{I \in \mathcal{I}} I$$

It remains now to show that $\mathcal{I}$ is countable. Let G_r be the set of all rational points in G. Clearly, G_r is non-empty. Define $f: G_r \rightarrow \mathcal{I}$ by $f(x) = I_x (x \in G_r)$. The function f is well defined since for each rational x in G_r, there is a unique I_x in $\mathcal{I}$ (the disjoint intervals in $\mathcal{I}$ cannot contain the same rational). Also f is surjective since each interval in $\mathcal{I}$ contains a rational in G_r. Since the set G_r is countable, the family $\mathcal{I}$ is countable (the range of a countable set is countable). □

2.4.9 Theorem

Let (X, d) be a metric space. Then, for every pair of distinct points x and y in X, $\exists$ disjoint open sets U and V such that $x \in U$ and $y \in V$.

Proof Let (X, d) be a metric space and let x, y be two distinct points of X. Since $x \neq y$, $d(x, y) > 0$. Let $d(x, y) = 3r$, $U = S_r(x)$ and $V = S_r(y)$. Then $x \in U$ and $y \in V$. Clearly, $U \cap V = \phi$. If not, let $p \in U \cap V$. Then $p \in U$ and $p \in V$. Therefore,

$$p \in U \quad \Rightarrow \quad d(x, p) < r$$

and

$$p \in V \quad \Rightarrow \quad d(y, p) < r$$

Now

$$d(x, y) \leq d(x, p) + d(y, p)$$

$$< r + r$$

$$= 2r < 3r$$

This is a contradiction. Hence the result follows. □

Note The result in Theorem 2.4.9 is popularly referred to *Hausdorff Property*.

Problems

11. Let (X, d) be a metric space, $a \in X$ and $0 < r < r'$. Prove that the set
$$\{x \in X : r < d(x, a) < r'\}$$
is open in (X, d).

12. Prove that in a metric space, the complement of any singleton set is open. Hence or otherwise, show that the complement of any finite set is open.

13. In a given metric space (X, d), prove that every subset of X is open if and only if every singleton set is open.

14. Prove that every singleton set, except $\{0\}$, in $\mathbb{R}^2$ is open with respect to the metric d^{**} given by
$$d^{**}(x, y) = |x_1 - y_1| + |x_2 - y_2|$$
for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$.
(This metric is known as *Postman metric* on $\mathbb{R}^2$).

2.5 EQUIVALENT METRICS

2.5.1 Definition

Two metrics d and d^* on a set X are said to be *equivalent* if the collection of all open sets in the metric space (X, d) is identical with the collection of all open sets in the metric space (X, d^*).

(In otherwords, two metrics d and d^* on a set X are said to be equivalent if they give rise to the same family of open sets).

We first establish a criteria for the two metrics on a set to be equivalent.

2.5.2 Theorem

Let d and d^* be two metrics on a set X and let $\mathcal{T}, \mathcal{T}^*$ be the families of open sets, respectively, in the metric spaces (X, d) and (X, d^*). Then, the metrics d and d^* are equivalent if and only if

(i) for each open set G in $\mathcal{T}$ and each point $x \in G$, there is a d^*-open sphere $S^*_{r^*}(x) \subset G$, and

(ii) for each open set G^* in $\mathcal{T}^*$ and each point $x^* \in G^*$, there is a d-open sphere $S_r(x^*) \subset G^*$.

Proof It is enough to prove that (i) is equivalent to $\mathcal{T} \subset \mathcal{T}^*$ and that (ii) is equivalent to $\mathcal{T}^* \subset \mathcal{T}$. Further, it is enough to establish the first assertion since the other one will follow by interchanging the roles of d and d^*.

Now, assume that $\mathscr{F} \subset \mathscr{F}^*$. If $G \in \mathscr{F}$, then $G \in \mathscr{F}^*$. Thus, for any point $x \in G$, $\exists$ some $r^* > 0$ such that

$$x \in S^*_{r^*}(x) \subset G$$

and as such (i) holds.

Conversely, if (i) holds and $G \in \mathscr{F}$, then for each $x \in G$, $\exists$ d^*-open sphere $S^*_{r^*}(x)$ such that

$$x \in S^*_{r^*}(x) \subset G$$

$$\Rightarrow \quad G \subset \bigcup_{x \in G} S^*_{r^*}(x) \subset G$$

$$\Rightarrow \quad G = \bigcup_{x \in G} S^*_{r^*}(x)$$

Thus G is an open set in the metric space (X, d^*) showing there by that $G \in \mathscr{F}^*$. Hence $\mathscr{F} \subset \mathscr{F}^*$. □

2.5.3 Corollary

The metrics d and d^* on a non-empty set X are equivalent if and only if

(i)′ for each d-open sphere $S_r(x)$, $\exists$ an $r^* > 0$ such that $S^*_{r^*}(x) \subset S_r(x)$, and

(ii)′ for each d^*-open sphere $S^*_{r^*}(x^*)$, $\exists$ an $r > 0$ such that $S_r(x^*) \subset S^*_{r^*}(x^*)$.

Proof The result follows if we establish that (i) and (ii) in Theorem 2.5.2 are equivalent, respectively, to (i)′ and (ii)′ of Corollary 2.5.3. It is enough to show that (i) $\Leftrightarrow$ (i)′ since the other one can be obtained similarly by interchanging the roles of d and d^*.

The assertion (i) $\Rightarrow$ (i)′ follows trivially since an open sphere is an open set in a metric space. In order to establish the reverse implication (i)′ $\Rightarrow$ (i), let us consider a $G \in \mathscr{F}$. If $x \in G$, then $\exists$ an $r > 0$ such that $S_r(x) \subset G$. Therefore, by (i)′, $\exists$ an $r^* > 0$ such that

$$S^*_{r^*}(x) \subset S_r(x) \subset G,$$

verifying (i).

Hence, the result is proved completely. □

We, now, give below a result giving sufficient condition for two metrics on a set to be equivalent.

2.5.4 Theorem

Let d and d^* be two metrics on a set X. If there exist two real numbers $k_1, k_2 > 0$ such that

$$k_1 d(x, y) \le d^*(x, y) \le k_2 d(x, y), \tag{1}$$

for all $x, y \in X$, then the metrics d and d^* on X are equivalent.

