Field :- An Integral domain $(F, +, \cdot)$ is said to be field if each non zero element of F has Multiplicative Inverse.

Q.No. $R = \mathbb{Z}$ is field?

Soln $R = \mathbb{Z}$ is an integral domain and $U(\mathbb{Z}) = \{1, -1\} \neq \mathbb{Z} - \{0\}$

then $\mathbb{Z}$ is not field.

Q. Show that $F = \mathbb{Q}$ is field?

Soln $F = \mathbb{Q}$ is an integral domain

& $U(\mathbb{Q}) = \mathbb{Q}^* = \mathbb{Q} - \{0\}$ —— ①

Equn ① Rep. that each non zero element of $\mathbb{Q}$ has Multiplicative inverse then $F = \mathbb{Q}$ is field.

Q.No. Show that $F = \mathbb{R} | \mathbb{C}$ is field.

$F = \mathbb{R}$ is an integral domain

& $U(\mathbb{R}) = \mathbb{R}^* = \mathbb{R} - \{0\}$ —①

Equn ① Rep. that each non zero element of $\mathbb{R}$ has Multiplicative Inverse then $F = \mathbb{R} | \mathbb{C}$ is field.

Ans

Minimum element are Required for field = 2
for Ideal = 1

Q. $R = \mathbb{Z}[i]$ is field.

Soln $\mathbb{Z}[i]$ is an integral domain and $U(\mathbb{Z}[i]) = \{1, -1, i, -i\} \neq \mathbb{Z}[i] - \{0\}$

then $\mathbb{Z}[i]$ is not field.

Q. $R = \mathbb{Q} \times \mathbb{R}$ is field?

Soln $\mathbb{Q} \times \mathbb{R}$ is not integral domain.
then $\mathbb{Q} \times \mathbb{R}$ is not field.

Q. $R = \{\mathbb{R} \times \{0\}\}$ is field?

Soln $\mathbb{R} \times \{0\}$ is an integral domain and $U(\mathbb{R} \times \{0\}) = \mathbb{R}^* \times \{0\}$

$= \mathbb{R} \times \{0\} - \{(0,0)\}$

then each non zero element of $\mathbb{R} \times \{0\}$ has Multiplicative inverse then $\mathbb{R} \times \{0\}$ is field.

Similarly $\mathbb{Q} \times \{0\}$, $\mathbb{C} \times \{0\}$, $\mathbb{R} \times \{0\} \times \{0\}$, $\mathbb{C} \times \{0\} \times \{0\} \times \{0\}$ etc are field.

Q.No. $R = \mathbb{Z}_5[i]$ is field?

Soln $\mathbb{Z}_5[i]$ is not an integral domain then $\mathbb{Z}_5[i]$ is not field.

Q.No. $R = \mathbb{Z}_{11}$ is field.

Soln $\mathbb{Z}_{11}$ is an integral domain and $U(\mathbb{Z}_{11}) = \{1, 2, \ldots, 10\}$

$= \mathbb{Z}_{11}^*$

$= \mathbb{Z}_{11} - \{0\}$

then $\mathbb{Z}_{11}$ is field.

Q.No. Show that $\mathbb{Z}_p$ is field.

Soln $\mathbb{Z}_p$ is an integral domain.
and $U(\mathbb{Z}_p) = \{1, 2, \ldots, p-1\} = \mathbb{Z}_p - \{0\}$
then each non zero element of $\mathbb{Z}_p$ has Multiplicative inverse.
Then $\mathbb{Z}_p$ is field.

Q.No. if R is field then R has exactly two ideals.

Soln If R is field then $o(R) \geq 2$ — ①
because, every field is an I.D.

integral domain is commutative Ring with unity.

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Case 1:- $I = \{0\}$ is an ideal of F

Case 2: Let I_2 is an ideal of F and $I_2 \neq \{0\}$, then-
since $\exists$ a non zero element of $a \in I_2$ i.e. $0 \neq a \in I_2$
$\uparrow$ I_2 is an ideal of F then
$$I_2 \subseteq F$$
$$\Rightarrow 0 \neq a \in F$$
Since F is field then $a^{-1} \in F$
Now, $a \in I_2$, $a^{-1} \in F$ and I_2 is an ideal of F then
$$a a^{-1} \in I_2$$
$$\Rightarrow 1 \in I_2$$
$$\Rightarrow I_2 = F.$$
Next, $\underbrace{I_1}_{\text{① element}} \neq \underbrace{I_2}_{\text{atleast ② element}}$

then F has exactly two ideals say $I = \{0\}$ and $I = F$

H.C.Q.
Q.No. Let I is an ideal of $\mathbb{C}$ and $I \neq \{0\}$ then
(i) $(1+i) \in I$
(ii) $(1+i) \notin I$
(iii) $(1+i)^{-1}$ is exist in I
(iv) $(1+i)^{-1}$ is not exist in I

Soln
I is an ideal of $\mathbb{C}$ and $I \neq \{0\}$
then $I = \mathbb{C}$ ($\because$ $\mathbb{C}$ is field $\Rightarrow$ 2 element are Required)
If $I \neq \{0\}$ then $I = \mathbb{C}$

$(1+i) \in \mathbb{C} \Rightarrow (1+i) \in I$
Since $\mathbb{C}$ is field ~~and~~ $1+i \neq 0$ then $(1+i)^{-1}$ is exist in $\mathbb{C}$.
$\Rightarrow (1+i)^{-1} \in I$.

Q. How many ideal in $F = \mathbb{Q} \,|\, \mathbb{R} \,|\, \mathbb{C} \,|\, \mathbb{Z}_p \,|\, \mathbb{Q} \times \{0\} \,|\, \mathbb{R} \times \{0\} \times \{0\}$

Sol^n:- $F = \mathbb{Q} \,/\, \mathbb{R} \,/\, \mathbb{Z}_p \,/\, \mathbb{Q}\times\{0\} \,/\, \mathbb{R}\times\{0\}\times\{0\}$ is field.
then F has exactly two ideals.
say $I = \{0\}$ and $I = F$.

Q. How many ideals in $\mathbb{Q}\times\mathbb{R}$?

Sol^n $R = \mathbb{Q}\times\mathbb{R}$

Ring के पास दो idea होते हैं $\{0\}$ & itself

Ideals of $\mathbb{Q}$ are $I_1' = \{0\}$ and $I_2' = \mathbb{Q}$
" " $\mathbb{R}$ are $I_1'' = \{0\}$ & $I_2'' = \mathbb{R}$

Ideals of $\mathbb{Q}\times\mathbb{R}$ are
$I_1 = \{0\}\times\{0\}$, $I_2 = \{0\}\times\mathbb{R}$, $I_3 = \mathbb{Q}\times\{0\}$, $I_4 = \{\mathbb{Q}\}\times\mathbb{R}$

since
(i) I_1 and I_4 are Ideal of $\mathbb{Q}\times\mathbb{R}$
Since $I = \{0\}$ and $I = R$ are always ideal of R
then I_1 & I_4 are ideal of $\mathbb{Q}\times\mathbb{R}$.

$(0,2) \in I$
$(1, \frac{1}{3}) \in \mathbb{Q}\times\mathbb{R}$
but
$(0,2)\cdot(1, \frac{1}{3})$
$= (0, \frac{2}{3})$ $\in$ $\{0\}\times\mathbb{R}$

(ii) Now, show that I_2 is an ideal of $\mathbb{Q}\times\mathbb{R}$

$\therefore I_2 = \{0\}\times\mathbb{R} = \{(0,a) \mid a \in \mathbb{R}\}$
Let $x = (0,a) \in \mathbb{Q}\times\mathbb{R}$
& $r = (r_1, r_2) \in \mathbb{Q}\times\mathbb{R}$, $r_1 \in \mathbb{Q}$, $r_2 \in \mathbb{R}$
s.t. $x\cdot r = (0,a)(r_1, r_2)$
$= (0, ar_2) \in \{0\}\times\mathbb{R}$
$\Rightarrow x\cdot r \in \{0\}\times\mathbb{R}$
Since $\mathbb{Q}\times\mathbb{R}$ is commutative Ring then $r\cdot x = x\cdot r$
$\Rightarrow r\cdot x \in \{0\}\times\mathbb{R}$
Hence $I_2 = \{0\}\times\mathbb{R}$ is an ideal of $\mathbb{Q}\times\mathbb{R}$.
Similarly $I_3 = \mathbb{Q}\times\{0\}$ is an ideal of $\mathbb{Q}\times\mathbb{R}$.

Hence $\mathbb{Q}\times\mathbb{R}$ has exactly 4 ideals.

Q.No. How many ideals in $\mathbb{R}\times\mathbb{Z}_{11}\times\mathbb{Q}$?

Soln $R = \mathbb{R} \times \mathbb{Z}_{11} \times \mathbb{Q}$

$$\begin{array}{ccc} \mathbb{R} & \mathbb{Z}_{11} & \mathbb{Q} \\ \downarrow & \downarrow \text{ideals} & \downarrow \\ \{0\}, \mathbb{R} & \{0\}, \mathbb{Z}_{11} & \{0\} \ \& \ \mathbb{Q} \end{array}$$

ideals of $\mathbb{R} \times \mathbb{Z}_{11} \times \mathbb{Q}$ are

$I_1 = \{0\} \times \{0\} \times \{0\}$

$I_2 = \{0\} \times \mathbb{Z}_{11} \times \{0\}$

$I_3 = \{0\} \times \{0\} \times \{\mathbb{Q}\}$

$I_4 = \mathbb{R} \times \{0\} \times \{0\}$

$I_5 = \{0\} \times \mathbb{Z}_{11} \times \mathbb{Q}$

$I_6 = \mathbb{R} \times \{0\} \times \mathbb{Q}$

$I_7 = \mathbb{R} \times \mathbb{Z}_{11} \times \{0\}$

$I_8 = \mathbb{R} \times \mathbb{Z}_{11} \times \mathbb{Q}$.

then $\mathbb{R} \times \mathbb{Z}_{11} \times \mathbb{Q}$ has exactly 8 ideals.

Q. $R = R_1 \times R_2$ and $I = I_1 \times S \subseteq R_1 \times R_2$ where if I_1 is an ideal of R_1, but S is not ideal of R_2 then $I_1 \times S$ is not ideal of $R_1 \times R_2$.

Soln:- If S is not ideal of R_2. then $\exists$ $a \in S$ and $r_1 \in R_2$ s.t. $a r_1 \notin S$. —— (*)

Now, Let $x = (x_1, a) \in I_1 \times S$, $x_1 \in I_1$, $a \in S$ and $r = (r_1', r_1) \in R_1 \times R_2$, $r_1' \in R_1$ and $r_1 \in R_2$

s.t. $x \cdot r = (x_1, a)(r_1', r_1) = (x_1 r_1', a r_1) \notin I_1 \times S$ $\Rightarrow$ since I_1 is the ideal then $x_1 r_1' \in I$ but $a \cdot r_1 \notin S$

then from (*) $\boxed{I_1 \times S \text{ is not ideal of } R_1 \times R_2}$

Q.No. $R = \mathbb{Z}_5 \times \mathbb{Z}_{10}$, then find all ideals.

Since $\mathbb{Z}_5$ is field then $\mathbb{Z}_5$ has exactly two ideals

$I_1' = \{0\}$ and $I_2' = \mathbb{Z}_5$

\# of ideals in $\mathbb{Z}_{10} = \tau(10) = 4$

$I_1'' = \{0\}$

$I_2'' = \langle 2 \rangle = 2\mathbb{Z}_{10}$

$I_3'' = \langle 5 \rangle = 5\mathbb{Z}_{10}$

$I_4'' = \mathbb{Z}_{10}$

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Ideals of $\mathbb{Z}_5 \times \mathbb{Z}_{10}$ are

$I_1 = \{0\} \times \{0\}$ $\quad$ $I_5 = \mathbb{Z}_5 \times \{0\}$

$I_2 = \{0\} \times 2\mathbb{Z}_{10}$ $\quad$ $I_6 = \mathbb{Z}_5 \times 2\mathbb{Z}_{10}$

$I_3 = \{0\} \times 5\mathbb{Z}_{10}$ $\quad$ $I_7 = \mathbb{Z}_5 \times 5\mathbb{Z}_{10}$

$I_4 = \{0\} \times \mathbb{Z}_{10}$ $\quad$ $I_8 = \mathbb{Z}_5 \times \mathbb{Z}_{10}$

H.W.
Q. Find all ideals of $\mathbb{Z}_8 \times \mathbb{Z}_{30}$

Since $\mathbb{Z}_8$ is field & also $\mathbb{Z}_{30}$.

# of ideals in $\mathbb{Z}_8 = \tau(8) = \tau(2^3) = 4$

$I_1 = \{0\}$

$I_2 = \langle 2 \rangle = 2\mathbb{Z}_8$

$I_3 = \langle 4 \rangle = 4\mathbb{Z}_8$

$I_4 = \mathbb{Z}_8$

# of ideal in $\mathbb{Z}_{30} = \tau(2 \times 3 \times 5) = 8$

$I_1' = \{0\}$ $\quad$ $I_5' = 6\mathbb{Z}_{30}$

$I_2' = \langle 2 \rangle = 2\mathbb{Z}_{30}$ $\quad$ $I_6' = 10\mathbb{Z}_{30}$

$I_3' = \langle 3 \rangle = 3\mathbb{Z}_{30}$ $\quad$ $I_7' = 15\mathbb{Z}_{30}$

$I_4' = \langle 5 \rangle = 5\mathbb{Z}_{30}$ $\quad$ $I_8' = \mathbb{Z}_{30}$

ideals of $\mathbb{Z}_8 \times \mathbb{Z}_{30}$ are total $\underline{\underline{32}}$.

