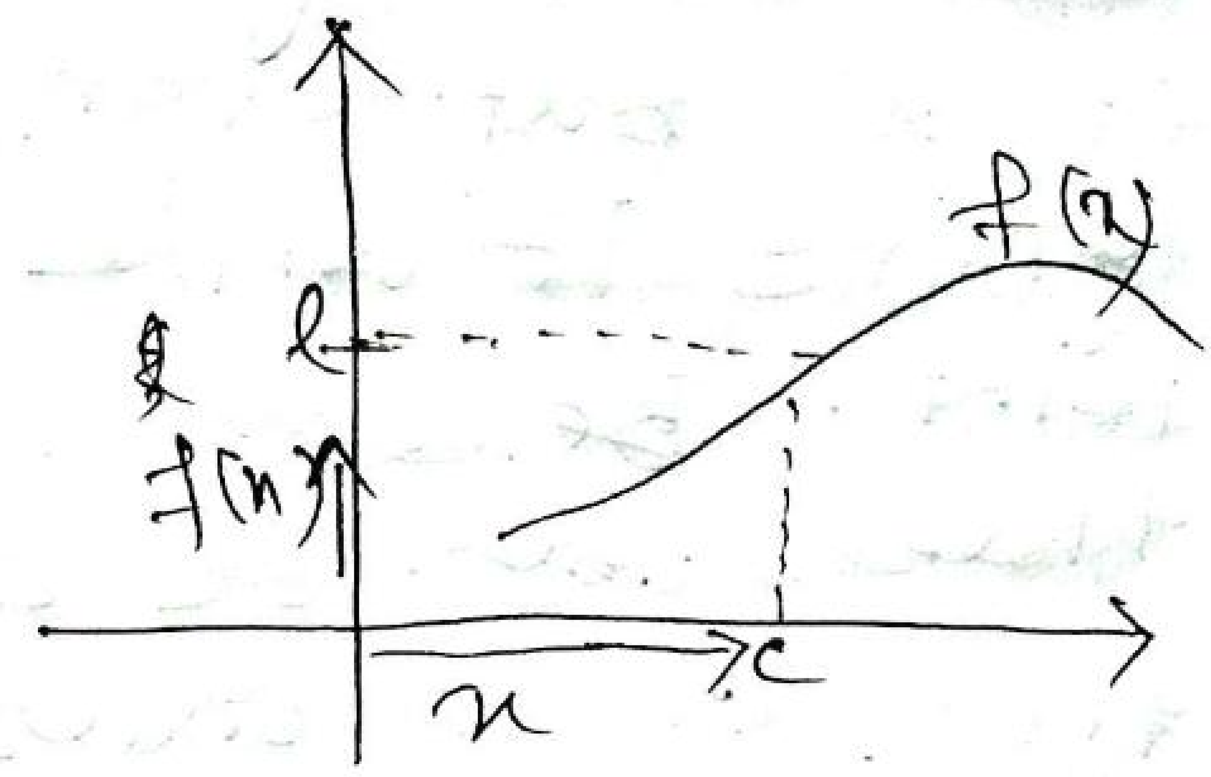
Limit of Functions

Let f is a real valued function. Let $c \in \mathbb{R}$.
$\lim_{x \to c} f(x) = l$ means $f(x)$ is closest to l when x is sufficiently close to c.



The important part or necessary part, in order to get the limit of ~~a~~ the fun^n $f(x)$ at a point c —

(i) The fun^n must be define at all points which is very very close to c. At c, $f(x)$ may or may not be defined.

(ii) ~~At~~ The point c may or may not be available in the set A (the domain of the fun^n) still we can find the limiting value of $f(x)$ when x appraches to c.

Note:- The point c may or may not be in the set, but still $\lim_{x \to c} f(x)$ can be defined, provided $x \in N_\delta(c) \cap A$.

Def^n (cluster point of a set A) :- Let $\underset{(\neq \phi)}{A} \subseteq \mathbb{R}$.
A point $c \in \mathbb{R}$ is said to be a cluster point of A if for every $\delta > 0$ there exists at least one point $x \in A$, $x \neq c$ s.t. $|x-c| < \delta$.

$N_\delta(c) = (c-\delta, c+\delta)$

OR

A point c is a cluster point (limit point) of the set A if every δ-neighbourhood (nbd)

$c-\delta \quad c \quad c+\delta$

$N_\delta(c)$, $(c-\delta, c+\delta)$ of c contains at least one point of A distinct from c.