Proof Let $r > 0$ be given. Take $r^* = k_1^{-1} r$. Then, by using the first inequality in (1), for each $x \in X$, we have

$$S_r(x) \subset S^*_{r^*}(x)$$

which, in fact, is Condition (ii)′ in Corollary 2.5.3. Similarly, by using the second inequality in (1), one can establish condition (i)′ in Corollary 2.5.3. Hence, the result follows. □

Notes

1. Theorem 2.5.4 is quite useful for establishing certain metrics on a set to be equivalent.
2. The condition in Theorem 2.5.4 is sufficient for two metrics on a set to be equivalent. However, this condition is not necessary.
3. We shall discuss more about the above condition in Chapter 4.

2.5.5 Examples

1. The three metrics defined on $\mathbb{R}^2$ given in Example 2.1.2(6), e.g.,

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$d^*(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

$$d^{**}(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

are equivalent.

Note that

$$d^*(x, y) \le d(x, y) \le \sqrt{2}\, d^*(x, y), \ \forall\, x, y \in \mathbb{R}^2$$

and

$$d^*(x, y) \le d^{**}(x, y) \le 2d^*(x, y), \ \forall\, x, y \in \mathbb{R}^2$$

Thus, the metrics d, d^* and d^{**} on $\mathbb{R}^2$ are equivalent in view of Theorem 2.5.4.

2. The metrics defined on $\mathbb{R}^n$ as in Examples 2.1.2(8), 2.1.2(10), e.g.,

$$d(x, y) = \left\{\sum_{i=1}^{n} (\alpha_i - \beta_i)^2\right\}^{\frac{1}{2}}$$

$$d^*(x, y) = \max_{1 \le i \le n} |\alpha_i - \beta_i|$$

$$d^{**}(x, y) = \sum_{i=1}^{n} |\alpha_i - \beta_i|$$

are equivalent.

The equivalence of the metrics d, d^* and d^{**} on $\mathbb{R}^n$ follows immediately in view of the following inequalities:

$$d^*(x, y) \le d(x, y) \le \sqrt{n}\, d^*(x, y), \ \forall\, x, y \in \mathbb{R}^n$$

and

$$d^*(x, y) \le d^{**}(x, y) \le n d^*(x, y), \ \forall\, x, y \in \mathbb{R}^n.$$

3. The usual metric and the discrete metric on $\mathbb{R}$ are not equivalent. Indeed, the set $\{0\}$ is not open in the usual metric space $\mathbb{R}_u$ while it is open in the discrete metric space $\mathbb{R}_d$.

Problems

15. Let X be a set and let $d_i (1 \le i \le n)$ be n metrics on X. Define

$$d^*(x, y) = \max_{1 \le i \le n} \{d_i(x, y)\}$$

and

$$d^{**}(x, y) = \sum_{i=1}^{n} d_i(x, y)$$

for all $x, y \in X$. Show that d^* and d^{**} are equivalent metrics on X.

16. Let (X, d) be a metric space and $k > 0$ be a fixed number. Prove that the metric d^* given in Problem 2 is equivalent to the metric d.

17. Let d and d^* be any two metrics defined on a finite set X. Prove that d and d^* are equivalent.

18. Let (X, d) be a metric space and let the metrics d^* and d^{**} defined on X as in Problem 3, e.g.,

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

and

$$d^{**}(x, y) = \min\{1, d(x, y)\}$$

Prove that each of the metrics d^* and d^{**} is equivalent to the metric d on X.

19. Prove that the two metric spaces (X, d) and (X, d^*) as described in Problem 18 have precisely the same family of open spheres with one exception. What is that exception?

20. Consider the set $C[0, 1]$ of all continuous functions defined on $[0, 1]$ and the two metrics d and d^* given by

$$d(x, y) = \max_{t \in [0,1]} | x(t) - y(t) |$$

and

$$d^*(x, y) = \int_0^1 | x(t) - y(t) | dt$$

Prove that d and d^* on $C[0, 1]$ are not equivalent.

2.6 INTERIOR POINTS

2.6.1 Definition

Let (X, d) be a metric space and $A \subset X$. A point $x \in A$ is said to be an *interior point* of A if A is a nbd of x.

(Equivalently, a point $x \in A$ is said to be an interior point of A if $\exists$ an $r > 0$ such that $S_r(x) \subset A$).

i.e,

Def: Interior of A :-

The interior of A, denoted by $A°$, is the set of all interior points of A; i.e.,

$$A° = \{x \in A : S_r(x) \subset A, \text{ for some } r > 0\}.$$

2.6.2 Examples

1. Let $\mathbb{R}_u$ be the usual metric space (Example 2.1.2(1)) and $A \subset \mathbb{R}$. Then:
   (a) If $A =]a, b[,]a, b], [a, b[$ or $[a, b]$, then $A° =]a, b[$
   (b) If $A = \mathbb{N}, \mathbb{Z}$ or $\mathbb{Q}$, then $A° = \phi$ R-Q
   (c) If $A = \mathbb{R}$, then $A° = \mathbb{R}$
   (d) If $A = C$, the Cantor set, then $A° = \phi$

2. Let X_d be the discrete metric space (Example 2.1.2(3)) and $A \subset X$. Then $A° = A$.

2.6.3 Theorem

Let (X, d) be a metric space and $A \subset X$. Then:

(a) $A°$ is an open set.
(b) $A°$ is the largest open subset of A.
(c) A is open $\Leftrightarrow A = A°$.
(d) $A°$ is the union of all open subsets of A.

Proof

(a) Let $x \in A°$ be arbitrary. Then, by definition, $\exists$ an open sphere $S_r(x) \subset A$. But $S_r(x)$ being an open set, each point of $S_r(x)$ is the centre of some open sphere contained in $S_r(x)$. as The $S_r(x)$ is a nbd of each of its points. Therefore each point of $S_r(x)$ is the interior point of A, i.e. $S_r(x) \subset A°$. Thus, x is the centre of an open sphere contained in $A°$. But x being arbitrary, it is true for each $x \in A°$. Hence $A°$ is an open set.

(b) Let $G \subset A$ be an open set and $x \in G$ be arbitrary. Then, $\exists$ an open sphere $S_r(x) \subset G \subset A$. By definition, $x \in A°$ and as such $G \subset A°$. Thus, given any open set $G \subset A$, we have $G \subset A° \subset A$ and $A°$ is open by (a). Hence $A°$ is the largest open subset of A.