Q.)) Note:- if R has exactly two ideals then R is field?

Hint:- $\boxed{\text{Need not}}$

$R = \mathbb{Z}_8$

$S = \langle 4 \rangle = 4\mathbb{Z}_8 = \{0, 4\}$ is subring of $\mathbb{Z}_8$

then $S = \{0, 4\}$ is ring w.r.t. Modulo 8

S has exactly two ideals.

$I_1 = \{0\}$

& $I_2 = S = \{0, 4\}$

But S is not field $\left(\because S \text{ is not integral domain} \atop 0 \neq 4 \in S \text{ but } 4 \cdot 4 = 0 \right)$

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Notes:- If F is field then If is an integral domain but converse need not be true?

Soln By defn of field, every field is an integral domain.

But converse need not be true

$\mathbb{Z}$ is an integral domain but not field.

Q. Show that every finite integral domain is field?

Soln:- Let F is finite integral domain say

$F = \{0, x_1, x_2, x_3, \text{---} x_n\}$, where $x_1, x_2, x_3,$ --- x_n are non zero element of F.

then $F - \{0\} = \{x_1, x_2, \text{---} x_n\}$

Now $\prod_{i=1}^{n} x_i = x_j$, $0 \neq x_j \in F$

$\Rightarrow x_1 \cdot x_2 \text{----} x_{j-1} x_j x_{j+1} \text{----} x_n = x_j$

$\Rightarrow (x_1 \cdot x_2, \text{---} \cancel{x_{j-1}} \text{---} x_n) - x_j = 0$

$x_j((x_1 x_2 \text{---} x_{j-1} x_{j+1} \text{---} x_n) - 1) = 0$

$\Rightarrow x_j = 0$ or $(x_1 \cdot x_2 \text{---} x_{j-1} x_{j+1} \text{---} x_n) - 1 = 0$

becoz If is an integral domain

Since $x_j \neq 0$

$\Rightarrow (x_1 \cdot x_2 \cdot \text{---} x_{j-1} x_{j+1} \text{--} x_n) - 1 = 0$

$x_1 \cdot x_2 \cdot \text{---} x_{j-1} x_{j+1} \text{--} x_n = 1$

$= x_1 \cdot x_2 \text{---} x_{k-1} x_k x_{k+1} \text{---} x_{j-1} x_{j+1} \text{---} x_n = 1$

$\Rightarrow x_k (x_1 x_2 \text{--} x_{k-1} x_{k+1} \text{--} x_{j-1} x_{j+1} \text{---} x_n) = 1$

$\Rightarrow x_k^{-1}$ is exist in F.

Hence each non zero element of F has Multiplicative inverse of F.

Note:- Let $0 \neq x_i \in F$ then $x_j \cdot x_i = x_r$ where $0 \neq x_r \in F$,

$\Rightarrow x_j = x_r x_i^{-1} \Rightarrow (x_j)^{-1} = (x_r \cdot x_i^{-1})^{-1}$

$= (x_i^{-1})^{-1} x_r^{-1}$

$= x_i x_r^{-1}$ — (*)

$x_i \in F$, $x_r^{-1} \in F$ then $x_i x_r^{-1} \in F$

then x_j^{-1} is exist in F

then every finite integral is field.

e.g.

(i) Z_p is finite integral domain then Z_p is field.

(ii) $Z_3[i]$ is finite integral domain then $Z_3[i]$ is field.

(iii) $Z_{23}[i]$ is finite I.D. then $Z_{23}[i]$ is field.

Q.No. How many ideals in $Z_{11}[i]$?

Soln $Z_{11}[i]$ is field then $Z_{11}[i]$ has exactly two ideals.

say $I_1 = \{0\}$

$I_2 = \langle 1 \rangle = 1 \cdot Z_{11}[i] = Z_{11}[i]$

Q.No. How many units in $Z_{11}[i]$.

Soln $Z_{11}[i]$ is field then each non zero element of $Z_{11}[i]$ has Multiplicative inverse.

$\Rightarrow U(Z_{11}[i]) = Z_{11}[i] - \{0\}$

$\Rightarrow o(U(Z_{11}[i])) = 11^2 - 1 = 120$

Q.No. $4/2$ in Z_6?

[If a/b in R $\exists x \in R$ s.t. $b = ax$]

If $4/2$ then $\exists$ some $x \in Z_6$

s.t. $4x \equiv 2 \pmod 6$ — (1)

$\boxed{x = 2, \ \& \ x = 5 \text{ are possible.}}$

$\gcd(4,6) = 2$ and $2/2$ then equn (1) has exactly 2 soln.

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e.g. $2x \equiv 1 \pmod 3$

$\Rightarrow x = 2$ and $x = 5$ are soln of

$4x \equiv 2 \pmod 6$

$x = 2$ and $x = 5$ are soln of $4x \equiv 2 \pmod 6$

Q.No. $4/3$ in Z_{100} ?

Soln If $4/3$ in Z_{100} then $4x \equiv 3 \pmod{100}$ —① has soln.
But $\gcd(4, 100) = 4$ But $4 \nmid 3$ then equn ① has no soln.
then $4 \nmid 3$.

H.W.
Q.No. $9/3$ in Z_{15} ?

If $9/3$ in Z_{15} then $9x \equiv 3 \pmod{15}$ — ①.
$\gcd(9, 15) = 3$ and $3/3$ then equn ① has exactly 3 soln.

Maximal Ideal:- Let R be a commutative Ring and ideal A of R is said to be maximal ideal of R ($A \neq R$), if $\exists$ an ideal B of R s.t.

$A \subseteq B \subseteq R$ then $A = B$ or $B = R$

Q.No.:- Find Maximal Ideal of Z_{10}.

Soln:- Ideals of $Z_{10} \Rightarrow I_1 = \{0\}$

$I_2 = \langle 2 \rangle = 2Z_{10} = \{0, 2, 4, 6, 8\}$

$I_3 = \langle 5 \rangle = 5Z_{10} = \{0, 5\}$

$I_4 = \langle 1 \rangle = ...$

$I_4 \neq Z_{10}$ (By def^n)

$\Rightarrow$ (i) $I_4 = Z_{10}$ is not Maximal ideal

(ii) $I_1 \subseteq I_2 \subseteq Z_{10}$ But $I_1 \neq I_2$ & $I_2 \neq Z_{10}$

$\Rightarrow I_1$ is not Maximal.

(iii) Let I is an ideal of Z_{10} s.t. $I_2 \subseteq I \subseteq Z_{10}$

$\Rightarrow I_2 = 2Z_{10} \subseteq nZ_{10} \subseteq Z_{10}$ $(mz \subseteq nz)$

$\Rightarrow n \mid 2$

$\Rightarrow n = 1$ and 2 are only possibility.

If $n=1$ then $I = Z_{10}$

If $n=2$ " $I = I_2$

then $I_2 = I$ or $I = Z_{10}$

then $2Z_{10}$ is Maximal ideal of (Z_{10})

(iv) Let I is an ideal of Z_{10} s.t.

$I_3 \subseteq I \subseteq Z_{10}$

$\Rightarrow 5Z_{10} \subseteq nZ_{10} \subseteq Z_{10}$

$\Rightarrow n/5$

$\Rightarrow n=1$ & 5 are only possibility

If $n=1$ then $I = Z_{10}$

If $n=5$ " $I = 5Z_{10} = I_3$

then $I = Z_{10}$ or $I = I_3$.

then $5Z_{10}$ also is maximal ideal of Z_{10}.

Q. How many maximal ideal in Z_{20}.

Soln Ideals of Z_{20}

$I_1 = \{0\}$

$I_2 = 2Z_{20} = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$

$I_3 = 4Z_{20} = \{0, 4, 8, 12, 16\}$

$I_4 = 5Z_{20} = \{0, 5, 15\}$

$I_5 = 10Z_{20} = \{0, 10\}$

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(1) I_6 is not maximal by def^n

(2) $I_1 \subseteq I_2 \subseteq Z_{20}$

But $I_1 \neq I_2$ and $I_2 \neq Z_{20}$

then I_1 is not maximal.

(3) $I_3 \subseteq I_2 \subseteq Z_{10}$ But $I_3 \neq I_2$ and $I_2 \neq Z_{20}$

then I_3 is not maximal.

(4) $I_5 \subsetneq I_4 \subseteq Z_{20}$ But $I_5 \neq I_4$ and $I_4 \neq Z_{20}$

then I_5 is not maximal ideal.

Hence I_2 and I_4 are maximal ideals of Z_{20}

<u>Maximal ideal of Z_8</u> :-

Ideals of Z_8 are $I_1 = \{0\}$

$I_2 = 2Z_8 = \{0, 2, 4, 6\}$

$I_3 = 4Z_8 = \{0, 4\}$

$I_4 = Z_8$

$\therefore I_2$ is maximal ideal of Z_8.

<u>Note</u>:- No. of Maximal. ideal in Z_n = No. of prime divisors of n.

<u>e.g</u> # of maximal in Z_{50} = No. of prime divisors of 50

$= 2$

Say $I_1 = 2Z_{50} = \langle 2 \rangle = \{0, 2, 4, 6, \text{———} 48\}$

$I_2 = 5Z_{50} = \langle 5 \rangle = \{0, 5, \text{———} 45\}$

<u>Q</u>: $I = m\mathbb{Z}$ is maximal ideal of $\mathbb{Z}$ iff $m = p$?

<u>Sol^n</u> Let $m = p$ then show that $p\mathbb{Z}$ is maximal ideal of $\mathbb{Z}$.

Let $I = n\mathbb{Z}$ is an ideal of $\mathbb{Z}$ s.t.

$p\mathbb{Z} \subseteq n\mathbb{Z} \subseteq \mathbb{Z}$

$\Rightarrow n \mid p$

$\Rightarrow n = 1$ or p.

If $n=1$ then $n\mathbb{Z} = 1\mathbb{Z} = \mathbb{Z} = R$

If $n = p$ then $n\mathbb{Z} = p\mathbb{Z}$

then $I = p\mathbb{Z}$ or $I = \mathbb{Z}$

then $p\mathbb{Z}$ is maximal ideal of $\mathbb{Z}$.

Conversely Now show that if $m \neq p$ then $m\mathbb{Z}$ is not maximal.

Case 1:- if $m=0$ then $0\mathbb{Z} = \{0\}$ and $\{0\} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$

$\{0\} \neq 2\mathbb{Z}$ & $2\mathbb{Z} \neq \mathbb{Z}$

then $\{0\}$ is not maximal ideal.

Case 2:- If $m=1$ then $m\mathbb{Z} = 1\mathbb{Z} = \mathbb{Z} = R$

By def^n $I = \mathbb{Z}$ is not maximal ideal.

Case 3. If $m > 1$ and $m \neq p$ then $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$

s.t. $m\mathbb{Z} \subseteq p_1\mathbb{Z} \subseteq \mathbb{Z}$

But $m\mathbb{Z} \neq p_1\mathbb{Z}$ and $p_1\mathbb{Z} \neq \mathbb{Z}$

then $m\mathbb{Z}$ is not maximal.

From case I, II, & III, we get $m\mathbb{Z}$ is not maximal if $m \neq p$.

$\Rightarrow m\mathbb{Z}$ is maximal then $m = p$.

e.g. $2\mathbb{Z}, 3\mathbb{Z}, 15\mathbb{Z}, 7\mathbb{Z}, 11\mathbb{Z}$ --- are maximal of $\mathbb{Z}$ and $4\mathbb{Z}, 6\mathbb{Z}, 8\mathbb{Z}, 10\mathbb{Z}$ --- are not maximal ideal of $\mathbb{Z}$.

Note!- $\mathbb{Z}$ has infinite no. of maximal ideals.

Q.No. Intersection of two maximal ideal of R is maximal ideal? $\boxed{\text{Need Not}}$

Sol^n $2\mathbb{Z}$ is maximal ideal of $\mathbb{Z}$ and $3\mathbb{Z}$ is maximal ideal of $\mathbb{Z}$

But $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ is not maximal ideal of $\mathbb{Z}$.

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then intersection of two maximal ideal need not be maximal.

Q.No. $R = Q \times \mathbb{R}$. Find Maximal Ideal of R.

Soln Ideal of $Q \times \mathbb{R}$

$I_1 = \{0\} \times \{0\}$

$I_2 = \{0\} \times \mathbb{R}$

$I_3 = Q \times \{0\}$

$I_4 = Q \times \mathbb{R}$

(i) I_4 is not maximal by defn

(ii) $I_1 \subseteq I_2 \subseteq Q \times \mathbb{R}$

But $I_1 \neq I_2$ or $I_2 \neq Q \times \mathbb{R}$

then I_1 is not maximal.

then I_2 and I_3 is maximal ideal of $Q \times \mathbb{R}$.