<u>Note</u>:- c may or may not be a member of A, but even if it is in A, still we ignore it while deciding c as a cluster point. Because we need to show that there ~~exist~~ be a point in $N_\delta(c) \cap A$ distinct from c in order for c to be a cluster point of A.

e.g., $A = \{1, 2\}$

Obviously 1 is not the cluster point of A.

Because $N_{1/2}(1) \cap A = \{1\}$

So there does not exist ~~$x \in A$~~ any x in $N_{1/2}(1) \cap A$.

Hence 1 can't be cluster pt.

Similarly 2 can't be.

<u>Theorem</u>:- A number $c \in \mathbb{R}$ is a cluster point (limit point) of a subset A of $\mathbb{R}$ if and only if (iff) there exist a sequence $\{a_n\}$ in A such that $\lim_{n\to\infty} a_n = c$ and $a_n \neq c \ \forall n \in \mathbb{N}$.

<u>Proof</u>:- If c is a cluster point of A, then for any $n \in \mathbb{N}$, $N_{1/n}(c)$ contains at least one point a_n in A distinct from c.

Then $a_n \in A$, $a_n \neq c$ and $|a_n - c| < 1/n$.

$$\Rightarrow \lim_{n\to\infty} a_n = c.$$

Coversely, given a sequence $\{a_n\}$ in $A \setminus \{c\}$ s.t. $\lim_{n\to\infty} a_n = c$.

For any $\varepsilon > 0$ $\exists$ K s.t. if $n \geq K$ then $a_n \in N_\delta(c)$.

$\therefore$ $N_\delta(c)$ contains a_n, for $n \geq K$. which belongs to A and distinct from c.

$\Rightarrow$ c is the cluster point of A.

Ex:-

1. $A = (0,1) \subseteq \mathbb{R}$.
   Every point including 0 & 1 are cluster pt.

2. A = finite set of real numbers $= \{a_1, a_2, \ldots, a_n\}$

   A has no cluster point.
   suppose a_2 is a cluster point.
   take $\delta = \min\left\{\frac{|a_2 - a_1|}{2}, \frac{|a_2 - a_3|}{2}\right\}$, then there is no point of A in $N_\delta(a_2)$.
   Finite set has no cluster point.

3. Infinite set $\mathbb{N}$ has no cluster point.

4. $A_n = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ has 0 as a cluster pt.

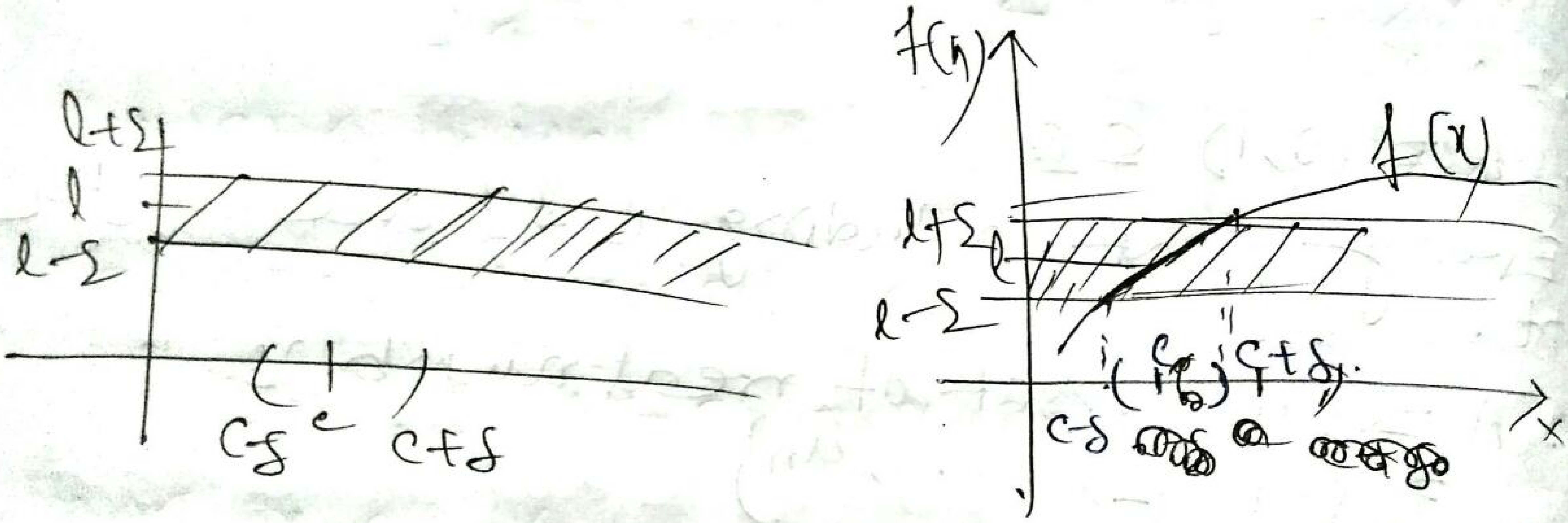
   $(0) \quad \frac{1}{3} \quad \frac{1}{2} \quad 1$