The assertions (c) and (d) follow obviously by (b). □

2.6.4 Theorem

Let (X, d) be a metric space and $A, B \subset X$. Then:

(a) $A \subset B \Rightarrow A° \subset B°$.
(b) $(A \cap B)° = A° \cap B°$. – W-18.
(c) $(A \cup B)° \supset A° \cup B°$.

Proof

(a) Let $x \in A^\circ$. Then,

$\exists$ an open sphere $S_r(x) \subset A$

$S_r(x) \subset B$ $\quad (\because A \subset B)$

$\Rightarrow$ $x \in B^\circ$

$\Rightarrow$ This verifies that $A^\circ \subset B^\circ$.

(b) Let $x \in (A \cap B)^\circ$. Then, $\exists$ an open sphere $S_r(x) \subset A \cap B$. This gives

$S_r(x) \subset A$ and $S_r(x) \subset B$

$\Rightarrow$ $x \in A^\circ$ and $x \in B^\circ$

$\Rightarrow$ $x \in A^\circ \cap B^\circ$

Hence, $(A \cap B)^\circ \subset A^\circ \cap B^\circ$.

To establish the reverse inclusion, let $y \in A^\circ \cap B^\circ$. Then, $y \in A^\circ$ and $y \in B^\circ$. Therefore, $\exists$

open spheres $S_{r_1}(y) \subset A$ and $S_{r_2}(y) \subset B$

$\Rightarrow$ $S_r(y) \subset A \cap B$, $\quad r = \min\{r_1, r_2\}$

$\Rightarrow$ $y \in (A \cap B)^\circ$

Consequently, $A^\circ \cap B^\circ \subset (A \cap B)^\circ$. This proves (b).

(c) Take some $x \in A^\circ \cup B^\circ$. Then, $S_r(x) \subset A$ or $S_r(x) \subset B$, for some $r > 0$ and therefore $S_r(x) \subset A \cup B$. This verifies that $x \in (A \cup B)^\circ$.

Hence $A^\circ \cup B^\circ \subset (A \cup B)^\circ$. □

Remark The inclusion relation in Theorem 2.6.4(c) is proper.

2.6.5 Example

In the usual metric space $\mathbb{R}_u$, consider the sets $A = [0, 1]$ and $B = [1, 2]$, $A \cup B = [0, 2]$. Note that

$$A^\circ =]0, 1[,\ B^\circ =]1, 2[,\ (A \cup B)^\circ =]0, 2[$$

$$A^\circ \cup B^\circ =]0, 1[\cup]1, 2[$$

This shows that $A^\circ \cup B^\circ \underset{\neq}{\subset} (A \cup B)^\circ$.

2.7 CLOSED SETS

2.7.1 Definition

Let (X, d) be a metric space and $A \subset X$. The set A is said to be *closed* if its complement $X - A$ is open.

2.7.2 Examples

1. In the usual metric space $\mathbb{R}_u$, the set

(a) $A = [1, 2]$ is closed since $\mathbb{R} - A =]-\infty, 1[\cup]2, \infty[$ is open.

(b) $A =]1, 2[$ is not closed since $\mathbb{R} - A =]-\infty, 1] \cup [2, \infty[$ is not open.

(c) $A = \mathbb{Q}$ is not closed since $\mathbb{R} - \mathbb{Q}$, the set of all irrational numbers, is not open.

(d) A, the set of all irrational numbers, is not closed since $\mathbb{R} - A = \mathbb{Q}$ is not open.

(e) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is not closed since $\mathbb{R} - A =]-\infty, 0] \cup \ldots \cup]\frac{1}{2}, 1[\cup]1, \infty[$ is not open.

(f) $A = \mathbb{R}$ is closed since $\mathbb{R} - A = \phi$ is open.

(g) $A = C$, the Cantor set is closed since by definition the complement of C is open (see Appendix A).

2. In a discrete metric space X_d, a subset $Y \subset X$ is closed.

2.7.3 Theorem

Let (X, d) be a metric space. Then, the empty set ϕ and the whole space X are closed sets.

Proof By Theorem 2.4.3, ϕ and X are open sets. Therefore, their complements X and ϕ are closed sets. □

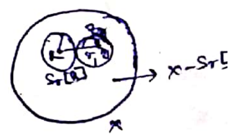
2.7.4 Theorem

Let (X, d) be a metric space. Then, each closed sphere in X is a closed set.

Proof Let $S_r[x]$ be a closed sphere in (X, d). It is sufficient to prove that $X - S_r[x]$ is an open set. Let $y \in X - S_r[x]$ be arbitrary. Then $y \notin S_r[x]$ and therefore $d(x, y) > r$.

Write $r_1 = d(x, y) - r$. Then $r_1 > 0$. Let $z \in S_{r_1}(y)$. Then

$$d(y, z) < r_1$$



By triangle inequality, we have

$$d(x, y) \le d(x, z) + d(z, y)$$

$\Rightarrow$ $d(x, z) \ge d(x, y) - d(z, y)$

$> d(x, y) - r_1 = r$

$\Rightarrow$ $z \notin S_r[x]$

$\Rightarrow$ $z \in X - S_r[x]$

Thus $S_{r_1}(y) \subset X - S_r[x]$ and hence $X - S_r[x]$ is a nbd of y. But $y \in X - S_r[x]$ is arbitrary. Therefore $X - S_r[x]$ is a nbd of each of its points. Hence $X - S_r[x]$ is an open set. □

2.7.5 Theorem

Let (X, d) be a metric space. Then:

(a) Arbitrary intersection of closed sets in X is closed. VH-12

(b) Finite union of closed sets in X is closed.

Proof

(a) Let $\{F_\alpha\}_{\alpha\in\Lambda}$ be an arbitrary family of closed sets in X. We claim that $\bigcap_{\alpha\in\Lambda} F_\alpha$ is closed. Since F_α is closed for each $\alpha\in\Lambda$, $X - F_\alpha$ is open for each $\alpha\in\Lambda$. Write $G_\alpha = X - F_\alpha$, $\alpha\in\Lambda$. Then

$$\bigcap_{\alpha\in\Lambda} F_\alpha = \bigcap_{\alpha\in\Lambda}(X - G_\alpha) = X - \bigcup_{\alpha\in\Lambda} G_\alpha \quad \text{(by De Morgan's law)}$$

In view of Theorem 2.4.6(a), $\bigcup_{\alpha\in\Lambda} G_\alpha$ is open. Therefore, $X - \bigcup_{\alpha\in\Lambda} G_\alpha$ is closed and hence $\bigcap_{\alpha\in\Lambda} F_\alpha$ is closed.