Q.No. find Maximal ideal of $Z_{11} \times Q \times Z_7[i]$

Soln. Ideals of $Z_{11} \times Q \times Z_7[i]$

$I_1 = \{0\} \times \{0\} \times \{0\}$

$I_2 = Z_{11} \times \{0\} \times \{0\}$

$I_3 = \{0\} \times Q \times \{0\}$

$I_4 = \{0\} \times \{0\} \times Z_7[i]$

$I_5 = Z_{11} \times Q \times \{0\}$

$I_6 = Z_{11} \times \{0\} \times Z_7[i]$

$I_7 = \{0\} \times Q \times Z_7[i]$

$I_8 = Z_{11} \times Q \times Z_7[i]$

(1) I_8 is not possible by defn

(2) $I_1 \subseteq I_2 \subseteq Z_{11} \times Q \times Z_7[i]$

But $I_1 \neq I_2$ and $I_2 \neq Z_{11} \times Q \times Z_7[i]$

I_2 is also not possible

(3) $I_3 \subseteq I_5 \subseteq Z_{11} \times Q \times Z_7[i]$, But $I_3 \neq I_5$ and $I_5 \neq Z_{11} \times Q \times Z_7[i]$

So I_3 is also not possible.

(4) $I_4 \subseteq I_6 \subseteq Z_{11} \times Q \times Z_7[i]$, But $I_4 \neq I_6$ & $I_6 \neq Z_{11} \times Q \times Z_7[i]$

So I_4 is also not.

I_5, I_6 and I_7 are 3 maximal ideal of $Z_{11} \times Q \times Z_7[i]$

Hence $Z_{11} \times Q \times Z_7[i]$ has exactly three maximal ideal.

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Q.No. If F is field then F has exactly 1 maximal ideal.

Soln We know that if F is field then F has exactly two ideals say $I_1 = \{0\}$ and $I_2 = F$

$I_2 = F$ is not maximal ideal (By defn)

Now Let I is an ideal of F s.t. $I_1 \subseteq I \subseteq F$

then $I = I_1$ or $I = F$

then $I_1 = \{0\}$ is maximal ideal.

Hence F has exactly one maximal ideal.

e.g. (i) $F = \mathbb{Q}$ has exactly one maximal ideal say $I = \{0\}$

(ii) $F = \mathbb{R}$ " " " " " " $= \{0\}$

(iii) $F = \mathbb{Z}_n[i]$ " " " " " $= \{0\}$

Note:- Converse of above statement need not be true.

e.g. $\mathbb{Z}_8$ has exactly one maximal ideal but $\mathbb{Z}_8$ is not field.

Q.No. $R = \mathbb{Z}_{11} \times \mathbb{Z}_{10}$, how many maximal ideals in R.

Soln Ideals of $\mathbb{Z}_{11} \times \mathbb{Z}_{10}$ are

$I_1 = \{0\} \times \{0\}$

$I_2 = \{0\} \times 2\mathbb{Z}_{10}$

$I_3 = \{0\} \times 5\mathbb{Z}_{10}$

✓ $I_4 = \{0\} \times \mathbb{Z}_{10}$

$I_5 = \mathbb{Z}_{11} \times \{0\}$

✓ $I_6 = \mathbb{Z}_{11} \times 2\mathbb{Z}_{10}$

✓ $I_7 = \mathbb{Z}_{11} \times 5\mathbb{Z}_{10}$

$I_8 = \mathbb{Z}_{11} \times \mathbb{Z}_{10}$

hence $R = \mathbb{Z}_{11} \times \mathbb{Z}_{10}$ has 3 Maximal ideals.

Q.No. How many maximal ideal of $\mathbb{Z}_8 \times \mathbb{Z}_{30}$.

Soln Maximal ideal of $\mathbb{Z}_8 \times \mathbb{Z}_{30}$

$I_1 = 2\mathbb{Z}_8 \times \mathbb{Z}_{30}$

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$I_2 = Z_8 \times 2Z_{30}$
$I_3 = Z_8 \times 3Z_{30}$
$I_4 = Z_8 \times 5Z_{30}$

Note:- Maximal ideal of $R_1 \times R_2$

$$= \begin{cases} \text{Maximal ideal of } R_1 \times R_2 \\ R_1 \times \text{Maximal ideal of } R_2. \end{cases}$$

H.w.

Q.No. How many maximal ideal of $Z_6 \times Z_{15}$?
prime divisors are 2, 3   3, 5

Maximal ideals are
$I_1 = 2Z_6 \times Z_{15}$
$I_2 = 3Z_6 \times Z_{15}$
$I_3 = Z_6 \times 3Z_{15}$
$I_4 = Z_6 \times 5Z_{15}$

Hence $Z_6 \times Z_{15}$ has 4 maximal ideal.

Q. Show that $4Z$ is maximal in $2Z$.

Soln :- Let $I = nZ$ is an ideal of $2Z$ s.t. $4Z \subseteq nZ \subseteq 2Z$
$\Rightarrow n \mid 4$
$\Rightarrow n = 1$ or 2 or 4

Put $n=1$ then $I = Z$
But $Z \not\subseteq 2Z$
then $n=1$ is not possible.
Put $n=2$, then $I = nZ = 2Z \Rightarrow I = 2Z = R$
Put $n=4$, " $I = nZ = 4Z \Rightarrow I = 4Z$
Hence $4Z$ is maximal ideal of $2Z$

Q.No. Show that $6Z$ is maximal ideal in $2Z$

Show Let $I = nZ$ is an ideal of $2Z$ s.t.
$6Z \subseteq nZ \subseteq 2Z \Rightarrow n \mid 6$

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$n=1$ and 3 is not possible only $n=2$ and 6 are possible

If $n=2$ then $I=2\mathbb{Z}=R$ $\quad (\because \mathbb{Z} \not\subseteq 2\mathbb{Z}$

If $n=6$ " $I=6\mathbb{Z}$ $\quad 3\mathbb{Z} \not\subseteq 2\mathbb{Z})$

then $I=6\mathbb{Z}$ is maximal ideal of $2\mathbb{Z}$.

($p m\mathbb{Z}$ is maximal ideal of $m\mathbb{Z}$)

Note/- $pm\mathbb{Z}$ is maximal ideal of $m\mathbb{Z}$, $m\geq 1$

Prime Ideal/-

Let R be a commutative ring.

An ideal P of R $(P\neq R)$ is said to be Prime ideal of R if $a\cdot b\in P$, $a\in R$, $b\in R$ then $a\in P$ or $b\in P$

Imp

Q.No. $I=\{0\}$ is prime ideal of $\mathbb{Z}$?

Soln Let $a\in\mathbb{Z}$ and $b\in\mathbb{Z}$

s.t. $a\cdot b\in\{0\}$

$\Rightarrow a\cdot b=0$

$\Rightarrow a=0$ or $b=0$ because $\mathbb{Z}$ is an integral domain.

If $a=0$ then $a\in I$

If $b=0$ then $b\in I$

if $a=0$ and $b=0$ then $a\in I$ and $b\in I$

then $I=\{0\}$ is prime ideal of $\mathbb{Z}$.

H.W.

Q. If R is an integral domain then $I=\{0\}$ is prime ideal of R.

Soln Since R is integral domain

Let $a,b\in R$ s.t. $a\cdot b=0$, then $a\cdot b\in\{0\}$

$a=0$ or $b=0$ $[\because R$ is I.D$]$

$\Rightarrow a\in\{0\}$ or $b\in\{0\}$

$I=\{0\}$ is prime ideal of R

$\left\{ \begin{array}{l} \because a\cdot b\in P, \ a\in R, b\in R \\ \Rightarrow a\in P \text{ or } b\in P \end{array} \right\}$

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Q. If F is field then F has exactly one prime ideal.

Soln If F is field then F has exactly two ideal say $I_1 = \{0\}$ & $I_2 = F$

By defn $I_2 = F$ is not prime ideal

If F is field then F is an integral domain then $I_1 = \{0\}$ is prime ideal.

then F has exactly one prime ideal.

e.g. $F = \mathbb{Q} | \mathbb{R} | \mathbb{C} | \mathbb{Z}_p | \mathbb{Z}_{11}[i] | \mathbb{Q} \times \{0\} | \mathbb{R} \times \{0\} \times \{0\}$ has exactly one prime ideal say $I = \{0\}$.

Q. $R = \mathbb{Z}_{15}$ then $I = \{0\}$ is prime ideal of R?

Soln $3 \in \mathbb{Z}_{15}$ and $5 \in \mathbb{Z}_{15}$ s.t. $3 \cdot 5 = 0 \in \{0\}$

$\Rightarrow 3 \cdot 5 \in \{0\}$

e.g. But $3 \notin \{0\}$ & $5 \notin \{0\}$

then $I = \{0\}$ is not prime ideal.

Q. Show that $p\mathbb{Z}$ is prime ideal of $\mathbb{Z}$.

Soln Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ s.t.

$a \cdot b \in p\mathbb{Z}$

if $a \cdot b \in p\mathbb{Z}$ then $\exists x \in \mathbb{Z}$ s.t. $ab = p \cdot x$

$\Rightarrow p | ab$ —— (1)

If $p | a$ then $a \in p\mathbb{Z}$

If $p \nmid a$ then $\gcd(p, a) = 1$ then $p | b \Rightarrow b \in p\mathbb{Z}$

Hence $a \in p\mathbb{Z}$ or $b \in p\mathbb{Z}$

then $p\mathbb{Z}$ is prime ideal of $\mathbb{Z}$.

e.g. $2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}$ are prime ideals of $\mathbb{Z}$.

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Q. if $m \geq 1$ and $m \neq p$ then $m\mathbb{Z}$ is not prime ideal.

Sol^n **Case 1** If $m=1$ then $1\mathbb{Z} = \mathbb{Z}$ is not prime ideal of $\mathbb{Z}$ (by def^n)

Case 2:- If $m > 1$ and $m \neq p$ then $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$
$= a \cdot b$
where $a = p_1^{r_1}$
$b = p_2^{r_2} p_3^{r_3} \cdots p_k^{r_k}$

$a = p_1^{r_1} \in \mathbb{Z}$ and $b = p_2^{r_2} p_3^{r_3} \cdots p_k^{r_k} \in \mathbb{Z}$

s.t. $a \cdot b = m \in m\mathbb{Z}$

$\Rightarrow a \cdot b \in m\mathbb{Z}$

But $a, b \notin m\mathbb{Z}$

then $m\mathbb{Z}$ is not prime ideal.

from Case (I) & (II), we get

$m\mathbb{Z}$ is not prime ideal if $m \neq p$, $m \geq 1$

e.g. $4\mathbb{Z}, 6\mathbb{Z}, 8\mathbb{Z}, 9\mathbb{Z}, 10\mathbb{Z} \cdots$ are not prime ideal of $\mathbb{Z}$.

Q.No Find all prime ideal of $\mathbb{Z}_{10}$.

Sol^n Ideals of $\mathbb{Z}_{10}$ are

$\times$ $I_1 = \{0\}$ $\Rightarrow 2 \in \mathbb{Z}_{10}, 5 \in \mathbb{Z}_{10}$ s.t. $2 \cdot 5 \in \{0\}$, but $2 \notin \{0\}, 5 \notin \{0\}$

$I_2 = \langle 2 \rangle = 2\mathbb{Z}_{10}$

$I_3 = \langle 5 \rangle = 5\mathbb{Z}_{10}$

$\times$ $I_4 = \langle 10 \rangle = \mathbb{Z}_{10}$

(1) $I_1 = \{0\}$ is not prime Ideal.

$I_2 = 2\mathbb{Z}_{10}$ is prime ideal of $\mathbb{Z}_{10}$

Now show that

Let $a \in \mathbb{Z}_{10}$ and $b \in \mathbb{Z}_{10}$ s.t. $a \cdot b \in 2\mathbb{Z}_{10}$

$\Rightarrow a \cdot b = 2 \cdot x, \; x \in \mathbb{Z}_{10}$

$\Rightarrow 2 \mid a \cdot b$

$\Rightarrow 2 \mid a$ or $2 \mid b$

If $2 \mid a$ then $a \in 2\mathbb{Z}_{10}$

if $2 \mid b$ then $b \in 2\mathbb{Z}_{10}$

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then $2\mathbb{Z}_{10}$ is prime ideal of $\mathbb{Z}_{10}$

Similarly $I_3 = 5\mathbb{Z}_{10}$ is prime ideal of $\mathbb{Z}_{10}$.

Note:- No. of Prime ideal in $\mathbb{Z}_n$ = No. of prime ~~ideal~~ divisor of n.

e.g. # of prime ideal of $\mathbb{Z}_{30} = 3$ (i.e. $2, 3, 5$)

Say $I_1 = \langle 2 \rangle = 2\mathbb{Z}_{30}$

$I_2 = \langle 3 \rangle = 3\mathbb{Z}_{30}$

$I_3 = \langle 5 \rangle = 5\mathbb{Z}_{30}$

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Q.No. $R = \mathbb{Z}_{p^2q}$, where p and q are distinct prime no. then how many prime ideal in $\mathbb{Z}_{p^2q}$.

Soln:- # of prime ideal in $\mathbb{Z}_{p^2q} = 2$

Q.No. Find prime ideals of $\mathbb{Q} \times \mathbb{R}$.