5. $I = [0,1]$
   consider $A = I \cap \mathbb{Q}$, $\mathbb{Q}$ is the set of rational numbers.

A = set of all rational no. lying in betⁿ $[0,1]$

Then every point in I is the cluster pt. of A.

<u>Defⁿ (Limit of function)</u> :- Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f: A \to \mathbb{R}$, a real ~~nuu~~ number l is said to be a limit of f at c if given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$ then $|f(x) - l| < \varepsilon$.



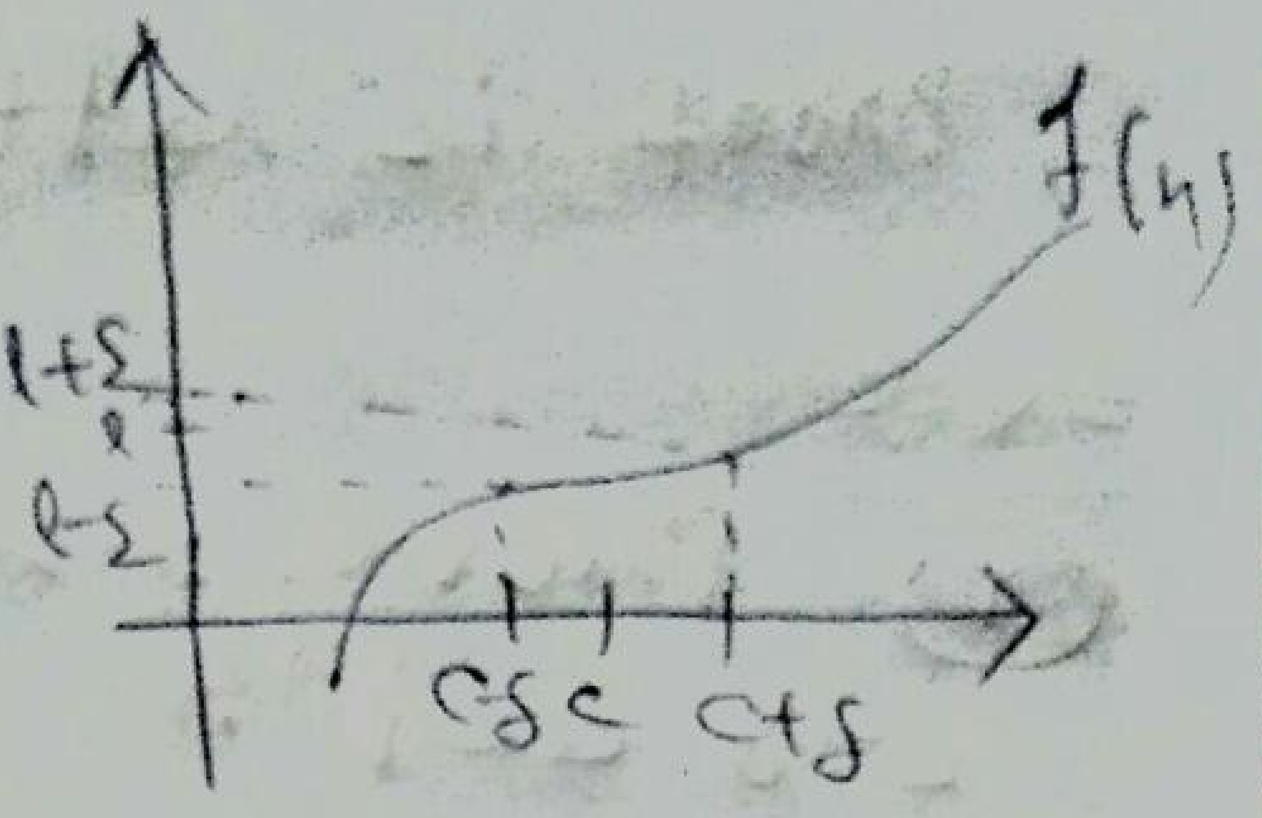
It can be easily converted into neighbourhood.