(b) Let $\{F_i : i = 1, 2,\ldots, n\}$ be a finite family of closed sets. Then, $X - F_i$ is an open set for $i = 1, 2,\ldots, n$. Write $G_i = X - F_i$. Then

or, $F_i = X - G_i$

$$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n}(X - G_i) = X - \bigcap_{i=1}^{n} G_i \quad \text{(by De Morgan's law)}$$

But $\bigcap_{i=1}^{n} G_i$ is open by Theorem 2.4.6(b). Therefore, $X - \bigcap_{i=1}^{n} G_i$ is closed and hence $\bigcup_{i=1}^{n} F_i$ is closed. □

Remark: Arbitrary union, or even countably infinite union, of closed sets need not be closed.

2.7.6 Example

Consider the family $\left\{\left[\frac{1}{n}, 2\right] : n\in\mathbb{N}\right\}$ of closed sets in the usual metric space $\mathbb{R}_u$.

Then

$$\bigcup\left\{\left[\frac{1}{n}, 2\right] : n\in\mathbb{N}\right\} =]0, 2]$$

Ex! $\{[-\frac{1}{n}, \frac{1}{n}] : n\in\mathbb{N}\}$

which is not a closed set.

Problems

21. Let (X, d) be a metric space. Prove that:
    (a) Every singleton set is closed.
    (b) Every finite set in X is closed.

22. Let (X, d) be a metric space. Prove that every subset of X is closed if and only if every singleton set is open. Deduce that every set in X is closed if and only if every set in it is open.

23. Let (X, d) be a metric space and $A, B \subset X$. If A is closed and B is open, prove that $A - B$ is closed and $B - A$ is open. State and prove the results in the case when A is open and B is closed.

$A - B = A \cap (X - B)$
$B - A = B \cap (X - A)$

2.8 LIMIT POINTS AND ISOLATED POINTS

2.8.1 Definition

Let (X, d) be a metric space and $A \subset X$. A point $x\in X$ is called a *limit point* (*accumulation point* or *cluster point*) of A if each open sphere centred on x contains at least one point of A other than x; in other words, if

$$(S_r(x) - \{x\}) \cap A \neq \phi \quad \text{for some } r > 0,$$

Derived set :-
The set of all limit points of A, denoted by A', is called the *derived set* of A.

2.8.2 Examples

1. Let $\mathbb{R}_u$ be the usual metric space and $A \subset \mathbb{R}$.
   (a) If $A = [a, b]$, $[a, b[$, $]a, b]$ or $]a, b[$, then $A' = [a, b]$.
   (b) If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$, then $A' = \{0\}$.
   (c) If $A = \mathbb{Z}$, then $A' = \phi$.
   (d) If $A = \mathbb{Q}$, then $A' = \mathbb{R}$.
   (e) If A is the set of all irrational numbers, then $A' = \mathbb{R}$.
   (f) If $A = \mathbb{R}$, then $A' = \mathbb{R}$.
   (g) If $A = C$, the Cantor set, then $A' = C$.
2. If A is any subset of a discrete metric space X_d, then $A' = \phi$. — HW-14

2.8.3 Theorem

Let (X, d) be a metric space and $A \subset X$. Then, A is closed if and only if A contains all its limit points, i.e., $A' \subseteq A$.

Proof Assume that A is closed. We shall prove that $A' \subset A$. Let $x \in A'$. Then, x is a limit point of A and hence every open sphere centred on x contains at least one point of A other than x. Suppose that $x \notin A$. Then $x \in X - A$. But A is closed. Therefore $X - A$ is open. Then $\exists$ an $r > 0$ such that $S_r(x) \subset X - A$. This shows that the open sphere $S_r(x)$ contains no any point of A which is a contradiction. Hence our assumption $x \notin A$ is wrong and thus $x \in A$. This proves that $A' \subset A$.

Conversely, assume that $A' \subset A$. We shall prove that A is closed. Let $x\in X - A$. Then $x\notin A$ and also $x\notin A'$ since $A' \subseteq A$. Therefore, we can find an $r > 0$ such that $S_r(x) \subset X - A$. This shows that $X - A$ is open and hence A is closed. This completes the proof. □

It follows, in view of Definition 2.8.1, that any open sphere centred on a limit point of A must contain infinitely many points of A. In particular, a point $x \in X$ is a limit point of A if $S_r(x) \cap A$ is an infinite set for each $r > 0$. On the other hand, if $x \in X$ is not a limit point of A, then $\exists$ an open sphere centred on x which contains no point of A, other than x itself; i.e., $S_r(x) \cap A = \{x\}$ for some $r > 0$. Such a point $x \in X$ is called an *isolated point* of X. Thus, every point of a metric space is either a limit point or an isolated point of X.

Th! Prove that a finite set has no limit point. [VH-15]
Th! Prove that the set $N(\mathbb{Z})$ has no limit points.

Th: Let (X, d) be a metric space and $A \subset X$. and x be a limit point of A. Then every open sphere centred at p and radius r contains infinitely many elements of A.

2.8.4 Example

Consider the metric space

$$X = \left\{0, \frac{1}{n} : n \in \mathbb{N}\right\}$$

with the usual metric given by the absolute value. Note that 0 is the only limit point of X while all other points are the isolated points of X.

In case $x \in X$ is an isolated point of the metric space (X, d), the set $\{x\}$ is an open sphere, and hence an open set in (X, d). If X consists of only isolated poins, then each singleton $\{x\}$, $x \in X$, and hence every subset of X is an open set in (X, d). That way the space (X, d) looks like discreted. More generally and precisely, we formulate the concept of a discrete metric space.

2.8.5 Definition

A metric space (X, d) is said to be *discrete* if every subset of X is open in (X, d) (and hence closed).

2.8.6 Examples

1. Let $X = \mathbb{N}$, the set of natural numbers. Define

$$d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|, \; m, n \in \mathbb{N}$$

Then (X, d) is a descrete metric space (Verify!).

2. Let $X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and consider the usual metric given by the absolute value. Then X becomes a discrete metric space.

3. Let $X \neq \phi$ be any set. Define

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Then (X, d), denoted by X_d, is a discrete metric space. This example has already been discussed, see Example 2.1.2(3).

4. Any two discrete metrics (Definition 2.8.5) on a set X are equivalent. In particular, every discrete metric on X is equivalent to the standard discrete metric on it.