Soln. Ideals of $\mathbb{Q} \times \mathbb{R}$

$I_1 = \{0\} \times \{0\}$

$I_2 = \{0\} \times \mathbb{R}$

$I_3 = \mathbb{Q} \times \{0\}$

$I_4 = \mathbb{Q} \times \mathbb{R}$

① $I_4 = \mathbb{Q} \times \mathbb{R}$ is not prime ideal (By defn)

② $x = (1,0) \in \mathbb{Q} \times \mathbb{R}$ and $y = (0,1) \in \mathbb{Q} \times \mathbb{R}$

s.t. $x \cdot y = (1,0) \cdot (0,1)$
$= (0,0) \in I_1$

$\Rightarrow x \cdot y \in I_1$ But $x = (1,0) \notin I_1$ and $y = (0,1) \notin I_1$

then I_1 is not prime ideal.

Now show that $I_2 = \{0\} \times \mathbb{R}$ is prime ideal of $\mathbb{Q} \times \mathbb{R}$.

let $x = (a,b) \in \mathbb{Q} \times \mathbb{R}$, $a \in \mathbb{Q}$, $b \in \mathbb{R}$

& $y = (c,d) \in \mathbb{Q} \times \mathbb{R}$, $c \in \mathbb{Q}$, $d \in \mathbb{R}$

s.t. $x \cdot y \in I_2 = \{0\} \times \mathbb{R}$

$\Rightarrow (a,b)\cdot(c,d) \in \{0\}\times\mathbb{R}$

$\Rightarrow (ac, bd) \in \{0\}\times\mathbb{R}$

$\Rightarrow a\cdot c = 0$

$\Rightarrow a=0$ or $c=0$

($\because$ $\mathbb{Q}$ is an integral domain)

Case 1: If $a=0$ then $x=(0,b)\in\{0\}\times\mathbb{R}$

Case 2: If $c=0$ then $y=(0,d)\in\{0\}\times\mathbb{R}$

Case 3: If $a=0$ & $c=0$ then $x=(0,b)\in\{0\}\times\mathbb{R}$ & $y=(0,d)\in\{0\}\times\mathbb{R}$

from (I), (II) and (III), we get

$x\in\{0\}\times\mathbb{R}$ or $y\in\{0\}\times\mathbb{R}$

then $I_2=\{0\}\times\mathbb{R}$ is prime ideal of $\mathbb{Q}\times\mathbb{R}$.

Similarly $I_3=\mathbb{Q}\times\{0\}$ is prime ideal of $\mathbb{Q}\times\mathbb{R}$

H.W. Q.No. (1) $R=\mathbb{Q}\times\mathbb{R}\times\mathbb{C}$, find prime ideal of R.

(1) Soln (1) $R=\mathbb{Q}\times\mathbb{R}\times\mathbb{C}$

$\quad\quad 2 \quad 2 \quad 2 = 8$

ideals are $\{0\},\mathbb{Q}$; $\{0\},\mathbb{R}$; $\{0\},\mathbb{C}$

$I_1=\{0\}\times\{0\}\times\{0\}$ $\quad I_3=\{0\}\times\mathbb{R}\times\{0\}$ $\quad I_5=\mathbb{Q}\times\mathbb{R}\times\{0\}$ $\quad \cancel{I_7}=\{0\}\times\mathbb{R}\times\mathbb{C}$

$I_2=\mathbb{Q}\times\{0\}\times\{0\}$ $\quad I_4=\{0\}\times\{0\}\times\mathbb{C}$ $\quad \cancel{I_6}=\mathbb{Q}\times\{0\}\times\mathbb{C}$ $\quad I_8=\mathbb{Q}\times\mathbb{R}\times\mathbb{C}$

(I) I_8 is not prime ideal (By def^n)

(II) $I_1=\{0\}\times\{0\}\times\{0\} \Rightarrow$ let $x=(1,0,0)\in\mathbb{Q}\times\mathbb{R}\times\mathbb{C}$, $y=(0,1,0)\in\mathbb{Q}\times\mathbb{R}\times\mathbb{C}$

s.t. $x\cdot y=(0,0,0)\in I_1$

but $x\notin I_1$ and $y\notin I_1 \Rightarrow I_1$ is not prime ideal

$I_2=\mathbb{Q}\times\{0\}\times\{0\}$ let $x=(a_1,b_1,c_1)\in\mathbb{Q}\times\mathbb{R}\times\mathbb{C}$ & $y=(a_2,b_2,c_2)\in\mathbb{Q}\times\mathbb{R}\times\mathbb{C}$

s.t. $x\cdot y\in I_2 \Rightarrow (a_1a_2, b_1b_2, c_1c_2)\in\mathbb{Q}\times\{0\}\times\{0\}$

$\Rightarrow b_1b_2=0$ and $c_1c_2=0 \Rightarrow b_1=0$ or $b_2=0$

$c_1=0$ or $c_2=0$

$x=(a_1,0,0)$, $y=(a_2,0,0)\in\mathbb{Q}\times\mathbb{R}\times\mathbb{C}$ [$\because \mathbb{R},\mathbb{C}$ are I.D]

$I_2=\mathbb{Q}\times\{0\}\times\{0\}$ prime ideal of $\mathbb{Q}\times\mathbb{R}\times\mathbb{C}$

② Q. No. Find all prime ideal of $Z_{11} \times Z_{10}$

③ Q. No. Find all prime ideal of $Z_8 \times Z_{10}$

[इसमे जितने Maximal ideal है उतने ही prime ideal]

Similarly, I_3 & I_4 are ideal of $Q \times IR \times \mathbb{C}$

Now $I_5 = Q \times IR \times \{0\} \Rightarrow x \cdot y = (a_1 a_2, b_1 b_2, c_1 c_2) \in Q \times IR \times \{0\}$

$\Rightarrow c_1 c_2 = 0 \Rightarrow c_1 = 0$ or $c_2 = 0$

If $c_1 = 0$ then $x = (a_1, b_1, 0) \in Q \times IR \times \{0\}$ and $c_1 = 0, c_2 = 0$

$c_2 = 0$ " $y = (a_2, b_2, 0) \in Q \times IR \times \{0\}$ $x, y \in Q \times IR \times \mathbb{C}$

Hence I_5 is prime ideal

Similarly I_6, I_7 is prime ideal

2 Sol^n $R = Z_{11} \times Z_{10}$

# of prime ideal $\to 2 \quad 2^2 = 8$

$\checkmark$ $\cancel{I_1} = \{0\} \times \{0\}$ $\cancel{I_2} = \{0\} \times Z_{10}$

$\checkmark$ $I_3 = \{0\} \times 2Z_{10}$ $I_4 = \{0\} \times 5Z_{10}$

$\times$ $I_5 = Z_{11} \times \{0\}$ $\cancel{I_6} = Z_{11} \times Z_{10}$

$\checkmark$ $I_7 = Z_{11} \times 2Z_{10}$ $\cancel{I_8} = Z_{11} \times 5Z_{10}$

1 case $I_6 = Z_{11} \times Z_{10}$ is not prime ideal (by def^n)

(ii) let $x = (1, 0) \in Z_{11} \times Z_{10}$

$y = (0, 1) \in Z_{11} \times Z_{10}$

s.t. $x \cdot y = (0, 0) \in \{0\} \times \{0\}$

but $x \notin \{0\} \times \{0\}$, $y \notin \{0\} \times \{0\}$

I_1 is not prime ideal

(iii) $x \cdot y = (a_1 a_2, b_1 b_2) \in \{0\} \times Z_{10}$

$\Rightarrow a_1 a_2 = 0 \Rightarrow a_1 = 0$ or $a_2 = 0$

$\Rightarrow x \in \{0\} \times Z_{10}$ or $y \in \{0\} \times Z_{10}$ [$\because Z_{11}$ is I.D]

I_2 is prime ideal.

(iv) Similarly I_3, I_4 are prime ideal.

(v) $x \cdot y = (a_1 a_2, b_1 b_2) \in Z_{11} \times \{0\}$

$\Rightarrow b_1 b_2 = 0$ but Z_{10} is not I.D.

So $x \notin Z_{11} \times \{0\}$ & $y \in Z_{11} \times \{0\}$

3 Sol $R = Z_8 \times Z_{10}$

$\downarrow 2 \quad \downarrow 2^2 = 8$

Ideal are

$\times$ $I_1 = \{0\} \times \{0\}$ $\times$ $I_5 = Z_8 \times \{0\}$

$\times$ $I_2 = \{0\} \times 2Z_{10}$ $\cancel{I_6} = Z_8 \times 2Z_{10}$

$\times$ $I_3 = \{0\} \times 5Z_{10}$ $I_7 = Z_8 \times 5Z_{10}$

$\times$ $I_4 = \{0\} \times I_{10}$ $I_8 = Z_8 \times Z_{10}$ $\times$

(i) It is not Prime ideal By def^n --

(ii) $I_2 = \{0\} \times 2Z_{10}$

$x \cdot y = (a_1 a_2, b_1 b_2) \in \{0\} \times 2Z_{10}$

$\Rightarrow a_1 a_2 = 0 \Rightarrow a_1 = 0$ or $a_2 = 0$

$\Rightarrow I_2$ is not prime ideal. [$\because Z_{10}$ is not I.D]

(iii) Similarly I_3, I_4, I_5 are not prime ideal.

(iv) $I_6 = Z_8 \times 2Z_{10}$

$x \cdot y = (a_1 a_2, b_1 b_2) \in Z_8 \times 2Z_{10}$

$\Rightarrow a_1 a_2 \in Z_8$, $b_1 b_2 \in 2Z_{10}$

$\Rightarrow a_1 \in Z_8, a_2 \in Z_8$ $b_1 \in 2Z_{10}, b_2 \in 2Z_{10}$

$\Rightarrow x = (a_1, b_1) \in Z_8 \times 2Z_{10}$

$y = (a_2, b_2) \in Z_8 \times 2Z_{10}$

Hence Z_6 is prime ideal.

$\Rightarrow I_7$ is prime ideal.

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Q.No: Every Prime ideal of R is maximal ideal of R?

Soln $\boxed{\text{Need Not}}$

$I=\{0\}$ is prime ideal in $\mathbb{Z}$ But $I=\{0\}$ is not maximal ideal in $\mathbb{Z}$.

Q.No: Every Maximal ideal of R is prime ideal of R?

Soln Need Not.

$4\mathbb{Z}$ is maximal ideal of $2\mathbb{Z}$
But not prime ideal of $2\mathbb{Z}$ becoz $2 \in 2\mathbb{Z}$ s.t.

$$2 \cdot 2 = 4 \in 4\mathbb{Z}$$
$$\text{But } 2 \notin 4\mathbb{Z}$$

Note:- (i) if R is commutative Ring with unity then every maximal ideal of R is prime ideal.

(ii) If R is finite commutative Ring with unity then every prime ideal of R is maximal ideal.

Q. Intersection of two prime ideals of R is prime ideal?

Soln Need Not.

$2\mathbb{Z}$ is prime ideal of $\mathbb{Z}$, $3\mathbb{Z}$ is prime ideal of $\mathbb{Z}$
But $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$ is not prime ideal of $\mathbb{Z}$.
s.t. $2 \cdot 3 = 6 \in 6\mathbb{Z}$ But $2 \notin 6\mathbb{Z}$ & $3 \notin 6\mathbb{Z}$

H.P.

13/08/16

Q.No: Show that $\mathbb{Z}[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b \in \mathbb{Z}\}$ is an integral domain but not field.

Soln:- $\mathbb{Z}[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b \in \mathbb{Z}\}$ —— (1)

$\phi \neq \mathbb{Z}\sqrt{2} \subseteq \mathbb{R}$

(i) Let $x = a+b\sqrt{2} \in \mathbb{Z}\sqrt{2}$ and $y = c+d\sqrt{2} \in \mathbb{Z}\sqrt{2}$ where $a,b,c,d \in \mathbb{Z}$

s.t. $x-y = (a+b\sqrt{2}) - (c+d\sqrt{2})$

$= (a-c)+(b-d)\sqrt{2}$

$= a'+b'\sqrt{2} \in \mathbb{Z}\sqrt{2}$

(ii) Let $x = a+b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$, $a,b \in \mathbb{Z}$

& $y = c+d\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$, $c,d \in \mathbb{Z}$

s.t. $x \cdot y = (a+b\sqrt{2}) \cdot (c+d\sqrt{2})$ where $a_1 = ac+2bc \in \mathbb{Z}$

$= (ac+2bd)+(ad+bc)\sqrt{2}$ $\quad b_1 = ad+bc \in \mathbb{Z}$

$= a_1 + b_1\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$

$\Rightarrow x \cdot y \in \mathbb{Z}[\sqrt{2}]$

then $\mathbb{Z}[\sqrt{2}]$ is subring of $\mathbb{R}$

Since $(\mathbb{R},+,\cdot)$ is commutative Ring and $(\mathbb{Z}[\sqrt{2}],+,\cdot)$ is subring of $(\mathbb{R},+,\cdot)$ then $(\mathbb{Z}[\sqrt{2}],+,\cdot)$ is commutative ring.