δ-nbd of c, $N_\delta(c) = \{x : |x - c| < \delta\}$

$0 < |x - c| < \delta$ means $x \in N_\delta(c)$ but $x \neq c$.

Similarly, $|f(x) - l| < \varepsilon$ means $f(x) \in N_\varepsilon(l)$

If we consider a ~~ε-bd~~ ε-nbd of l, $N_\varepsilon(l)$, then corresponding to this nbd, we can find a δ-nbd of c; $(c-\delta, c+\delta)$, where c may not be included in it.



Then the limit says that for any arbitrary point x which lies between the interval $(c-\delta, c+\delta)$ and different from c, the corresponding image $f(x)$ will always fall within the interval $(l-\varepsilon, l+\varepsilon)$.

If it does not happen, then obviously the limit of the funⁿ $f(x)$ when x tends to c will not exist.

It means, there will exists some ε-nbd of l such that whatever the δ-nbd we choose, there will definitely one point whose image will not fall inside $(l-\varepsilon, l+\varepsilon)$.

<u>Theorem</u>:- Let ~~f: ℝ~~ $f: A \to \mathbb{R}$ and let c be a cluster pt. of A. Then the following statements are equivalent —

(i) $\lim_{x \to c} f(x) = l$.

(ii) Given any ε-nbd of l, i.e., $N_\varepsilon(l)$, there exists a δ-nbd $N_\delta(c)$ of c s.t. if $x \neq c$ is any point in $N_\delta(c) \cap A$, then ~~the~~ $f(x) \in N_\varepsilon(l)$.

Proof is left for you!

Ex:-

(1) Prove that $\lim_{x\to c} x^2 = c^2$ using $\epsilon-\delta$ defⁿ.

Here $f(x)=x^2$, $l=c^2$

Given $\epsilon>0$, we have to identify δ so that the condⁿ given earlier ~~hod~~ holds.

So, $|f(x)-l| = |x^2-c^2| = |x+c||x-c|$

Let us take

$|x-c|<1$ —(*) then $|x|<|c|+1$ | $1>|x-c| \geq |x|-|c| \Rightarrow |x|<|c|+1$

$|x+c| \leq |x|+|c|$
$< 2|c|+1$

We are interested to get

$|f(x)-l| = |x^2-c^2| = |x+c||x-c| < \epsilon$.

$|x+c||x-c| < (2|c|+1)|x-c|$ when $|x-c|<1$.

Choose $|x-c| < \frac{\epsilon}{2|c|+1}$ —(**)

& $\delta = \min\left\{1, \frac{\epsilon}{2|c|+1}\right\}$ → for this particular δ both the results (*) & (**) will hold together

$\Rightarrow |x^2-c^2|<\epsilon$, whenever $0<|x-c|<\delta$.

So, for a given ϵ, we can identify a $\delta = \min\left\{1, \frac{\epsilon}{2|c|+1}\right\}$ such that whenever

$0<|x-c|<\delta$, $|f(x)-l|<\epsilon$

$\Rightarrow \lim_{x\to c} f(x) = l \Rightarrow \lim_{x\to c} x^2 = c^2$

(2) To show $\lim_{x\to 2} \frac{x^3-4}{x^2+1} = \frac{4}{5}$ using ε-δ defn.

Here $f(x) = \frac{x^3-4}{x^2+1}$, $l = \frac{4}{5}$

Let given $\varepsilon > 0$,

$$|f(x) - l| = \left|\frac{x^3-4}{x^2+1} - \frac{4}{5}\right|$$

$$= \left|\frac{5x^3-4x^2-24}{5(x^2+1)}\right| = \left|\frac{5x^2+6x+12}{5(x^2+1)}\right| |x-2|$$

Let us take

$|x-2| < 1 \Rightarrow 1 < x < 3$

first bound

then, $5x^2+6x+12 < 5.9+6.3+12 = 75$

$5(x^2+1) > 5(1+1) = 10$

$$|f(x) - l| < \frac{75}{10}|x-2| = \frac{15}{2}|x-2|$$

choose $|x-2| < \frac{2}{15}\varepsilon$, $\delta = \min\{1, \frac{2}{15}\varepsilon\}$

$\therefore |f(x)-l| = \left|\frac{x^3-4}{x^2+1} - \frac{4}{5}\right| < \varepsilon$ provided $|x-2| < \delta$

therefore $\lim_{x\to 2} \frac{x^2-4}{x^2+1} = \frac{4}{5}$