Remark In Example 3, the set X is arbitrary and it is the metric d which makes every point of X isolated. We call the space X_d the *standard discrete metric space*. However, a metric space can be discrete even if the matric d is not discrete, for instance, see Examples 1 and 2.

Note. In our subsequent work, if there is no confusion likely to occur, we shall be referring the space X_d, or $\mathbb{R}_d$, simple discrete metric space instead of standard discrete metric space.

2.9 CLOSURE OF A SET

2.9.1 Definition

Let (X, d) be a metric space and $A \subset X$. The *closure* of A, denoted by $\overline{A}$, is the union of A and the set of all its limit points, i.e., $\overline{A} = A \cup A'$.

(In other words, $\overline{A}$ is the set of all points of A together with all those points which are arbitrarily close to A; i.e. $x \in \overline{A} \Leftrightarrow S_r(x) \cap A \neq \phi$, for every $r > 0$.)

Adherent points:- The points of $\overline{A}$ are called adherent points of A

2.9.2 Theorem

Let (X, d) be a metric space and $A \subset X$. Then:

(a) $\overline{A}$ is a closed set.

(b) A is closed if and only if $A = \overline{A}$.

(c) $\overline{A}$ is the smallest closed subset of X containing A.

(d) $\overline{A}$ is the intersection of all closed subsets of X containing A.

Proof

(a) Let x (arbitrary) be a limit point of $\overline{A}$. Then, for a given $\varepsilon > 0$, $\exists\, y \in \overline{A}$ such that $d(x, y) < \frac{\varepsilon}{2}$. Further, since $y \in \overline{A}$, i.e., either $y \in A$ or y is a limit point of A, $\exists$ $z \in A$ such that $d(y, z) < \frac{\varepsilon}{2}$. Now

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \quad \text{(by triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This shows that x is a limit point of A and hence $x \in \overline{A}$. This verifies that $\overline{A}$ is a closed set.

(b) It follows from the fact that $\overline{A} = A \cup A'$ and that A is closed if and only if $A' \subset A$.

(c) Let B be any closed subset of X with $A \subset B$ and let $x \in \overline{A}$. If $x \in A$, then $x \in B$. In case $x \notin A$, it is a limit point of A. Then, for a given $\varepsilon > 0$, $\exists\, y \in A$ such that $d(x, y) < \varepsilon$. But $y \in B$ also since $A \subset B$ and therefore x is a limit point of B. The set B being closed, $x \in B$ and as such $\overline{A} \subset B$. Thus, given any closed set $B \supset A$, we have

$$B \supset \overline{A} \supset A$$

and $\overline{A}$ is closed by (a).

Hence $\overline{A}$ is the smallest closed subset of X containing A.

(d) Let $M = \cap \{B \subset X : B \text{ is closed and } B \supset A\}$. Then, by Theorem 2.7.5(a), M is closed. Clearly, M is the smallest closed subset of X containing A. Therefore, by (c), $\overline{A} = M$ which completes the proof. □

2.9.3 Theorem

Let (X, d) be a metric space and $A, B \subset X$. Then:

(a) $\overline{\phi} = \phi$.

(b) $\overline{X} = X$.

(c) $\overline{(\overline{A})} = \overline{A}$.

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(d) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$. V+I-II

(e) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(f) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(g) $A' = (\overline{A})'$.

Proof Straightforward. □

Remark $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

2.9.4 Example

In the usual metric space $\mathbb{R}_u$, consider the sets $A =]0, 1[$ and $B =]1, 2[$. Then, $A \cap B = \phi$. Note that

$$\overline{A} = [0, 1],\ \overline{B} = [1, 2]$$
$$\overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\}$$
$$\overline{A \cap B} = \phi$$

This shows that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

Problems

24. Is the converse implication in Theorem 2.9.3(d) true? Justify your answer.
25. In a metric space (X, d), prove that

$$\overline{S_r(x)} \subset S_r[x]$$

[Hint: Use Theorems 2.7.4 and 2.9.2(b)].

26. Give an example to show that the inclusion relation in Problem 25 is proper.
27. Let (X, d) be a metric space and $A \subset X$. Prove that

$$\overline{X - A} = X - A^\circ$$

2.10 BOUNDARY POINTS

2.10.1 Definition

Let (X, d) be a metric space and $A \subset X$. A point $x \in X$ is said to be a *boundary point* of A if x is neither an interior point of A nor of $X - A$.

(In other words, $x \in X$ is said to be a boundary point of A if every open sphere centred on x intersects both A and $X - A$.)

The set of all boundary points of A, denoted by ∂A, is called the *boundary of A*. The boundary of A is also denoted by $b(A)$ or A^b.

2.10.2 Examples

1. Let $\mathbb{R}_u$ be the usual metric space and $A \subset \mathbb{R}$. Then:
   (i) If $A = [a, b], [a, b[,]a, b]$ or $]a, b[$, then $\partial A = \{a, b\}$.
   (ii) If $A = \mathbb{N}$ (resp.$\mathbb{Z}$), then $\partial A = \mathbb{N}$ (resp.$\mathbb{Z}$).
   (iii) If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$, then $\partial A = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.
   (iv) If $A = \mathbb{Q}$, then $\partial A = \mathbb{R}$.
   (v) If A is the set of all irrational numbers, then $\partial A = \mathbb{R}$.
   (vi) If $A = C$, the Cantor set, then $\partial A = C$.
2. Let X_d be a discrete metric space and $A \subset X$. Then $\partial A = \phi$.

2.10.3 Theorem

Let (X, d) be a metric space and $A \subset X$. Then:

(a) $\partial A = \partial(X - A)$.
(b) $\partial A = \overline{A} \cap \overline{(X - A)}$.
(c) $\partial A = \overline{A} - A^\circ = \overline{X - A} - (X - A)^\circ$.
(d) $X - \partial A = A^\circ \cup (X - A)^\circ$.
(e) $A = A^\circ \cup \partial A = A \cup \partial A$.
(f) $A^\circ \cap \partial A = \phi$.
(g) $A^\circ = A - \partial A$.
(h) A is closed $\Leftrightarrow \partial A \subset A$.
(i) A is open $\Leftrightarrow A \cap \partial A = \phi$.

Proof

(a) Let $x \in \partial A$. Then, by the definition, x is neither interior point of A nor of $X - A$. Equivalently, x is neither interior point of $X - A$ nor of $X - (X - A)$ $(= A)$. This means that

$$x \in \partial(X - A)$$

Hence

$$\partial A \subset \partial(X - A)$$

The reverse inclusion follows by replacing A with $X - A$ in the above.