Now $1 = 1+0\sqrt{2} \in \mathbb{Z}\sqrt{2}$ s.t. $x \cdot 1 = 1 \cdot x = x$, $\forall\ x \in \mathbb{Z}[\sqrt{2}]$

then $(\mathbb{Z}[\sqrt{2}],+,\cdot)$ is commutative Ring with unity 1.

Next

$\mathbb{Z}[\sqrt{2}] \subseteq \mathbb{R}$ and $\mathbb{R}$ is an integral domain then $\mathbb{Z}[\sqrt{2}]$ is an integral domain.

Now show that $\mathbb{Z}[\sqrt{2}]$ is not field.

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$2 \in \mathbb{Z}[\sqrt{2}]$ But $\frac{1}{2} \notin \mathbb{Z}[\sqrt{2}]$ s.t. $2 \cdot \frac{1}{2} = 1$
then $\mathbb{Z}[\sqrt{2}]$ is not field.

Similarly $\mathbb{Z}[\sqrt{3}]$, $\mathbb{Z}[\sqrt{5}]$, $\mathbb{Z}[\sqrt{6}]$, $\mathbb{Z}[\sqrt{-1}]$, $\mathbb{Z}[\sqrt{-2}]$ ... are integral domain but not field.

Note:- $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$, where d is not perfect square then $\mathbb{Z}[\sqrt{d}]$ is an integral domain but not field.

Q.No. How many Units in $\mathbb{Z}[\sqrt{2}]$?

Soln

$1+\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ and $-1+\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ s.t.

$$(1+\sqrt{2})(-1+\sqrt{2}) = 1 \quad \text{---} \quad (\mathrm{I})$$

then $(1+\sqrt{2})$ & $(-1+\sqrt{2})$ also units of $\mathbb{Z}[\sqrt{2}]$

Now squaring eqn (I), we get

$$(1+\sqrt{2})^2(-1+\sqrt{2})^2 = 1 \quad \text{---} \quad (\mathrm{II})$$

Since $\mathbb{Z}[\sqrt{2}]$ is an integral domain then $(1+\sqrt{2})^2 \neq (1+\sqrt{2})$

Similarly,

$((1+\sqrt{2})^2)^2 \neq (1+\sqrt{2})^2 \neq (1+\sqrt{2})$ & $((1+\sqrt{2})^2)^2((-1+\sqrt{2})^2)^2 = 1$ and so on.

then $\mathbb{Z}[\sqrt{2}]$ has infinite no. of units.

Similarly $\mathbb{Z}[\sqrt{3}]$, $\mathbb{Z}[\sqrt{5}]$, $\mathbb{Z}[\sqrt{6}]$, ... has infinite no. of units.

H.W. Imp

Q: $\mathbb{Z}[\sqrt{-2}] = \{a+b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$, how many units in $\mathbb{Z}[\sqrt{-2}]$?

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Ques: Show that $Q[\sqrt{2}] = \{a+b\sqrt{2} \mid a, b \in Q\}$ is field.

Solⁿ:- $Q[\sqrt{2}] = \{a+b\sqrt{2} \mid a, b \in Q\}$

Q से छोटा field नहीं होगा

$\phi \neq Q[\sqrt{2}] \subseteq \mathbb{R}$

(i) Let $x = a+b\sqrt{2} \in Q[\sqrt{2}]$ and $y = c+d\sqrt{2} \in Q[\sqrt{2}]$

s.t. $x-y = (a-c)+(b-d)\sqrt{2}$

$= a'+b'\sqrt{2} \in Q[\sqrt{2}]$

$\Rightarrow x-y \in Q[\sqrt{2}]$

(ii) $x \cdot y = (a+b\sqrt{2})(c+d\sqrt{2})$

$= (ac+2bd)+(ad+bc)\sqrt{2} \in Q[\sqrt{2}]$

$\Rightarrow x \cdot y \in Q[\sqrt{2}]$

then $Q[\sqrt{2}]$ is subring of $\mathbb{R}$

Since $\mathbb{R}$ is commutative Ring and $Q[\sqrt{2}]$ is subring of $\mathbb{R}$ then $Q[\sqrt{2}]$ is commutative.

Now $1 = 1+0\sqrt{2} \in Q[\sqrt{2}]$ then $Q[\sqrt{2}]$ is commutative ring with unity 1.

Next, $\mathbb{R}$ is an integral domain and $Q[\sqrt{2}]$ is subring of $\mathbb{R}$.

then $Q[\sqrt{2}]$ is an integral domain.

Let $x \neq 0$, $0 \neq x = a+b\sqrt{2} \in Q[\sqrt{2}]$ then $x^{-1} = (a+b\sqrt{2})^{-1}$

$= \frac{1}{(a+b\sqrt{2})}$

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$$\frac{1}{(a+b\sqrt{2})}\frac{(a-b\sqrt{2})}{(a-b\sqrt{2})} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \left(\frac{a}{a^2-2b^2}\right) + \left(\frac{-b}{a^2-2b^2}\right)\sqrt{2} \in Q\sqrt{2}$$

where $\frac{a}{a^2-2b^2} \in Q$ and $\left(\frac{-b}{a^2-2b^2}\right) \in Q$

s.t. $xx^{-1}=1$ then each non zero elements of $Q[\sqrt{2}]$ has multiplicative inverse.

then $Q[\sqrt{2}]$ is field.

Q.No. $F = Q(\sqrt{2})$, Find ideals of $Q[\sqrt{2}]$

Soln $Q(\sqrt{2})$ is field then $Q(\sqrt{2})$ has exactly two ideal say

$I_1 = \{0\}$

$I_2 = Q(\sqrt{2})$

Similarly $Q\sqrt{2}$, $Q[\sqrt{3}]$, $Q[\sqrt{5}]$, $Q[\sqrt{7}]$ - - - are field.

H.W. Q.No. Show that $Q[i] = \{a+ib \mid a,b \in Q\}$ is field.

Hint:- $\phi \neq Q[i] \subseteq C$

$Q[i]$ is commutative ring with unity 1.

$a+ib \in Q[i]$ s.t. $x^{-1} = (a+ib)^{-1} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$

$$\Rightarrow x^{-1} = \frac{a}{a^2+b^2} + i\left(\frac{-b}{a^2+b^2}\right) \in Q[i]$$

then each non zero element of $Q[i]$ has multiplicative inverse then $Q[i]$ is field.

Q.No. $k = Q \times Q \times Q[i]$, how many prime ideal and maximal Ideal.

Soln

$I_1 = \{0\} \times Q \times Q[i]$

$I_2 = Q \times \{0\} \times Q[i]$ are maximal ideal & prime ideals.

$I_3 = Q \times Q \times \{0\}$

Note:- $Q[\sqrt{d}] = \{a+b\sqrt{d} \mid a,b \in Q\}$, where d is not perfect square then $Q[\sqrt{d}]$ is field.

H.W.

Q.No. (i) Show that $Q[\sqrt{2},\sqrt{3}] = \{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\cdot\sqrt{3} \mid a,b,c,d \in Q\}$ 1

(ii) Show that $Q[\sqrt{2},\sqrt{3},\sqrt{5}]$ is field. $\Rightarrow$ (on last Page)

Sol^n $Q[\sqrt{2},\sqrt{3}] = \{a+\overset{8\text{variables}}{b\sqrt{2}+c\sqrt{3}}+d\sqrt{2}\sqrt{3} \mid a,b,c,d \in Q\}$

let $\phi \neq Q[\sqrt{2},\sqrt{3}] \subseteq \mathbb{R}$

1stly, show that $Q[\sqrt{2},\sqrt{3}]$ is subring of $\mathbb{R}$

(I) let $x = a_1+b_1\sqrt{2}+c_1\sqrt{3}+d_1\sqrt{2}\sqrt{3} \in Q[\sqrt{2},\sqrt{3}]$

$y = a_2+b_2\sqrt{2}+c_2\sqrt{3}+d_2\sqrt{2}\sqrt{3} \in Q[\sqrt{2},\sqrt{3}]$

S.t. $x-y = (a_1-a_2)+(b_1-b_2)\sqrt{2}+(c_1-c_2)\sqrt{3}+(d_1-d_2)\sqrt{2}\sqrt{3}$

$= a'+b'\sqrt{2}+c'\sqrt{3}+d'\sqrt{2}\sqrt{3} \in Q[\sqrt{2},\sqrt{3}]$

(II) $x\cdot y = (a_1+b_1\sqrt{2}+c_1\sqrt{3}+d_1\sqrt{2}\sqrt{3})\cdot(a_2+b_2\sqrt{2}+c_2\sqrt{3}+d_2\sqrt{2}\sqrt{3})$

$\Rightarrow x\cdot y \in Q[\sqrt{2},\sqrt{3}]$

then $Q[\sqrt{2},\sqrt{3}]$ is subring of $\mathbb{R}$

Since $\mathbb{R}$ is common ring and $Q[\sqrt{2},\sqrt{3}]$ is subring of R then $Q[\sqrt{2},\sqrt{3}]$ is common ring.

Now, $1 = 1+0\sqrt{2}+0\sqrt{3}+0\sqrt{2}\sqrt{3} \in Q[\sqrt{2},\sqrt{3}]$

$\Rightarrow 1 \in Q[\sqrt{2},\sqrt{3}]$

i.e. $Q[\sqrt{2},\sqrt{3}]$ is common ring with unity

Since $\mathbb{R}$ is and ID and $Q[\sqrt{2},\sqrt{3}]$ is subring of $\mathbb{R}$ then $Q[\sqrt{2},\sqrt{3}]$ is integral domain.

Now let $0 \neq x = a+b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3} \in Q[\sqrt{2},\sqrt{3}]$

then $x^{-1} = (a+b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3})^{-1}$

$$= \frac{1}{(a+b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3})} \times \frac{(a-(b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3}))}{(a-(b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3}))}$$

$$= \frac{a-(b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3})}{a^2-(b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3})} \in Q[\sqrt{2},\sqrt{3}]$$

S.t. $xx^{-1} = 1$

then each non zero element of $Q[\sqrt{2}]$ has multiplicative

Factor Ring :- let R be a ring and I is an ideal of R. then the set $\frac{R}{I} = \{a+I \mid a \in R\}$ is ring with respect to

(i) $(a+I)+(b+I) = (a+b)+I$

(ii) $(a+I)\cdot(b+I) = a\cdot b+I$

Q.No. Construct factor ring $\frac{\mathbb{Z}}{6\mathbb{Z}}$.

Soln $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}$ and $6\mathbb{Z} = \{0, \pm6, \pm12, \ldots\}$ is an ideal of $\mathbb{Z}$ then $\frac{\mathbb{Z}}{6\mathbb{Z}}$ is exist.

Now $\frac{\mathbb{Z}}{6\mathbb{Z}} = \{0+6\mathbb{Z}, 1+6\mathbb{Z}, 2+6\mathbb{Z}, \ldots 5+6\mathbb{Z}\}$

$\approx \mathbb{Z}_6$ ($\because \mathbb{Z}_6$ is not integral domain)

Q. If R is commutative Ring then $\frac{R}{I}$ is also commutative.

Solnº- Let R be a commutative Ring and I is an ideal of R. Now $\frac{R}{I} = \{a+I \mid a \in R\}$

let $x = a+I \in \frac{R}{I}$ and $y = b+I \in \frac{R}{I}$ where $b \in R$

S.t. $x\cdot y = (a+I)\cdot(b+I) = ab+I$ ($a, b \in R$ and R is commutative then $a\cdot b = b\cdot a$)

$= ba+I$

$= (b+I)(a+I)$

$= y\cdot x$

$\Rightarrow x\cdot y = y\cdot x$

then $\frac{R}{I}$ is commutative Ring.

Q.No. Construct $\frac{\mathbb{Z}}{4\mathbb{Z}}$.

Soln $\frac{\mathbb{Z}}{4\mathbb{Z}} = \{0+4\mathbb{Z}, 1+4\mathbb{Z}, 2+4\mathbb{Z}, 3+4\mathbb{Z}\}$

$\mathbb{Z}$ is commutative Ring then $\frac{\mathbb{Z}}{4\mathbb{Z}}$ is commutative.

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$\Rightarrow 1+4\mathbb{Z} \in \frac{\mathbb{Z}}{4\mathbb{Z}}$ s.t. $(a+4\mathbb{Z})\cdot(1+4\mathbb{Z}) = a+4\mathbb{Z} \in \frac{\mathbb{Z}}{4\mathbb{Z}}$

then $1+4\mathbb{Z}$ is unity of $\frac{\mathbb{Z}}{4\mathbb{Z}}$.

Now $2+4\mathbb{Z} \in \frac{\mathbb{Z}}{4\mathbb{Z}}$ s.t. $(2+4\mathbb{Z})(2+4\mathbb{Z}) = 0+4\mathbb{Z}$, then $\frac{\mathbb{Z}}{4\mathbb{Z}}$ is not integral domain.

$$\Rightarrow \boxed{\frac{\mathbb{Z}}{4\mathbb{Z}} \approx \mathbb{Z}_4}$$

Note:-

(i) $\frac{\mathbb{Z}}{m\mathbb{Z}} \approx \mathbb{Z}_m$, $m>0$

(ii) $\frac{\mathbb{Z}}{\{0\}} \approx \mathbb{Z}$

(iii) $\frac{\mathbb{Z}}{\mathbb{Z}} = \{0+\mathbb{Z}\} \approx \{0\}$

Q.No. Construct $\frac{3\mathbb{Z}}{9\mathbb{Z}}$.

Soln:- Now show that $9\mathbb{Z}$ is an ideal in $3\mathbb{Z}$

$9\mathbb{Z} \subseteq 3\mathbb{Z} \subseteq \mathbb{Z}$

$9\mathbb{Z}$ is an ideal in $\mathbb{Z}$ and $3\mathbb{Z} \subseteq \mathbb{Z}$ then $9\mathbb{Z}$ is an ideal in $3\mathbb{Z}$.

then $\frac{3\mathbb{Z}}{9\mathbb{Z}}$ is exist

$\therefore \frac{3\mathbb{Z}}{9\mathbb{Z}} = \{0+9\mathbb{Z}, 3+9\mathbb{Z}, 6+9\mathbb{Z}\}$

$\Rightarrow o\left(\frac{3\mathbb{Z}}{9\mathbb{Z}}\right) = 3$

Now $3+9\mathbb{Z}$ is not zero element

$0 \neq 3+9\mathbb{Z} \in \frac{3\mathbb{Z}}{9\mathbb{Z}}$ and $0 \neq 6+9\mathbb{Z} \in \frac{3\mathbb{Z}}{9\mathbb{Z}}$

s.t. $(3+9\mathbb{Z})(6+9\mathbb{Z}) = 18+9\mathbb{Z} = 0+9\mathbb{Z} = 0$

then $\frac{3\mathbb{Z}}{9\mathbb{Z}}$ is not integral domain but $o\left(\frac{3\mathbb{Z}}{9\mathbb{Z}}\right)=3$

Note:- Let R be a Ring and $o(R)=p$ then R need not be a field.

Q.No. Construct Ring of order 5 which is not integral domain.

Soln:-

Hint:- $\frac{5\mathbb{Z}}{25\mathbb{Z}} = \{0+25\mathbb{Z},\ 5+25\mathbb{Z},\ 10+25\mathbb{Z},\ 15+25\mathbb{Z},\ 20+25\mathbb{Z}\}$

is ring of order 5 but not integral domain.

Q. Construct factor ring of Q.

Soln Since Q is field then Q has exactly two ideals.

$I_1=\{0\}$

$I_2=Q$

then factor ring of Q

(I) $\frac{Q}{\{0\}} \approx Q$

(ii) $\frac{Q}{Q} = \{0+Q\} \approx \{0\}$

Note: if F is field then F has exactly two factor rings. say $\frac{F}{\{0\}} \approx F$ and $\frac{F}{F} \approx \{0\}$

Q.No. Construct factor ring $\frac{\mathbb{Z}[i]}{\langle 1-i \rangle}$

Soln

$\frac{\mathbb{Z}[i]}{\langle (1-i) \rangle} = \{x+\langle (1-i) \rangle \mid x \in \mathbb{Z}[i]\}$

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$= \{a+ib+\langle(1-i)\rangle \mid a,b \in \mathbb{Z}\}$ —— (*)

Now $1-i+\langle(1-i)\rangle = 0+\langle(1-i)\rangle$

$\Rightarrow 1-i=0$ in $\frac{\mathbb{Z}[i]}{\langle(1-i)\rangle}$ i.e. $\frac{\mathbb{Z}[i]}{\langle(1-i)\rangle}$

$\Rightarrow i=1$ —— (1)

$\Rightarrow i^2=1^2$

$\Rightarrow -1=1$

$\Rightarrow 2=0$ —— (11)

from equn (*), (1) and (11), we get

$$\frac{\mathbb{Z}[i]}{\langle(1-i)\rangle} = \{0+\langle(1-i)\rangle, 1+\langle(1-i)\rangle\}.$$

$$\approx \mathbb{Z}_2$$

Q. No. Construct $\frac{\mathbb{Z}[i]}{\langle(-1-i)\rangle}$.

Soln

$$\frac{\mathbb{Z}[i]}{\langle(-1-i)\rangle} = \{x+\langle(-1-i)\rangle \mid x \in \mathbb{Z}[i]\}$$

$$= \{a+ib+\langle(-1-i)\rangle \mid a,b \in \mathbb{Z}\}$$

Now

$(-1-i)+\langle(-1-i)\rangle = 0+\langle(-1-i)\rangle$

$\Rightarrow -1-i=0$

$\Rightarrow i=-1$ —— (1) from equn (1) we get $i=1$ —— (1)'

$\Rightarrow i^2=(-1)^2$

$\Rightarrow -1=1$

$\Rightarrow 2=0$ —— (11)

Now from equn (*), (1)' and (11) we get

$$\frac{\mathbb{Z}[i]}{\langle(-1-i)\rangle} = \{0+\langle(-1-i)\rangle, 1+\langle(-1-i)\rangle\} \approx \mathbb{Z}_2$$

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Note: (i) $O\left(\frac{\mathbb{Z}[i]}{\langle(a+ib)\rangle}\right) = a^2+b^2$

(ii) $\frac{\mathbb{Z}[i]}{\langle a+ib\rangle} \approx \frac{\mathbb{Z}[i]}{\langle a-ib\rangle} \approx \frac{\mathbb{Z}[i]}{\langle -a+ib\rangle} \approx \frac{\mathbb{Z}[i]}{\langle -a-ib\rangle}$

(iii) $\frac{\mathbb{Z}[i]}{\{0\}} \approx \mathbb{Z}[i]$

(iv) $\frac{\mathbb{Z}[i]}{\mathbb{Z}[i]} \approx \{0\} = \{0+\mathbb{Z}[i]\}$

Q.No. Construct $\frac{\mathbb{Z}[i]}{\langle(3-i)\rangle} = ?$ Ans $\mathbb{Z}_{10}$

Soln:- $\frac{\mathbb{Z}[i]}{\langle(3-i)\rangle} = \{a+ib+\langle(3-i)\rangle \mid a,b\in\mathbb{Z}\}$ —— ✱

Now $(3-i)+\langle(3-i)\rangle = 0+\langle(3-i)\rangle$

$\Rightarrow 3-i=0$

$\Rightarrow i=3$ —— (i)

$\Rightarrow i^2=3^2$

$\Rightarrow -1=9$

$\Rightarrow 10=0$ —— (ii)

$i=3$

$\{a+3b \mid a,b\in\mathbb{Z}\}$

from equn (i), (ii) and ✱ we get

$\frac{\mathbb{Z}[i]}{\langle(3-i)\rangle} = \{0+\langle(3-i)\rangle, 1+\langle(3-i)\rangle, 2+\langle(3-i)\rangle, 3+\langle(3-i)\rangle, 4+\langle(3-i)\rangle, 5+\langle(3-i)\rangle, 7+\langle(3-i)\rangle, 9+\langle(3-i)\rangle\}$

$\approx \mathbb{Z}_{10}$

Q.No. Construct $\frac{\mathbb{Z}[i]}{\langle(2-i)\rangle}$

Soln Ans. $\left(\frac{\mathbb{Z}[i]}{\langle(2-i)\rangle} \approx \mathbb{Z}_5\right)$

$\frac{\mathbb{Z}[i]}{\langle(2-i)\rangle} = \{a+ib+\langle(2-i)\rangle \mid a,b\in\mathbb{Z}\}$ —— ✱

Now $(2-i)+\langle(2-i)\rangle = 0+\langle(2-i)\rangle$

$\Rightarrow 2-i=0$

$\Rightarrow i=2$ —— (i)

$i^2 = 2^2$

$-1=4$

$5=0$ —— (ii)

from (i), (ii) & (*)

$$\Rightarrow \frac{\mathbb{Z}[i]}{\langle(2-i)\rangle} \approx \mathbb{Z}_5$$

Note:- $\boxed{\dfrac{\mathbb{Z}[i]}{\langle a+ib\rangle} \approx \mathbb{Z}_{a^2+b^2} \text{ if } \gcd(a,b)=1}$

e.g. $\dfrac{\mathbb{Z}[i]}{\langle(2-i)\rangle} \approx \mathbb{Z}_{2^2+1^2} = \mathbb{Z}_5$

$$\frac{\mathbb{Z}[i]}{\langle(3-i)\rangle} \approx \mathbb{Z}_{3^2+1^2} = \mathbb{Z}_{10}$$

Q.No. Construction of $\dfrac{\mathbb{Z}[i]}{\langle 2\rangle}$

Sol^n $\dfrac{\mathbb{Z}[i]}{\langle 2\rangle} = \left\{ x+\langle 2\rangle \mid x\in \mathbb{Z}[i] \right\}$

$$\frac{\mathbb{Z}[i]}{\langle 2\rangle} = \left\{ a+ib+\langle 2\rangle \mid a,b\in \mathbb{Z} \right\} \text{ —— (*)}$$

Now $2+\langle 2\rangle = 0+\langle 2\rangle$

$\Rightarrow 2=0$ —— (i)

from equ^n (*) & (i), we get

$$\frac{\mathbb{Z}[i]}{\langle 2\rangle} = \left\{ 0+\langle 2\rangle,\ 1+\langle 2\rangle,\ i+\langle 2\rangle,\ 1+i+\langle 2\rangle \right\}$$

$\approx \mathbb{Z}_2[i]$ which is not field.

Q.No. Construct $\frac{\mathbb{Z}[i]}{<3>} \approx$

Soln

$$\frac{\mathbb{Z}[i]}{<3>} = \{a+ib+<3> \mid a,b \in \mathbb{Z}\}$$

$$= \{0+<3>, 1+<3>, 2+<3>, i+<3>, 2i+<3>, 1+i+<3>, 1+2i+<3>, 2+2i+<3>, 2+i+<3>\}$$

Note:- $\boxed{\frac{\mathbb{Z}[i]}{<n>} = \frac{\mathbb{Z}[i]}{n\mathbb{Z}[i]} \approx \mathbb{Z}_n[i]} \neq \frac{\mathbb{Z}[i]}{<ni>}$

$$\left[\begin{aligned} <2i> &= 2i\,\mathbb{Z}[i] \\ &= 2\,\mathbb{Z}[i] \\ &= <2> \end{aligned} \right.$$

e.g. $<2i> = 2i\,\mathbb{Z}[i]$ $i \in \mathbb{Z}[i]$

then $i\mathbb{Z}[i] = \mathbb{Z}[i]$ because $(-i) \in \mathbb{Z}[i]$ s.t.

$$i(-i) \in i\mathbb{Z}[i]$$
$$\Rightarrow 1 \in i\mathbb{Z}[i]$$
$$\Rightarrow i\mathbb{Z}[i] = \mathbb{Z}[i]$$

$$\begin{aligned} <2i> &= 2i\,\mathbb{Z}[i] \\ &= 2\,\mathbb{Z}[i] \\ &= <2> \end{aligned}$$

$\Rightarrow <2i> = <2>$ in $\mathbb{Z}[i]$

Q.No. which of the $\frac{\mathbb{Z}}{n\mathbb{Z}[i]}$ is an integral domain for which value of

(I) $n=5$ (II) $n=7$ (III) $n=13$

(IV) $n=19$

Soln $\frac{\mathbb{Z}[i]}{n\mathbb{Z}[i]} \approx \mathbb{Z}_n[i]$

(I) $n=5$ then

$\frac{\mathbb{Z}[i]}{5\mathbb{Z}[i]} \approx \mathbb{Z}_5[i]$ $4 \times 5\text{-}3$ then $\frac{\mathbb{Z}[i]}{5\mathbb{Z}[i]}$ is not integral domain.

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(ii) $n=7$ then

$Z_7[i]$ is an integral domain.

(iv) $Z_{19}[i]$ is an integral domain.

(iii) $Z_{13}[i]$ is not an integral domain

$\because 4 \nmid 13-3$

H.W.

Q.No. $R = \frac{Z[i]}{\langle 2+3i \rangle}$, how many maximal ideal and prime ideal and also find all ideal of $\frac{Z[i]}{\langle (2+3i) \rangle}$

Soln

$$\frac{Z[i]}{\langle (2+3i) \rangle} = \{ a+ib+\langle (2+3i) \rangle \mid a,b \in Z \} \approx Z_{13}$$

Z_{13} is field then $\frac{Z[i]}{\langle (2+3i) \rangle}$ is also field then $\frac{Z[i]}{\langle (2+3i) \rangle}$ has exactly two ideals say $I_1 = \{0+\langle 2+3i \rangle\} \approx \{0\}$

and $I_2 = \frac{Z[i]}{\langle (2+3i) \rangle}$

then maximal and prime ideal of $\frac{Z[i]}{\langle (2+3i) \rangle} = \{a+\langle (2+3i) \rangle\}$

Q.No. Find Maximal and prime ideal of $\frac{Z[i]}{\langle (3+i) \rangle}$

Soln:-

$$\frac{Z[i]}{\langle 3+i \rangle} = \{ 0+\langle 3+i \rangle, 1+\langle (3+i) \rangle, 2+\langle (3+i) \rangle \cdots 9+\langle (3+i) \rangle \}$$

$$\approx Z_{10}$$

Z_{10} has 4 ideals then

$\frac{Z[i]}{\langle (3+i) \rangle}$ has 4 ideals

$I_1 = 0+\langle 3+i \rangle$

$I_2 = \langle 2+\langle 3+i \rangle \rangle = \{ 0+\langle 3+i \rangle, 2+\langle (3+i) \rangle, 3+\langle (3+i) \rangle, 4+\langle (3+i) \rangle, 6+\langle (3+i) \rangle, 8+\langle (3+i) \rangle$

$I_3 = \langle 5+\langle 3+i \rangle \rangle = \{ 0+\langle 3+i \rangle, 5+\langle (3+i) \rangle$

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(iv): $I_4 = \dfrac{\mathbb{Z}[i]}{\langle(3+i)\rangle} = \langle 1+\langle(3+i)\rangle\rangle$

then maximal & prime ideal I_2 and I_3.