(b) Let $x \in \partial A$. Then, x is neither an interior point of A and nor of $X - A$. Therefore, $S_r(x) \cap (X - A) \neq \phi$ and $S_r(x) \cap A \neq \phi$, for every $r > 0$. Hence $x \in \overline{X - A}$ as well as $x \in \overline{A}$. This proves that $\partial A \subset \overline{A} \cap (\overline{X - A})$.

On the otherhand, if $x \in \overline{A} \cap (\overline{X - A})$, then $x \in \overline{A}$ and $x \in \overline{X - A}$ and as such $S_r(x) \cap A \neq \phi$ and $S_r(x) \cap (X - A) \neq \phi$, for every $r > 0$. Thus, x is neither an interior point of A nor of $X - A$. Therefore $x \in \partial A$. Hence $\overline{A} \cap (\overline{X - A}) \subset \partial A$. This verifies (b).

(c) It follows by using (b) and the fact that $\overline{X - A} = X - A°$.

(d)
$$\begin{aligned} X - \partial A &= X - (\overline{A} \cap \overline{X - A}) && \text{(by (b))} \\ &= (X - \overline{A}) \cup (X - \overline{X - A}) && \text{(by De Morgan's law)} \\ &= (X - A)° \cup (X - (X - A°)) && \text{(by Problem 27)} \\ &= (X - A)° \cup A° \end{aligned}$$

(e) We have
$$\begin{aligned} A \cup \partial A &= A \cup (\overline{A} \cap \overline{X - A}) && \text{(by (b))} \\ &= (A \cup \overline{A}) \cap (A \cup \overline{X - A}) && \text{(by distributive law)} \\ &= \overline{A} \cap X = \overline{A} \end{aligned}$$

(f) Straightforward.

(g) We know that for any two subsets A and B of X,
$$A - B = A \cap (X - B)$$
Taking $B = \partial A$, we get
$$\begin{aligned} A - \partial A &= A \cap (X - \partial A) \\ &= A \cap (A° \cup (X - A)°) && \text{(by (d))} \\ &= (A \cap A°) \cup (A \cap (X - A)°) && \text{(by distributive law)} \\ &= A° \cup \phi = A° \end{aligned}$$

(h) Since in view of (e) above and Theorem 2.9.2 (b), A is closed if and only if $A = A \cup \partial A$, the result follows.

(i) We have
$$\begin{aligned} A \text{ is open} &\Leftrightarrow X - A \text{ is closed} \\ &\Leftrightarrow \partial(X - A) \subset X - A && \text{(by (h))} \\ &\Leftrightarrow \partial A \subset X - A && \text{(by (a))} \\ &\Leftrightarrow A \cap \partial A = \phi. \ \square \end{aligned}$$

2.11 DISTANCE BETWEEN SETS AND DIAMETER OF A SET

2.11.1 Definition

Let (X, d) be a metric space and let $A, B \subset X$ be non-empty sets.

(a) The *distance* of a point $x \in X$ from the set A, denoted by $d(x, A)$, is given by
$$d(x, A) = \inf \{d(x, y) : y \in A\}$$

(b) The *distance between the sets* A and B, denoted by $d(A, B)$, is given by
$$d(A, B) = \inf \{d(x, y) : x \in A, y \in B\}$$

(c) The *diameter* of A, denoted by $d(A)$, is given by
$$d(A) = \sup \{d(x, y) : x, y \in A\}$$

Note One may not confuse as we use the same symbol d for the metric (distance between two points), distance of a point from a set A, distance between the sets A and B and diameter of a set A.

2.11.2 Theorem

Let (X, d) be a metric space and $A, B \subset X$. Then:

(a) $x \in A, y \in B \Rightarrow d(A, B) \leq d(x, y)$.
(b) $x \in \overline{A} \Leftrightarrow d(\{x\}, A) = 0$.
(c) $d(\overline{A}, \overline{B}) = d(A, B)$.
(d) $d(A) = 0 \Leftrightarrow A$ contains atmost one point.
(e) $A \subset B \Rightarrow d(A) \leq d(B)$.
(f) $d(\overline{A}) = d(A)$.
(g) $A \cap B \neq \phi \Rightarrow d(A \cup B) \leq d(A) + d(B)$.
(h) $x \in A, y \in B \Rightarrow d(x, y) \leq d(A \cup B)$.
(i) $d(A \cup B) \leq d(A) + d(A, B) + d(B)$.

2.11.3 Definition

Let (X, d) be a metric space and $A \subset X$. The set A is said to be *bounded* if $d(A) \leq \eta < \infty$. In other words, A is bounded if its diameter is finite, otherwise *unbounded*.

In particular, the metric space (X, d) is bounded if the set X is bounded.

2.11.4 Examples

1. Let $\mathbb{R}_u$ be the usual metric space and $A \subset \mathbb{R}$.
   (a) If $A = [a, b]$, $]a, b[$, $[a, b[$ or $]a, b]$, then A is bounded and $d(A) \leq b - a$.
   (b) If $A = C$, the Cantor set, then A is bounded and $d(A) \leq 1$.
   (c) If $A = [a, \infty[$ or $]-\infty, a]$, $a \in \mathbb{R}$, then A is not bounded.
2. Let (X, d) be a metric space. If $A = S_r(x)$ or $S_r[x]$, where $x \in X$ and $r > 0$, then A is bounded and $d(A) \leq 2r$.
3. Every set in a discrete metric space is bounded (Verify!).

2.11.5 Theorem

Let (X, d) be a metric space and $A \subset X$. Then, the following statements are equivalent:

(a) A is bounded.
(b) $\exists\ \eta > 0$ such that $d(x, y) \le \eta,\ \forall x, y \in A$.
(c) $\exists\ x_0 \in X$ and $r_0 > 0$ such that $A \subset S_{r_0}[x_0]$.
(d) For every point $x \in X$, $\exists$ an $r > 0$ such that $A \subset S_r[x_0]$.

2.11.6 Theorem

Let (X, d) be a metric space. Define the metrics

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

and

$$d^{**}(x, y) = \min\{1, d(x, y)\}$$

Then, the metric spaces (X, d^*) and (X, d^{**}) are both bounded irrespective of whether the metric space (X, d) is bounded or not.