H.W.
Q.No. (1) $R = \dfrac{\mathbb{Z}[i]}{\langle 2+i\rangle} \times \mathbb{Q}$, find maximal ideal of R

(ii) $R = \dfrac{\mathbb{Z}[i]}{\langle 3-i\rangle} \times \dfrac{\mathbb{Z}}{2\mathbb{Z}} \times \dfrac{\mathbb{Z}[i]}{\langle 7\rangle}$, find maximal ideal of R

1 Soln :- Since $\mathbb{Q}$ is field and field has 1 maximal ideal say $\{0\}$

and $\dfrac{\mathbb{Z}[i]}{\langle 2+i\rangle} = \left\{ 0+\langle 2+i\rangle, 1+\langle 2+i\rangle, 2+\langle 2+i\rangle, 3+\langle 2+i\rangle, 4+\langle 2+i\rangle \right\}$

$\approx \mathbb{Z}_5$ $\quad [\mathbb{Z}_5 \Rightarrow \text{field} \rightarrow 1 \text{ maximal ideal}]$

and $\mathbb{Z}_5$ is field. and field has 1 maximal ideal.

Hence $\dfrac{\mathbb{Z}[i]}{\langle 2+i\rangle}$ is maximal ideal say $\langle 2+i\rangle$

" $\dfrac{\mathbb{Z}[i]}{\langle 2+i\rangle} \times \mathbb{Q}$ having Maximal ideal are

$I_1 = \{0+\langle 2+i\rangle\} \times \mathbb{Q}$, $I_2 = \dfrac{\mathbb{Z}[i]}{\langle 2+i\rangle} \times \{0\}$

(2) $R = \dfrac{\mathbb{Z}[i]}{\langle 3-i\rangle} \times \dfrac{\mathbb{Z}}{2\mathbb{Z}} \times \dfrac{\mathbb{Z}[i]}{\langle 7\rangle}$

Since $\dfrac{\mathbb{Z}[i]}{\langle 3-i\rangle} = \dfrac{\mathbb{Z}[i]}{\langle(3+i)\rangle} \Rightarrow \dfrac{\mathbb{Z}[i]}{\langle 3-i\rangle} \approx \mathbb{Z}_{10}$ and

$\mathbb{Z}_{10}$ has 4 ideals, in which 2 ideals are maximal

say $I_1 = \langle 2+\langle 3-i\rangle\rangle$, $I_2 = \langle 5+\langle 3-i\rangle\rangle$

& $\dfrac{\mathbb{Z}}{2\mathbb{Z}} \approx \mathbb{Z}_2$ and $\mathbb{Z}_2$ is field $\Rightarrow \mathbb{Z}_2$ has 1 maximal ideal say $2\mathbb{Z}$.

& $\dfrac{\mathbb{Z}[i]}{\langle 7\rangle} \approx \mathbb{Z}_7[i]$ which is also field $\Rightarrow \mathbb{Z}_7[i]$ has one maximal ideal say $\langle 7\rangle$.

Hence maximal ideal are :-

$I_1' = \langle 2+\langle 3-i\rangle\rangle \times \mathbb{Z} \times \mathbb{Z}[i]$

$$I'' = \langle 5 + \langle 3-i\rangle\rangle \times \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}[i]}{\langle 7\rangle}$$

$$I''' = \frac{\mathbb{Z}[i]}{\langle 3-i\rangle} \times 2\mathbb{Z} \times \frac{\mathbb{Z}[i]}{\langle 7\rangle}$$

$$I^{iv} = \frac{\mathbb{Z}[i]}{\langle 3-i\rangle} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \times \langle 7\rangle$$

H.P.

Note:- $\frac{\mathbb{Z}[i]}{\langle a+ib\rangle}$ is not integral domain if $\gcd(a,b) \neq 1$ and $a \neq 0, b \neq 0$

e.g. $\frac{\mathbb{Z}[i]}{\langle 2+2i\rangle} \approx \mathbb{Z}_2 \times \mathbb{Z}_2[i]$

and $\mathbb{Z}_2 \times \mathbb{Z}_2[i]$ is not integral domain

$\frac{\mathbb{Z}[i]}{\langle 3+9i\rangle} \approx \mathbb{Z}_3[i] \times \mathbb{Z}_{10}$ is not Integral domain.

Imp

Theorem 1:- Let R be a commutative Ring with unity and A is an ideal of R. $\frac{R}{A}$ is an integral domain iff A is prime ideal.

Thm 2 Let R be a commutative Ring with unity and A is an ideal of R. $\frac{R}{A}$ is field iff A is maximal ideal.

e.g. $I = 2\mathbb{Z}$ is maximal ideal of $\mathbb{Z}$ and $\mathbb{Z}$ is commutative ring with unity then $\frac{\mathbb{Z}}{2\mathbb{Z}} \approx \mathbb{Z}_2$ is field.

$\frac{\mathbb{Z}}{m\mathbb{Z}} \approx \mathbb{Z}_m$ iff $m = p$

$\mathbb{Z}$ is field iff $m = p$ $\Rightarrow m\mathbb{Z}$ is maximal iff $m = p$

Q.No. $m\mathbb{Z}$ is maximal ideal of $\mathbb{Z}$ iff $m=p$

Soln $\frac{\mathbb{Z}}{m\mathbb{Z}} \approx \mathbb{Z}_m$

$\mathbb{Z}_m$ is field iff $m=p$

$\Rightarrow \frac{\mathbb{Z}}{m\mathbb{Z}}$ is field iff $m=p$

$\Rightarrow m\mathbb{Z}$ is maximal ideal of $\mathbb{Z}$ iff $m=p$.

Q. Show that $I=\{0\}$ is prime ideal but not maximal ideal in $\mathbb{Z}$.

Soln $\frac{\mathbb{Z}}{\{0\}} \approx \mathbb{Z}$

$\mathbb{Z}$ is an integral domain but not field.

then $I=\{0\}$ is prime Ideal but not maximal.

Q.No. If R is commutative Ring with Unity then every maximal ideal of R is prime ideal.

Soln: Let R be a commutative Ring with unity and A is any maximal ideal of R then $\frac{R}{A}$ is field.

$\Rightarrow \frac{R}{A}$ is an integral domain.

$\Rightarrow A$ is prime ideal of R.

Q.No. $I=\langle 2+i\rangle$ is maximal and prime ideal of $\mathbb{Z}[i]$?

Soln

$\frac{\mathbb{Z}[i]}{\langle 2+i\rangle} \approx \mathbb{Z}_5$

$\mathbb{Z}_5$ is field then $\frac{\mathbb{Z}[i]}{\langle(2+i)\rangle}$ is field

$\Rightarrow I=\langle(2+i)\rangle$ is maximal and prime ideal.

Q.No. $I=\langle 2\rangle$ is maximal & prime ideal of $\mathbb{Z}[i]$?

Soln $\frac{\mathbb{Z}[i]}{\langle 2\rangle} \approx \mathbb{Z}_2[i]$

$\mathbb{Z}_2[i]$ is not integral domain $\Rightarrow \mathbb{Z}_2[i]$ is not field.

$\Rightarrow I=\langle 2\rangle$ is not Maximal & prime ideal of $\mathbb{Z}[i]$

Q.No. Show that $I = \langle 11 \rangle$ is maximal and prime ideal of $\mathbb{Z}[i]$.

Soln: $\frac{\mathbb{Z}[i]}{\langle 11 \rangle} \simeq \mathbb{Z}_{11}[i]$

$\mathbb{Z}_{11}[i]$ is field then $I = \langle 11 \rangle$ is maximal and prime ideal.

Q.No. Show that $I = \langle 2+2i \rangle$ is not prime ideal and maximal ideal in $\mathbb{Z}[i]$.

Soln: $I = \langle 2+2i \rangle$ is an Ideal of $\mathbb{Z}[i]$

Let $a = 2 \in \mathbb{Z}[i]$ and $b = (1+i) \in \mathbb{Z}[i]$ s.t. $a \cdot b = 2(1+i)$

$= 2+2i \in I$

$= a \cdot b \in I$

But $2 \notin I$ and $(1+i) \notin I$

then $I = \langle (2+2i) \rangle$ is not prime ideal

$\Rightarrow \frac{\mathbb{Z}[i]}{\langle 2+2i \rangle}$ is not integral domain

$\Rightarrow \frac{\mathbb{Z}[i]}{\langle 2+2i \rangle}$ is not field

$\Rightarrow \frac{\mathbb{Z}[i]}{\langle 2+2i \rangle}$ is not maximal ideal in $\mathbb{Z}[i]$

Q.No. Show that $\mathbb{Q} \times \{0\}$ is maximal and prime ideal of $\mathbb{Q} \times \mathbb{R}$.

Soln $\frac{\mathbb{Q} \times \mathbb{R}}{\mathbb{Q} \times \{0\}} \approx \mathbb{R}$. $\mathbb{R}$ is field then $I = \mathbb{Q} \times \{0\}$ is maximal & prime ideal.

Q.No. $I = \{0\} \times \{0\}$ is neither maximal nor prime ideal of $\mathbb{Q} \times \mathbb{R}$.

Soln: $\frac{\mathbb{Q} \times \mathbb{R}}{\{0\} \times \{0\}} \approx \mathbb{Q} \times \mathbb{R} \to \mathbb{Q} \times \mathbb{R}$ is not integral domain

then $I = \{0\} \times \{0\}$ is not maximal and prime

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Q.No. if F is field then $I=\{0\}$ is maximal and prime ideal of F.

Soln $\frac{F}{\{0\}} \approx F \rightarrow F$ is field then $I=\{0\}$ is maximal and prime Ideal.

Characterstic of Ring :-

Characterstic of Ring $(R,+,\cdot)$ is the least +ve integer n s.t. $n\cdot a=0, \forall a\in R$

if such n does not exist then $char(R)=0$

Q.No. $char(\mathbb{Z})=?$

Soln $\mathbb{Z}=\{0,\pm1,\pm2,\ldots\}$

if $0\neq a\in\mathbb{Z}$ then $n\cdot a=0$ is not possible for any +ve integer n.

because $\mathbb{Z}$ is an integral domain

then $char(\mathbb{Z})=0$

Similarly, $char(\mathbb{Q})=char(\mathbb{R})=char(\mathbb{C})=char(\mathbb{Q}(\sqrt{2}))=$
$=char(\mathbb{Q}[\sqrt{2},\sqrt{3}])=0$

Q.No. $char(\mathbb{Z}_8)=?$

Soln: $n=8$ s.t. $8\cdot a=0, \forall a\in\mathbb{Z}_8$

then $char(\mathbb{Z}_8)=8$

Q.No. $char(\mathbb{Z}_1)=?$

Soln $\mathbb{Z}_1=\{0\}$

$n=1$ s.t. $1\cdot 0=0$ then $char(\mathbb{Z}_1)=1$

Note: $\boxed{char(\mathbb{Z}_n)=n}$

Q.No: char $(\mathbb{Z}[i]) = ?$

Sol^n If $0 \neq a \in \mathbb{Z}[i]$ then $n \cdot a = 0$ is not possible for any +ve integer n.

because $\mathbb{Z}[i]$ is an integral domain

then char $(\mathbb{Z}[i]) = 0$

Q.No. $R = \mathbb{Q} \times \mathbb{R}$ then char $(R) = ?$

Sol^n:- $(1,1) \in \mathbb{Q} \times \mathbb{R}$ s.t. $n(1,1) = 0$ is not possible.

then for any +ve integer n then

$\boxed{\text{char } (\mathbb{Q} \times \mathbb{R}) = 0}$

Note:-

let $1 \in R$ and $o(1) = n$ under addition then $\boxed{\text{char } (R) = n}$

if $o(1) = \infty$ under addition then $\boxed{\text{char}(\mathbb{R}) = 0}$

e.g.:-

(i) $1 \in \mathbb{Z}$ and $o(1) = \infty$ in $\mathbb{Z}$ under addition then char $(\mathbb{Z}) = 0$

(ii) $1 \in \mathbb{Q}$ and $o(1) = \infty$ under addition then char$(\mathbb{Q}) = 0$

" " " " " " char$(\mathbb{R}) = 0$

" " " " " " char$(\mathbb{C}) = 0$

Q.No. char$(\mathbb{Z}_n[i])$?