Proof Straightforward. □

Remark For a given metric space (X, d) which is not necessarily bounded, it is always possible to define equivalent metrics in which X becomes a bounded metric space, e.g., the metric spaces (X, d^*) and (X, d^{**}) as described in Theorem 2.11.6 are bounded and each of the metrics d^* and d^{**} is equivalent to the metric d (see Problem 18).

Problems

28. Prove that in a metric space, every finite set is bounded.
29. Let (X, d) be a metric space and $A \subset X$ be non-empty. Prove that
$$d(x, A) \le d(x, y) + d(y, A), \qquad x, y \in X$$
30. Let (X, d) be a metric space and $A \subset X$ be any set. Prove that
$$x \in \bar{A} \Leftrightarrow d(x, A) = 0$$
and hence
$$x \in (X - A)^\circ \Leftrightarrow d(x, A) > 0$$
31. Let (X, d) be a metric space, $x \in X$ and $A \subset X$ be non-empty. Prove that $d(x, A) = 0$ if and only if every neighbourhood of x contains a point of A.
32. Use Problem 31 to solve Problem 30.
33. Prove Theorem 2.11.5.

2.12 SUBSPACE OF A METRIC SPACE

2.12.1 Definition

Let (X, d) be a metric space and $Y \subset X$. The mapping d_Y: $Y \times Y \to \mathbb{R}$ given by

$$d_Y(x, y) = d(x, y),\ \forall\ x, y \in Y$$

is a metric on Y. The metric d_Y is called the *relative metric* induced (or simply the metric induced) on Y by d. The space (Y, d_Y) is called the metric *subspace* of the metric space (X, d).

The above method of forming subspace of a given metric space enables us to construct several examples of metric spaces.

2.12.2 Examples

1. The intervals $[0, 1]$, $]0, 1[$, $[0, 1[$, the set $\mathbb{Q}$ etc. are the subspaces of the metric space $\mathbb{R}_u$. The metric space $\mathbb{R}_u$ itself is a subspace of the metric space $\mathbb{C}_u$.
2. The space $\mathbb{R}$ is a subspace of $\mathbb{R}^2$.
3. The set $\mathscr{P}[a, b]$ of all polynomials defined on $[a, b]$ is a subspace of the metric space $C[a, b]$ with the uniform metric d_∞ (Example 2.1.2(15)).

Remark If Y is a subspace of a metric space X, a set which is open (resp. closed) in Y is not necessarily open (resp. closed) in X.

2.12.3 Examples

Let Y be a subspace of the usual metric space $\mathbb{R}_u$.

1. If $Y = [0, 1]$, then the set $[0, \frac{1}{2}[$ is open in Y but not in $\mathbb{R}_u$.
2. If $Y =]0, 1[$, then the set $]0, \frac{1}{2}]$ is closed in Y but not in $\mathbb{R}_u$.

2.12.4 Lemma

Let (Y, d_Y) be a subspace of a metric space (X, d). If $a \in Y$ and $r > 0$, then

$$S'_r(a) = Y \cap S_r(a)$$

where $S_r(a)$ and $S'_r(a)$ are open spheres, respectively, in (X, d) and (Y, d_Y).

2.12.5 Theorem

Let (Y, d_Y) be a subspace of a metric space (X, d) and $A \subset Y$. Then:

(a) A is open in Y if and only if $\exists$ an open set G in X such that $A = G \cap Y$.
(b) A is closed in Y if and only if $\exists$ a closed set F in X such that $A = F \cap Y$.

Proof
(a) We will use the symbols $S_r(x)$ and $S'_r(x)$, for the open spheres centred at x with radius $r > 0$, respectively, for the spaces (X, d) and (Y, d_Y).

Suppose first that $A = G \cap Y$. In order to prove that A is open in Y, let $x \in A$ be arbitrary. Then $x \in G$ and $x \in Y$. Since G is open in X, $\exists\ r > 0$ such that $S_r(x) \subset G$. Also, since $x \in Y$, we have

$$S'_r(x) = S_r(x) \cap Y \subset G \cap Y = A$$

This verifies that x is an interior point of A as a subset of the metric space (Y, d_Y). Since $x \in A$ being arbitrary it follows that $A^\circ = A$ in (Y, d_Y). Hence A is open in Y.

Conversely, let A be open in Y and let $x \in A$ be arbitrary. Then $\exists$ an open sphere $S'_{r_x}(x)$ such that $S'_{r_x}(x) \subset A$. Now

$$A = \bigcup_{x \in A} S'_{r_x}(x) = \left(\bigcup_{x \in A} S_{r_x}(x)\right) \cap Y = G \cap Y,\ G = \bigcup_{x \in A} S_{r_x}(x)$$

But G being an arbitrary union of open spheres in X is an open set in X. Hence $A = G \cap Y$, where G is open in X.

(b) We have

$$\begin{aligned} A \text{ is closed in } Y &\Leftrightarrow Y - A \text{ is open in } Y \\ &\Leftrightarrow Y - A = G \cap Y, \quad \text{(by part (a))} \\ &\quad \text{where } G \text{ is open in } X \\ &\Leftrightarrow A = Y - G \cap Y \\ &\Leftrightarrow A = X \cap Y - G \cap Y \\ &\Leftrightarrow A = (X - G) \cap Y \\ &\Leftrightarrow A = F \cap Y, \end{aligned}$$

where $F = X - G$ is a closed set. □

2.12.6 Corollary

Let (Y, d_Y) be a subspace of a metric space (X, d) and $A \subset Y$. Then:

(a) A is open in Y and Y is open in $X \Rightarrow A$ is open in X.

(b) A is closed in Y and Y is closed in $X \Rightarrow A$ is closed in X.

2.12.7 Theorem

Let (Y, d_Y) be a subspace of a metric space (X, d) and $A \subset Y$. Then:

(a) $x \in Y$ is a limit point of A in Y if and only if x is a limit point of A in X.

(b) The closure of A in Y is $\bar{A} \cap Y$, where $\bar{A}$ is the closure of A in X.

Proof

(a) Let $x \in Y$ be a limit point of A in Y. Then

$$(S'_r(x) - \{x\}) \cap A \neq \phi \tag{1}$$

for each $r > 0$, where $S'_r(x)$ denotes an open sphere in Y.

Now, for any given $r > 0$, we have

$$\begin{aligned} (S_r(x) - \{x\}) \cap A &= (S_r(x) \cap Y - \{x\}) \cap A && (\because\ A \subset Y) \\ &= (S'_r(x) - \{x\}) \cap A \\ &\neq \phi && \text{(by (1))} \end{aligned}$$

This proves that x is a limit point of A in X. The converse can be established by retracing the above steps.

(b) Let B denote the closure of A in Y. The set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed in Y (Theorem 2.12.5). Since $\bar{A} \cap Y$ contains A, and since by Theorem 2.9.2, B equals the intersection of all closed subsets of Y containing A, we must have $B \subset \bar{A} \cap Y$.

In order to have the reverse inclusion, note that B is closed in Y. As such, by Theorem 2.12.5, $B = F \cap Y$, for some set F closed in X. Then F is a closed set of X containing A. Since $\bar{A}$ is the intersection of all such closed sets, we conclude that $\bar{A} \subset F$. Hence $\bar{A} \cap Y \subset F \cap Y = B$. □

Problem

34. Prove Lemma 2.12.4.

2.13 PRODUCT METRIC SPACES

Let X and Y be two non-empty sets. The *Cartesian product* of X and Y, denoted by $X \times Y$, is defined by

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

In case X and Y are metric spaces, equipped with the metrics d_X and d_Y, respectively, then a metric d is immediately available on $X \times Y$, namely,

$$d(x, y) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

for all $x = (x_1, y_1)$ and $y = (x_2, y_2)$ in $X \times Y$.

2.13.1 Definition

Let (X, d_X) and (Y, d_Y) be two metric spaces. Then, the metric d defined as above on $X \times Y$ is called, the *product metric* of the metrics d_X and d_Y, and is denoted by $d_X \times d_Y$.

The metric space $(X \times Y, d_X \times d_Y)$ is called the *product metric space* of the metric spaces (X, d_X) and (Y, d_Y).

Further, we can also define other metrics on $X \times Y$ given by

$$d^*(x, y) = \left\{(d_X(x_1, x_2))^2 + (d_Y(y_1, y_2))^2\right\}^{\frac{1}{2}}$$

and

$$d^{**}(x, y) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

Problem

35. Verify that d^* and d^{**} defined above are metrics on $X \times Y$.

2.13.2 Theorem

The metrics d, d^* and d^{**} defined on $X \times Y$ are equivalent.

Proof We observe that

$$d(x, y) \le d^*(x, y) \le \sqrt{2}\, d(x, y), \qquad \forall\, x, y \in X \times Y$$

and

$$d(x, y) \le d^{**}(x, y) \le 2d(x, y), \qquad \forall\, x, y \in X \times Y$$

By Theorem 2.5.4, it follows that the metrics d, d^* and d^{**} are equivalent. □

Note In the sequel, we shall consider the product metric on $X \times Y$ to be any one of the equivalent metric d, d^* or d^{**} (unless specified) depending on the suitability of the situation.

Problems

36. If A and B are open (closed) sets in the usual metric space $\mathbb{R}_u$, prove that $A \times B$ is an open (resp., closed) set in Euclidean metric space $\mathbb{R}^2$.
37. Let (X_i, d_i), $(i = 1, 2, \ldots, n)$, be metric spaces and $X = X_1 \times X_2 \times \ldots \times X_n$. Define d, d^* and d^{**} on X by

$$d(x, y) = \left\{\sum_{i=1}^{n} \left(d_i(x_i, y_i)\right)^2\right\}^{\frac{1}{2}}$$

$$d^*(x, y) = \max_{1 \le i \le n} d_i(x_i, y_i)$$

and

$$d^{**}(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i),$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in X. Prove that (X, d), (X, d^*) and (X, d^{**}) are metric spaces.

38. Extend the results of Problem 36 to the Euclidean space $\mathbb{R}^n$.
39. Extend the result of Theorem 2.13.2 to the Euclidean metric space $\mathbb{R}^n$.

2.14 BASES

2.14.1 Definition

Let (X, d) be a metric space and $\mathscr{C}$ be the family of all open subsets of X. A subfamily $\mathscr{B}$ of $\mathscr{C}$ is said to be a *base* (or *basis*) for $\mathscr{C}$ if for each $x \in X$ and for each open set G containing x, $\exists$ a set $B \in \mathscr{B}$ such that $x \in B \subset G$.

2.14.2 Definition

A metric space (X, d) is said to be a *first countable space* (or *first axiom space*) if for every point $x \in X$, $\exists$ a countable family $\{B_n(x)\}$ of open sets containing x such that whenever x belongs to an open set G, $B_n(x) \subset G$ for some n.

2.14.3 Definition

A metric space (X, d) is said to be a *second countable space* (or *second axiom space*) if $\exists$ a countable base for the family of all open subsets of X.

2.14.4 Examples

1. The collection of all open intervals forms a base for the family of all open sets in $\mathbb{R}_u$.
2. The collection of all open spheres forms a base for the family of all open sets in a metric space (X, d).
3. The usual metric space $\mathbb{R}_u$ is a first countable space. Indeed, we may take

$$B_n(x) = \left]x - \frac{1}{n}, x + \frac{1}{n}\right[\quad \text{for each } x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

4. The usual metric space $\mathbb{R}_u$ is a second countable space. In fact, the collection of all open intervals $]a, b[$ with a and b as rational points forms a base for the family of all open subsets of $\mathbb{R}$.

Remark Every second countable metric space is first countable but not conversely.

2.14.5 Example

Let X_d be a discrete metric space, where X is an uncountable set. Then, X is a first countable space but not second countable (Verify !).

In fact, these two types of spaces, first countable and second countable, have been introduced in general topological spaces of which the metric spaces are the special cases. In case of metric spaces, we have

2.14.6 Theorem

Every metric space (X, d) is a first countable space.

Proof Let $x \in X$ and $n \in \mathbb{N}$. Write $B_n(x) = S_{1/n}(x)$. Then $\{B_n(x)\}$ is a countable collection of open subsets of X each of which contains x. Now, suppose x is in some open set G. Then $S_\varepsilon(x) \subset G$, for some $\varepsilon > 0$. In that case $B_n(x) \subset G$ for each $n > \dfrac{1}{\varepsilon}$ and hence X is a first countable space. □

It is clear that every metric space is not necessarily a second countable space (Example 2.14.5). A characterisation for such spaces will be discussed in Theorem 3.4.7.

2.14.7 Theorem

Let (X, d) be a metric space and $Y \subset X$. If (X, d) is second countable space, then (Y, d_Y) is so.