Sol^n Case I if $n = 1$ then $\mathbb{Z}_1[i] = \{0\}$

then char $[\mathbb{Z}_1[i]] = 1$

Case II: if $n > 1$ then $1 \in \mathbb{Z}_n[i]$ s.t. $n \cdot 1 = 0$

$\Rightarrow o(1) = n$

then char $(\mathbb{Z}_n[i]) = n$

from case (I) and (II)

$\therefore$ $char(Z_n[i]) = n$

Q. $R = Z_2 \times Z_6$ then $char(R) = ?$

Soln $(1,1) \in Z_2 \times Z_6$ s.t. $o(1,1) = LCM(o(1) \text{ in } Z_2, o(1) \text{ in } Z_6)$

$= LCM(2,6)$

$= 6$

then $char(Z_2 \times Z_6) = 6$

Note:

$$char(R \times S) = \begin{cases} 0 & \text{if } char(R) = 0 \text{ or } char(S) = 0 \\ K & \text{if } \underset{\text{where } K =}{LCM(char(R), char(S))} \end{cases}$$

For Example $R = Z_4 \times Z'$

$char(Z_4) = 4$ and $char(Z') = 0$

then $char(Z_4 \times Z') = 0$

$(1,1) \in Z_4 \times Z'$
s.t. $o(1,1) = \infty$
then $char(Z_4 \times Z') = 0$

Q.No. $char(R) = p$ then R is field?

Soln $char(Z_5[i]) = 5$ But $Z_5[i]$ is not field

Need not be field.

C.S.I.R 2013 if S is subring of R and $1 \in S$ then $char(S) = char(R)$

Soln $1 \in S$ and S is subring of R then $1 \in R$

if $o(1)$ in $R = n$ then $o(1) = n$ in S

then $char(R) = char(S) = n$

Note $\Rightarrow$ If $o(1)$ in $R = \infty$ then $o(1) = \infty$ in S then $char(S) = char(R)$

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Ring Homomorphism :- let $(R,+,\cdot)$ and $(S,+,\cdot)$ are two rings. A mapping $f:(R,+,\cdot) \to (S,+,\cdot)$ is said to be Ring homomorphism if

(1) $f(x+y) = f(x) + f(y)$

(2) $f(xy) = f(x) f(y)$, $x, y \in R$

Q.No. $f: Z_4 \to Z_{10}$ defined by $f(x) = 5x$ is ring homo. ?

Soln $f: Z_4 \to Z_{10}$ defined by

$f(x) = 5x$

(1) let $x, y \in Z_4$ s.t. $f(x+y) = 5(x+y) = 5x + 5y$

$= f(x) + f(y)$

$= f(x+y) = f(x) + f(y)$, $\forall\ x, y \in Z_4$

(II) Let $x, y \in Z_4$ s.t. $f(x \cdot y) = 5 \cdot xy$

$= 25xy$

$= 5x \cdot 5y$

$= f(x) \cdot f(y)$, $\forall\ x, y \in Z_4$

then it is ring homo-morphism.

Q.No. $f: Z_5 \to Z_{10}$ defined by

Soln $f(x) = 5x$

(1) $f(4+1) = f(5) = f(0) = 5 \cdot 0 = 0$

Now $f(4) + f(1) = 5 \cdot 4 + 5 \cdot 1 = 25 = 5$

$0 \neq 5$ in Z_{10}

then $f(x) = 5x$ is not ring homomorphism.

Q.No. $f: Z \to Z$ defined by

$f(x) = 0x$ is ring homomorphism ?

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Solⁿ: let $x, y \in \mathbb{Z}$ s.t. $f(x+y) = 0(x+y)$
$= 0(x) + 0(y)$
$= f(x) + f(y)$

(ii) $f(x \cdot y) = 0 x \cdot y$ then $f(x) = 0 \cdot x$ is Ring homomorphism.

Q. $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 1x$ is ring homomorphism.

Solⁿ: Let $x, y \in \mathbb{Z}$ s.t. $f(x+y) = x+y$
$= f(x) + f(y)$

(ii) $f(x \cdot y) = 1(x \cdot y) = f(x) f(y)$ is ring homo.

Q: $f: \mathbb{Z} \to \mathbb{Z}$ defined by
$f(x) = 2x$ is ring homomorphism?

Solⁿ: $2 \in \mathbb{Z}$ But $2^2 \neq 2$ then $f(x) = 2x$ is not ring homomorphism.

Note: $f: \mathbb{Z} \to \mathbb{Z}$ has exactly two ring homomorphism.
Say $f(x) = 0x$ and $f(x) = 1x$.

Trivial Ring Homomorphism:- A mapping $f: R \to S$ is defined by $f(x) = 0$, is called Trivial Ring Homomorphism.

For Example:-
$f: \mathbb{Z}_8 \to \mathbb{Z}_{12}$ defined by $f(x) = 0x$ is trivial ring homomorphism.

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Q.No. $f: \mathbb{Z}_{12} \to \mathbb{Z}_{28}$, how many non trivial Ring Homomorphism.

(I) 1 (II) 3 (III) 4 (iv) 5

Soln:

Z_{28} has exactly 4 idemptotent elements. then

$f: Z_{12} \to Z_{28}$ has atmost 4 ring homomorphism.

⇒ " " " 3 " "

then (III) & (iv) are not possible

$1 \in Z_{28}$ s.t. $1^2 = 1$ then 1 is idemptotes element of Z_{28}

$f(x) = 1x$, $o(1)$ in $Z_{28} = 28$ But Z_{12} has no element of order 28 then $f(x) = 1x$ is not ring homomorphism.

then (II) is also not possible.

Hence ① is true.

Q.No. $f: Z_4 \to Z_{10}$, how many Ring homomorphism?

Soln Idemptotets element of Z_{10} are 0, 1, 5 and 6

then possible ring homomorphism

$f(x) = 0x$ is ring homomorphism

$f(x) = 1x$ is not ring homo.

$f(x) = 5x$ is ring homo.

$f(x) = 6x$ is not ring homo.

then $f(x) = 6x$ and $f(x) = 5x$ are ring homomorphism.

then $f: Z_4 \to Z_{10}$ has exactly 2 ring homomorphism.

How

Q.No. (i) $f: Z_5 \to Z_{10}$

(ii) $f: Z_{20} \to Z_{30}$

how many ring homomorphism.

(i) Soln:- No. of Idemptent = 4

Possible ring homo. $f(x) = 0.x$, $f(x) = 1x$, $f(x) = 2x$, $f(x) = 5x$.

① $f(x) = 0.x \to$ ring homo.

(ii) $f(x) = 1.x \to o(1)$ in $Z_{10} = 10$

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but Z_5 has no element of order 10. So it not ring homo.

(iii) $f(x) = 2x \to$ ring homo.

(iv) $f(x) = 5x \to$ not ring homo. ($\because o(5) = 2$ in Z_{10} but Z_5 has no element of order 2.

So, $f: Z_5 \to Z_{10}$ has exactly 2 ring homo.

Sol^n (ii) No. of idemp. element in $Z_{30} = 2^3 = 8$

Possible ring homo.

(i) $f(x) = 0 \cdot x \longrightarrow$ ring homo.

(ii) $f(x) = 1 \cdot x \longrightarrow$ " "

(iii) $f(x) = 6x \longrightarrow$ " " $\quad$ ($\because 6^2 = 36 = 6$

(iv) $f(x) = 10x \longrightarrow$ " " $\quad$ ($1 - 6 = -5$, $30 - 5 = 25$

(v) $f(x) = 15x \to$ " " $\quad$ ($10^2 = 100 = 10$, $1 - 10 = -9$, $30 - 9 = 21$

(vi) $f(x) = 16x \to$ " " $\quad$ ($15^2 = 225 = 15$, $1 - 15 = -14 = 16$

(vii) $f(x) = 21x \to$ " "

(viii) $f(x) = 25x \to$ " "

H.P.

Q: $f: Z_{12} \to Z_{12}$, how many Ring homomorphism?

Sol^n:- Possible ring homo. of $f: Z_{12} \to Z_{12}$ are

$f(x) = 0x \to$ trivial ring homo.

$f(x) = 1x \to o(1)$ in $Z_{12} = 12$ and Z_{12} has Elements of order 12.

$f(x) = 4x$ $\quad$ then $f(x) = 1x$ is ring homo.

$f(x) = 9x$

(ii) $o(4)$ in Z_{12} is 3. and Z_{12} has element of order 3.

(iii) $o(9) = 4$ in Z_{12} and Z_{12} has element of order 4

then $f(x) = 4x, 9x$ is ring homo.

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Note: $f: Z_m \to Z_n$ has exactly 2^d ring homo. where d is prime no.

For e.g. $f: Z_{20} \to Z_{20}$ how many ring homo.?

Soln # of ring homo. = # of idempotents in Z_{20}

$= 2^2 = 4$

Part B
H. Imp.

Q.No. $f: Z_p \to Z_{p^m}$, $m > 1$ then

1. exactly two ring homo.
2. exactly one ring homo.
3. At most two ring homo.
4. Atleast two ring homo.

Soln idempotents elements of Z_{p^m} are 0 and 1

then possible ring homo. are $f(x) = 0x$

$f(x) = 1x$

$o(1) = p^m$ in Z_{p^m} if $m > 1$

then $p^m > p$ then Z_p has no elements of order p^m.

then $f(x) = 1x$ is not ring homomorphism.

then $f: Z_p \to Z_{p^m}$, $m > 1$ has exactly one ring homo.

H.W. e.g: $f: Z_2 \to Z_{25}$, then it has " " " "

Q.No. $f: Z_{p^m} \to Z_p$, $m > 1$, how many ring homo.

Ans. $\boxed{\text{exactly 2}}$

Since Z_p is field, then idempotent element of Z_p are {0} and 1.

then possible ring homo.

$f(x) = 0 \cdot x$ & $f(x) = 1x$

then $o(1) = p$ in Z_p and $p^m > p$ then Z_{p^m} has an elements of order p. and Z_{p^m} has also an elements of order 1.

So exactly 2 ring homo. $\Rightarrow$ So exactly 2 ring homo.

Q.No. $f: \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$, how many ring homo.?

Solⁿ:- Idempotent elements of $\mathbb{Z}\times\mathbb{Z}$ are $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$

then $f(x) = (0,0)x$ $\quad f(x) = (1,0)x$
$f(x) = (0,1)x$ $\quad f(x) = (1,1)x$ are ring homo.

Q.No. $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, how many ring homo.?

Solⁿ Idemptotes elements of $\mathbb{Z}$ are 0 and 1

$f(x,y)=0$ $\quad f(x,y) = 1x \Rightarrow \ker f = \{0\}\times\mathbb{Z}$
$\Downarrow$ $\quad f(x,y) = 1y \Rightarrow \ker f = \mathbb{Z}\times\{0\}$
$\ker f = \mathbb{Z}\times\mathbb{Z}$

then $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ has exactly three ring homo.

Q.No. Find ring homomorphism $f: \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$

Solⁿ $f(x,y,z)=0 \to \ker f = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$
$f(x,y,z) = 1x \to \ker f = \{0\}\times\mathbb{Z}\times\mathbb{Z}$
$f(x,y,z) = 1y \Rightarrow \ker f = \mathbb{Z}\times\{0\}\times\mathbb{Z}$
$f(x,y,z) = 1z \Rightarrow \ker f = \mathbb{Z}\times\mathbb{Z}\times\{0\}$

Q.No. If R is commutative Ring with characteristic 2 then $f: R\to R$ defined by $f(x) = x^2$ is ring homo.

Solⁿ $f: R\to R$ defined by
$f(x) = x^2$ and R is commutative Ring with
$\text{char}(R) = 2$

① $f(x+y) = (x+y)^2$
$= x^2+y^2+2xy$ $\qquad 2xy = 0 \quad \therefore \text{char}(R) = 2$

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$\therefore f(x+y) = f(x) + f(y)$

(ii) $f(x\cdot y) = (x\cdot y)^2 = x^2 y^2 = f(x)\cdot f(y)$ ($\because$ R is commutative Ring.

$\Rightarrow \boxed{f(x\cdot y) = f(x)\cdot f(y)}$

then f is ring homo.

<u>Q.No.</u> $f: Z_2 \to Z_2$ is defined by
$f(x) = x^2 - x$, $\forall x \in Z_2$ is ring homo.

<u>Soln</u> $f(x) = x^2 - x = 0$, $\forall x \in Z_2$
then $f(x) = x^2 - x$ is trivial ring homo.

<u>Another Method</u>

(i) Let $x, y \in Z_2$

S.t. $f(x+y) = (x+y)^2 - (x+y)$

$= x^2 + y^2 + 2xy - x - y$

$= (x^2 - x) + (y^2 - y)$ ($\because 2xy = 0$ char $(Z_2) = 2$

$= f(x) + f(y)$

(ii) $f(xy) = (xy)^2 - xy$

$= x^2 y^2 - xy$

$= (x^2 - x)(y^2 - y) + x^2 y + xy^2 - 2xy$ [$\because x^2y^2 - x^2y - xy^2 + xy$]

$= (x^2 - x)(y^2 - y) + xy(x+y) + 0$

$= (x^2 - x)(y^2 - y) + 0$ ($\because$ $x+y$ is always zero.)

$= f(x)\, f(y)$

then f is ring homo.

$x = 0, 1$
$y = 0, 1$
($xy(x+y) = 0$, $\forall x, y \in Z_2$

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